6 Insurances on Joint Lives

6.1 Introduction

It is common for life insurance policies and annuities to depend on the death or survival of **more than one life**. For example:

(i) A policy which pays a monthly benefit to a wife or other dependents after the death of the husband **(widow's or dependent's pension).**

(ii) A policy which pays a lump-sum on the second death of a couple **(often to meet inheritance tax liability).**

We will confine attention to policies involving **two lives**but the same approaches can be extended to any number of lives. We assume that policies are sold to a life age x and a life age y, denoted (x) and (y).

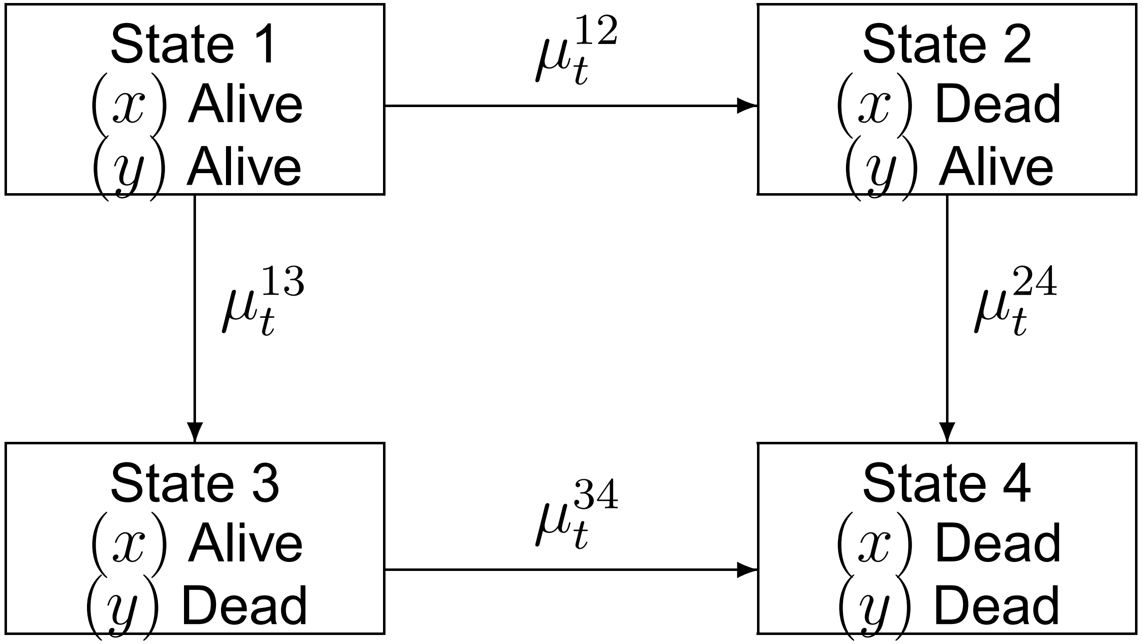
The basic contract types we will consider are:

- Assurances paying out *on* first death, and annuities payable *until* first death.

- Assurances paying out *on* second death, and annuities payable *until* second death.
- Assurances and annuities whose payment depends on the *order* of deaths.

6.2 Multiple-state model for two lives

The following multiple-state model represents the joint mortality of two lives, (x) and (y).



Notes:

- We just index the transition intensities by time t. Other notations are possible.
- We implicitly assume that the simultaneous death of (x) and (y) is impossible: there is no direct transition from state 1 to state 4.

Here we list the main EPVs met in practice, giving their symbols in the standard actuarial notation. In the first place, we assume all insurance contracts (assurance- or annuity-type) to be for **all of life**and to be of **unit amount, i.e. $1 sum assured or $1 annuity per annum.**

Joint-life assurances

- $\bar{A}_{xy}$ is the EPV of $1 paid immediately on the **first death of (x) or (y).**
- $\bar{A}_{\overline{xy}}$ is the EPV of $1 paid immediately on the **second death of (x) and (y).**

Contingent assurances

- $\bar{A}^{1}_{xy}$ is the EPV of \$1 paid immediately on the **death of** (x)*provided* (y) is then alive, i.e. provided (x) is the first to die.

- $\bar{A}_{xy}^{\;\,1}$ is the EPV of \$1 paid immediately on the **death of** (y)*provided* (x) is then alive, i.e. provided (y) is the first to die.

- $\bar{A}^{2}_{xy}$ is the EPV of \$1 paid immediately on the **death of** (x)*provided* (y) is then dead, i.e. provided (x) is the second to die.

- $\bar{A}_{xy}^{\;\,2}$ is the EPV of \$1 paid immediately on the **death of** (y)*provided* (x) is then dead, i.e. provided (y) is the second to die.

Joint-life annuities

- $\bar{a}_{xy}$ is the EPV of an annuity of \$1 per annum, payable continuously, until the **first of** (x) **or** (y) **dies.**

- $\bar{a}_{\overline{xy}}$ is the EPV of an annuity of \$1 per annum, payable continuously, until the **second of (x) or (y) dies.**

Reversionary annuities

- $\bar{a}_{x|y}$ is the EPV of an annuity of \$1 per annum, payable continuously, to (y) as long as (y) is alive and (x) is dead.
- $\bar{a}_{y|x}$ is the EPV of an annuity of \$1 per annum, payable continuously, to (x) as long as (x) is alive and (y) is dead.

The notation for reversionary annuities helpfully suggests (taking $\bar{a}_{x|y}$ as an example) **an annuity payable to (y) *deferred* until (x) is dead. Think of ${}_{m|}\bar{a}_{\overline{n}|}$ from Financial Mathematics.**

Limited terms

The above contracts may all be written for a limited term of n years, in which case the usual $\overline{n}|$ is appended to the subscript.

Evaluation of EPVs

In the multiple-state, continuous-time model, we can compute all the above EPVs simply by appropriate choices of **assurance benefits b_{ij} or annuity benefits b_i** in Thiele's differential equations. The following table lists these choices for whole-life contracts.

EPV	b_1	b_2	b_3	b_{12}	b_{13}	b_{24}	b_{34}
$\bar{A}_{xy}$	0	0	0	1	1	0	0
$\bar{A}_{\overline{xy}}$	0	0	0	0	0	1	1
$\bar{A}^{1}_{xy}$	0	0	0	1	0	0	0
$\bar{A}_{xy}^{\;\;1}$	0	0	0	0	1	0	0
$\bar{A}_{xy}^{\;\;2}$	0	0	0	0	0	1	0
$\bar{A}^{2}_{xy}$	0	0	0	0	0	0	1
$\bar{a}_{xy}$	1	0	0	0	0	0	0
$\bar{a}_{x\|y}$	0	1	0	0	0	0	0
$\bar{a}_{y\|x}$	0	0	1	0	0	0	0
$\bar{a}_{\overline{xy}}$	1	1	1	0	0	0	0

Example: Consider $\bar{A}_{xy:\overline{n}|}$. Thiele's equations are:

$$\begin{aligned}
\frac{d}{dt}V^1(t) &= V^1(t)\,\delta - (\mu_t^{12} + \mu_t^{13})(1 - V^1(t)) \\
\frac{d}{dt}V^2(t) &= \frac{d}{dt}V^3(t) = \frac{d}{dt}V^4(t) = 0
\end{aligned}$$

with boundary conditions $V^1(n) = 1$ **and all other** $V^i(n) = 0$**.**

Composite benefits

Many joint life contracts can be built up out of the above EPVs and the EPVs of single life benefits. This is generally the simplest way to compute them, especially if using tables rather than a spreadsheet.

Example: Consider a pension of $10,000 per annum, payable continuously as long as (x) and (y) are alive, reducing by half on the death of (x) if (x) dies before (y). **(This would be a typical retirement pension with a spouse's benefit.)** Its EPV, denoted $\bar{a}$ say, is most easily computed by noting that $10,000 p.a. is payable as long as (x) is alive, and in addition, \$5,000 p.a. is payable if (y) is alive but (x) is dead, hence:

$$\bar{a} = 10,000\,\bar{a}_x + 5,000\,\bar{a}_{x|y}.$$

By similar reasoning, an annuity of \$1 p.a. payable to (y) for life can be decomposed into an annuity of \$1 p.a. payable until the first death of (x) and (y), plus a reversionary annuity of \$1 p.a. payable to (y) after the prior death of (x); hence the useful:

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}.$$

Computing contingent assurance EPVs

The one type of joint life benefit whose EPV is not easily written in terms of first-death, second-death and single-life EPVs is the contingent assurance. We defer discussion until

Section 6.4.

6.3 Random joint lifetimes

It is also possible to specify a joint lives model *via* random future lifetimes. We have the random variables:

$$T_x = \textbf{the future lifetime of } (x)$$
$$T_y = \textbf{the future lifetime of } (y)\textbf{.}$$

Now define the random variables:

$$T_{\text{min}} = \text{min}(T_x, T_y) \text{ and}$$
$$T_{\text{max}} = \text{max}(T_x, T_y).$$

T_{min} is the random time until the first death occurs, and T_{max} is the random time until the second death occurs.

The distribution of $T_{\min}$

Define:

$$
\begin{aligned}
{}_tq_{xy} &= \mathsf{P}\left\{T_{\min} \leq t\right\} \quad \text{(c.d.f.)} \\
{}_tp_{xy} &= \mathsf{P}\left\{T_{\min} > t\right\}
\end{aligned}
$$

So that: ${}_tq_{xy} + {}_tp_{xy} = 1$.

Now if T_x and T_y are independent:

$$
\begin{aligned}
{}_tp_{xy} &= \mathsf{P}\left\{T_{\min} > t\right\} \\
&= \mathsf{P}\left\{T_x > t \text{ and } T_y > t\right\} \\
&= \mathsf{P}\left\{T_x > t\right\} \times \mathsf{P}\left\{T_y > t\right\} \\
&= {}_tp_x \, {}_tp_y
\end{aligned}
$$

and (useful result) the joint life survival function ${}_tp_{xy}$ is the product of the single life survival functions.

If T_x and T_y are not independent then we **cannot write** ${}_tp_{xy} = {}_tp_x \, {}_tp_y$.

If T_x and T_y are independent then also:

$$\begin{aligned}
{}_tq_{xy} &= 1 - {}_tp_{xy} \\
&= 1 - {}_tp_x \, {}_tp_y \\
&= 1 - \{(1 - {}_tq_x)(1 - {}_tq_y)\} \\
&= {}_tq_x + {}_tq_y - {}_tq_x \, {}_tq_y.
\end{aligned}$$

Example: Given ${}_nq_x = 0.2$ and ${}_nq_y = 0.4$

calculate ${}_nq_{xy}$ and ${}_np_{xy}$.

Solution:

$$ {}_nq_{xy} = {}_nq_x + {}_nq_y - {}_nq_x \, {}_nq_y $$

$$
\begin{aligned}
&= 0.2 + 0.4 - 0.2 \times 0.4 \\
&= 0.52
\end{aligned}
$$

$$
\begin{aligned}
{}_n p_{xy} &= {}_n p_x \times {}_n p_y \\
&= 0.8 \times 0.6 \\
&= 0.48
\end{aligned}
$$

To simplify the evaluation of probabilities, like ${}_t p_{xy}$, we can develop a life table function, l_{xy}, **associated with** T_{min}**.**

If T_x and T_y are independent then:

$$
{}_t p_{xy} = {}_t p_x \, {}_t p_y = \frac{l_{x+t}}{l_x} \frac{l_{y+t}}{l_y} = \frac{l_{x+t:y+t}}{l_{x:y}}
$$

where:

$$
l_{xy} = l_x \, l_y .
$$

We obtain the p.d.f. of T_{min}, denoted $f_{xy}(t)$, by differentiation when T_x and T_y are independent:

$$
\begin{aligned}
f_{xy}(t) &= \frac{d}{dt}\,{}_tq_{xy} \\
&= -\frac{d}{dt}\,{}_tp_{xy} \\
&= -\frac{d}{dt}\,{}_tp_x \cdot {}_tp_y \\
&= -\left[{}_tp_x \frac{d}{dt}\,{}_tp_y + {}_tp_y \frac{d}{dt}\,{}_tp_x\right] \\
&= -\left[{}_tp_x \left(-{}_tp_y \mu_{y+t}\right) + {}_tp_y \left(-{}_tp_x \mu_{x+t}\right)\right] \\
&= {}_tp_{xy} \left(\mu_{x+t} + \mu_{y+t}\right).
\end{aligned}
$$

Compare this to the single life case:

$$f_x(t) = {}_tp_x\, \mu_{x+t}.$$

Now, define $\mu_{xy}(t)$ as the 'force of mortality' associated with T_{min}. We can show directly that

$$\mu_{xy}(t) = \mu_{x+t} + \mu_{y+t}$$

when T_x and T_y are independent because:

$$\mu_{xy}(t) = \frac{f_{xy}(t)}{{}_tp_{xy}} = \mu_{x+t} + \mu_{y+t}.$$

However, using the multiple-state formulation of the model, we see that T_{min} is just the

time when state 1 is left. The survival function associated with this event is, by definition, ${}_tp_{\bullet}^{\overline{11}}$ (where the bullet represents policy inception) and we know that:

$$ {}_tp_{\bullet}^{\overline{11}} = \exp\left(-\int_0^t \mu_s^{12} + \mu_s^{13}\, ds \right). $$

By differentiating this, the 'force of mortality' associated with leaving state 1 is the sum of the intensities out of state 1 **without assuming that T_x and T_y are independent.**

The distribution of T_{max}

Define:

$$ \begin{aligned} {}_tq_{\overline{xy}} &= \mathrm{P}\left\{T_{\text{max}} \le t\right\} \quad \text{(c.d.f.)} \\ {}_tp_{\overline{xy}} &= \mathrm{P}\left\{T_{\text{max}} > t\right\} \end{aligned} $$

So that ${}_tq_{\overline{xy}} + {}_tp_{\overline{xy}} = 1$. We have:

$$
\begin{aligned}
{}_t p_{\overline{xy}} &= \mathsf{P}\{T_{max} > t)\} \\
&= \mathsf{P}\{T_x > t \quad \text{or} \quad T_y > t\} \\
&= \mathsf{P}\{T_x > t\} + \mathsf{P}\{T_y > t\} \\
&\qquad\qquad - \mathsf{P}\{T_x > t \text{ and } T_y > t\} \\
&= {}_t p_x + {}_t p_y - {}_t p_{xy}
\end{aligned}
$$

which does *not* require T_x and T_y to be independent. If they are independent then:

$$
{}_t p_{\overline{xy}} = {}_t p_x + {}_t p_y - {}_t p_x \, {}_t p_y
$$

and also:

$$
{}_t q_{\overline{xy}} = \mathsf{P}\{T_{max} \le t\}
$$

$$= \quad \mathsf{P}\left\{T_x \le t \text{ and } T_y \le t\right\}$$
$$= \quad \mathsf{P}\{T_x \le t\}\mathsf{P}\{T_y \le t\} = {}_tq_x\, {}_tq_y.$$

The p.d.f. of T_{max} and the 'force of mortality' $\mu_{\overline{xy}}(t)$associated with T_{max} are left as tutorial questions.

Expectations of joint lifetimes

Define:

$$\mathring{e}_{xy} = \mathsf{E}[T_{\text{min}}] \qquad \text{and} \qquad \mathring{e}_{\overline{xy}} = \mathsf{E}[T_{\text{max}}].$$

Now, consider the following identities:

Identity 1 $\quad T_{\textbf{min}} + T_{\textbf{max}} = T_x + T_y.$

Identity 2 $\quad T_{\textbf{min}}T_{\textbf{max}} = T_xT_y.$

These can be verified by considering the three exhaustive and exclusive cases:

$$T_x > T_y, \; T_x < T_y \quad \text{and} \quad T_x = T_y.$$

Taking expectations of the first identity gives:

$$\begin{aligned} & \mathsf{E}\left[T_{\mathsf{min}} + T_{\mathsf{max}}\right] && = \mathsf{E}\left[T_x + T_y\right] \\ \Longrightarrow \quad & \mathsf{E}\left[T_{\mathsf{min}}\right] + \mathsf{E}\left[T_{\mathsf{max}}\right] && = \mathsf{E}\left[T_x\right] + \mathsf{E}\left[T_y\right] \\ \Longrightarrow \quad & \overset{\circ}{e}_{xy} + \overset{\circ}{e}_{\overline{xy}} && = \overset{\circ}{e}_x + \overset{\circ}{e}_y. \end{aligned}$$

¿From the second identity we have:

$$\mathsf{E}\left[T_{\mathsf{min}}\, T_{\mathsf{max}}\right] = \mathsf{E}\left[T_x\, T_y\right]$$

and **if** T_x and T_y are independent we have:

$$\mathsf{E}\left[T_{\mathsf{min}}\, T_{\mathsf{max}}\right] = \mathsf{E}\left[T_x\right]\, \mathsf{E}\left[T_y\right].$$

Evaluation of EPVs

We can evaluate EPVs using the distributions of T_{min} and T_{max}, just as for a single life, as an alternative to solving Thiele's equations. For example, consider an assurance with a sum assured of \$1 payable immediately on the first death of (x) and (y).

$$\text{The PV of benefits} = v^{T_{\text{min}}}$$

$$\text{with p.d.f.} = {}_tp_{xy}\,\mu_{xy}(t)$$

so the expected present value $\bar{A}_{xy}$ is:

$$\bar{A}_{xy} = \mathsf{E}\left[v^{T_{\text{min}}}\right] = \int_0^\infty v^t \, {}_tp_{xy}\,\mu_{xy}(t)\,dt.$$

Note that evaluating an integral numerically is not significantly easier than solving Thiele's equations numerically.

6.4 Curtate Future Joint Lifetimes

Basic definitions

We now introduce discrete random variables. Let:

$$K_{\text{min}} = \text{Integer part of } T_{\text{min}}$$
$$K_{\text{max}} = \text{Integer part of } T_{\text{max}}.$$

We can then define the **curtate expectation of life as:**

$$e_{xy} = \mathsf{E}\left[K_{\text{min}}\right]$$
$$e_{\overline{xy}} = \mathsf{E}\left[K_{\text{max}}\right].$$

For more properties of e_{xy} and $e_{\overline{xy}}$ see tutorial.

The associated deferred probabilities ${}_{k|}q_{xy}$ and ${}_{k|}q_{\overline{xy}}$ **(i.e. the distribution functions of**

$K_{\textbf{min}}$ **and** $K_{\textbf{max}}$ are given by:

$$
\begin{aligned}
{}_{k|}q_{xy} &= \mathsf{P}\{k \le T_{\text{min}} < k+1\} \\
&= \mathsf{P}\{K_{\text{min}} = k\} \\
{}_{k|}q_{\overline{xy}} &= \mathsf{P}\{k \le T_{\text{max}} < k+1\} \\
&= \mathsf{P}\{K_{\text{max}} = k\}.
\end{aligned}
$$

Example: Given:

$$
{}_tq_x^{\text{non-smoker}} = \frac{70 - x + t}{80 - x}
$$

valid for $0 \le x \le 70$ and $0 \le t \le 10$, and that the force of mortality for smokers is twice that for non-smokers, calculate the expected time to first death of a (70) smoker and a (70) non-smoker. You are given that the lives are independent.

Solution: **We want:**

$$\overset{\circ}{e}{}^{s\ \ n-s}_{70:70} = \int_0^{10} {}_tp^{s\ \ n-s}_{70:70}\, dt = \int_0^{10} {}_tp^{s}_{70}\, {}_tp^{n-s}_{70}\, dt.$$

Now let μ_x be the force of mortality for non-smokers, then:

$$\begin{aligned} {}_tp^s_x &= \exp\left(-\int_0^t 2\,\mu_{x+r}\,dr\right) \\ &= \left[\exp\left(-\int_0^t \mu_{x+r}\,dr\right)\right]^2 \\ &= \left({}_tp^{n-s}_x\right)^2 \\ &= \left(\frac{10-t}{10}\right)^2. \end{aligned}$$

Since:

$${}_tp^{n-s}_x = 1 - \left(\frac{70-70+t}{80-70}\right) = \frac{10-t}{10}$$

therefore:

$$\overset{\circ}{e}{}^{s\ \ n-s}_{70:70} = \int_0^{10} \left[\frac{10-t}{10}\right]^3 dt = 2.5 \textbf{ years}.$$

Discrete joint life annuities

Any annuity or assurance we can define as a function of the single lifetime K_x, we can define using K_{min}, therefore **depending on the first death**or K_{max}, therefore **depending on the second death.**

Example: Consider an annuity of \$1 per annum, payable yearly in advance in advanceas long as both (x) and (y) are alive **(i.e. payable until the first death).**The same reasoning as in the single-life case shows that the present value of this is:

$$\text{PV} = \ddot{a}_{\overline{K_{\text{min}}+1}|}.$$

We denote the EPV of this benefit $\ddot{a}_{xy}$ so:

$$\begin{aligned}\ddot{a}_{xy} &= \mathsf{E}[\ddot{a}_{\overline{K_{min}+1}|}] \\ &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_{xy} \\ &= \sum_{k=0}^{\infty} v^k\ {}_k p_{xy}\end{aligned}$$

Since the distribution of the present value is known, the variance, Var$[\ddot{a}_{\overline{K_{min}+1}|}]$, and other moments can be calculated.

Example: Consider an annuity of \$1 per annum, payable yearly in advance in advanceas long as at least one of (x) and (y) is alive **(i.e. payable until the second death).**The same reasoning as before leads to:

$$\text{PV} = \ddot{a}_{\overline{K_{\text{max}}+1}|}.$$

We denote the EPV of this benefit $\ddot{a}_{\overline{xy}}$so:

$$\begin{aligned}
\ddot{a}_{\overline{xy}} &= \mathsf{E}[\ddot{a}_{\overline{K_{max}+1}|}] \\
&= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\, {}_{k|}q_{\overline{xy}} \\
&= \sum_{k=0}^{\infty} v^k \; {}_kp_{\overline{xy}}.
\end{aligned}$$

To help evaluate $\ddot{a}_{\overline{xy}}$ note that:

$$\begin{aligned}
\ddot{a}_{\overline{xy}} &= \sum_{k=0}^{\infty} v^k \; {}_kp_{\overline{xy}} \\
&= \sum_{k=0}^{\infty} v^k \; \left({}_kp_x + {}_kp_y - {}_kp_{xy}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} v^k \, {}_kp_x + \sum_{k=0}^{\infty} v^k \, {}_kp_y - \sum_{k=0}^{\infty} v^k \, {}_kp_{xy} \\
&= \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}.
\end{aligned}
$$

Hence also:

$$\ddot{a}_{xy} + \ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y.$$

Computing $\ddot{a}_{xy}$ etc. using tables

It is easy to compute EPVs of discrete benefits using a spreadsheet if ${}_tp_{xy}$ is known, which is particularly simple if T_x and T_y are independent so ${}_tp_{xy} = {}_tp_x \, {}_tp_y$.

However, other methods can be used if **joint life tables**are available, as they will be in examinations.

(i) Given tabulated values of $\ddot{a}_{xy}$ directly.

e.g. a(55) tables, but note that the values of a_{xy} are given not $\ddot{a}_{xy}$, and they are only given for even ages — may need to use linear interpolation.

For example:

$$a_{66:61} \approx \frac{1}{2}(a_{66:60} + a_{66:62}).$$

(ii) Given commutation functions. e.g. A1967–70 tables, but note that they are only given for $x = y$ **and at 4% interest.**

Note that (assuming independence):

$$\begin{aligned} \ddot{a}_{xy} &= \sum_{t=0}^{\infty} v^t \, {}_tp_{xy} = \sum_{t=0}^{\infty} v^t \, {}_tp_x \, {}_tp_y \\ &= \sum_{t=0}^{\infty} \left\{ v^t \, \frac{l_{x+t} \, l_{y+t}}{l_x \, l_y} \right\} \end{aligned}$$

$$= \sum_{t=0}^{\infty} \left\{ \frac{v^{\frac{1}{2}(x+y)+t}\, l_{x+t}\, l_{y+t}}{v^{\frac{1}{2}(x+y)}\, l_x\, l_y} \right\}$$

$$= \frac{N_{xy}}{D_{xy}}$$

where:

$$D_{xy} = v^{\frac{1}{2}(x+y)}\, l_x\, l_y$$

$$N_{x+t:y+t} = \sum_{r=0}^{\infty} D_{x+t+r:\,y+t+r}.$$

Warning:

$$\begin{aligned}\ddot{a}_{xy:\overline{n|}} &= \ddot{a}_{xy} - {}_{n|}\ddot{a}_{xy} \\ &= \ddot{a}_{xy} - v^n \, {}_np_x \, {}_np_y \, \ddot{a}_{x+n:y+n} \\ &= \ddot{a}_{xy} - \frac{1}{v^n} \frac{D_{x+n} \, D_{y+n}}{D_x \, D_y} \, \ddot{a}_{x+n:y+n}.\end{aligned}$$

Example: Using a(55) mortality and 6% interest calculate the following:

(i) $\ddot{a}_{80:75}$

(ii) ${}_{10|}\ddot{a}_{70:65}$.

Solution:

$$\begin{aligned}\ddot{a}_{80:75} &= 0.5(\ddot{a}_{80:74} + \ddot{a}_{80:76}) \\ &= 0.5(1 + a_{80:74} + 1 + a_{80:76}) \\ &= 0.5(4.395 + 4.538) = 4.467.\end{aligned}$$

$$
\begin{aligned}
{}_{10|}\ddot{a}_{70:65} &= v^{10}\,{}_{10}p_{70}\,{}_{10}p_{65}\,\ddot{a}_{80:75} \\
&= \frac{1}{v^{10}}\left(v^{10}\,{}_{10}p_{70}\,v^{10}\,{}_{10}p_{65}\right)\ddot{a}_{80:75} \\
&= \frac{1}{v^{10}}\left(\frac{D_{80}}{D_{70}}\right)\left(\frac{D_{75}}{D_{65}}\right)\ddot{a}_{80:75} \\
&=(1.06)^{10}\left(\frac{3440.5}{11559}\right)\left(\frac{8536.2}{19281}\right)4.467 \\
&= 1.054.
\end{aligned}
$$

We can extend many of the formulations for single life annuities to joint life annuities. This applies to:

- Deferred annuities (e.g. previous example)
- Temporary annuities, $\ddot{a}_{xy:\overline{n}|}$ etc.

- Increasing annuities, $(I\ddot{a})_{xy}$ etc.
- Annuities paid monthly. For example:

$$\ddot{a}^{(m)}_{xy} \approx \ddot{a}_{xy} - \frac{m-1}{2m}.$$

- Approximations for annuities paid continuously. For example:

$$\overline{a}_{xy} \approx \ddot{a}_{xy} - 0.5 = a_{xy} + 0.5.$$

Example (Question No 8 Diploma Paper June 1997):

A certain life office issues a last survivor annuity of $2,000 per annum, payable annually in arrear, to a man aged 68 and a woman aged 65.

(a) Using the a(55) ultimate table (male/female as appropriate) and a rate of interest of 4% per annum, estimate the expected present value of this benefit.

(b) Using the basis of (a) above, derive an expression (which you need NOT evaluate) for the standard deviation of the present value of this benefit, in terms of single life and joint life annuity functions.

Solution:

(a) **We want:** $2000\, a^{m\ \ f}_{\overline{68:65}}$

$$
\begin{aligned}
&= 2000\left(a^{m}_{68} + a^{f}_{65} - a^{m\ \ f}_{68:65}\right) \\
&\approx 2000\left(a^{m}_{68} + a^{f}_{65} - \frac{1}{2}\left(a^{m\ \ f}_{68:64} + a^{m\ \ f}_{68:66}\right)\right) \\
&= 2000\,(8.688 + 11.497 \\
&\qquad - \frac{1}{2}(7.418 + 7.186)\Big) \\
&= 25,766.00.
\end{aligned}
$$

(b) **Present Value** $= 2000\, a_{\overline{K_{\textbf{max}}}|}$

So we want: Var $\left[2000\, a_{\overline{K_{\textbf{max}}}|}\right]$

$$\begin{aligned} &= 2000^2 \textbf{ Var} \left[\ddot{a}_{\overline{K_{\textbf{max}}+1}|} - 1\right] \\ &= 2000^2 \textbf{ Var} \left[\ddot{a}_{\overline{K_{\textbf{max}}+1}|}\right] \\ &= 2000^2 \textbf{ Var} \left[\frac{1 - v^{K_{\textbf{max}}+1}}{d}\right] \end{aligned}$$

$$\begin{aligned} &= \left(\frac{2000}{d}\right)^2 \textbf{ Var} \left[v^{K_{\textbf{max}}+1}\right] \\ &= \left(\frac{2000}{d}\right)^2 \left({}^{*}A^{m\ \ f}_{\overline{68:65}} - \left(A^{m\ \ f}_{\overline{68:65}}\right)^2\right) \end{aligned}$$

where * means evaluated at:

$$j = i^2 + 2i = 8.16\%.$$

We now use:

$$A_{\overline{xy}} = 1 - d\,\ddot{a}_{\overline{xy}}$$

and:

$$\ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}$$

and:

$$SD[PV] = \sqrt{Var[PV]}$$

to get:

$$\frac{2000}{d}\left\{1 - {}^*d\left({}^*\ddot{a}^m_{68} + {}^*\ddot{a}^f_{65} - {}^*\ddot{a}^{m\ \ f}_{68:65}\right)\right.$$

$$\left. - \left[1 - d\left(\ddot{a}^m_{68} + \ddot{a}^f_{65} - \ddot{a}^{m\ \ f}_{68:65}\right)\right]^2\right\}^{\frac{1}{2}}.$$

Joint life assurances

Consider an assurance with sum assured $1, payable at the end of the year of death of the first of (x) and (y) to die. The benefit is payable at time $K_{\text{min}} + 1$ so has present value:

$$v^{K_{\text{min}}+1}.$$

The expected present value is denoted A_{xy}, and:

$$A_{xy} = \mathsf{E}\left[v^{K_{\text{min}}+1}\right] = \sum_{k=0}^{\infty} v^{k+1}\ {}_{k|}q_{xy}.$$

Relationships: Recall from single life:

$$A_x = 1 - d\ddot{a}_x.$$

It can be shown by similar reasoning that:

$$\begin{aligned} A_{xy} &= 1 - d\ddot{a}_{xy} \\ \bar{A}_{xy} &= 1 - \delta\,\bar{a}_{xy} \\ A_{xy:\overline{n}|} &= 1 - d\ddot{a}_{xy:\overline{n}|} \\ \bar{A}_{xy:\overline{n}|} &= 1 - \delta\,\bar{a}_{xy:\overline{n}|} \end{aligned}$$

and so on.

Associated with the second death we have:

$$A_{\overline{xy}} = \mathsf{E}[v^{K_{max}+1}] = \sum_{k=0}^{\infty} v^{k+1}\ {}_{k|}\,q_{\overline{xy}}$$

and similar relationships can be derived, e.g:

$$A_{\overline{xy}} = 1 - d\ddot{a}_{\overline{xy}}$$
$$A_{\overline{xy}:\overline{n}|} = 1 - d\ddot{a}_{\overline{xy}:\overline{n}|}.$$

Reversionary Annuities

We noted before that:

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$$

just by noticing that the following define identical cashflows no matter when (x) and (y) should die:

(1) a reversionary annuity of \$1 per annum payable continuously while (y) is alive, following the death of (x).

(2) an annuity of \$1 per annum payable continuously to (y) for life, *less* an annuity of \$1

per annum payable continuously until the first death of (x) and (y).

Hence the cashflows have identical present values, and the same EPVs. We can apply similar reasoning to other variants of reversionary annuities, for example:

$$
\begin{aligned}
\ddot{a}_{x|y} &= \ddot{a}_y - \ddot{a}_{xy} \\
&= \sum_{k=0}^{\infty} v^k \, {}_kp_y (1 - {}_kp_x) \\
a_{x|y} &= a_y - a_{xy} \\
&= \sum_{k=1}^{\infty} v^k \, {}_kp_y (1 - {}_kp_x) \\
\ddot{a}^{(m)}_{x|y} &= \ddot{a}^{(m)}_y - \ddot{a}^{(m)}_{xy} \\
&= \frac{1}{m} \sum_{k=0}^{\infty} v^{\frac{k}{m}} \, {}_{\frac{k}{m}}p_y (1 - {}_{\frac{k}{m}}p_x)
\end{aligned}
$$

$$
\begin{aligned}
a_{x|y}^{(m)} &= a_y^{(m)} - a_{xy}^{(m)} \\
&= \frac{1}{m}\sum_{k=1}^{\infty} v^{\frac{k}{m}} \, {}_{\frac{k}{m}}p_y \left(1 - {}_{\frac{k}{m}}p_x\right).
\end{aligned}
$$

We can take advantage of some relationships to simplify calculations.

For example:

$$
\begin{aligned}
\ddot{a}_{x|y} &= \ddot{a}_y - \ddot{a}_{xy} \\
&= (1 + a_y) - (1 + a_{xy}) \\
&= a_y - a_{xy}.
\end{aligned}
$$

Hence:

$$
\ddot{a}_{x|y} = a_{x|y}.
$$

Similarly, we can derive other relationships like:

(a) $\ddot{a}_{x|y} \approx \bar{a}_{x|y}$

(b) $\ddot{a}^{(m)}_{x|y} \approx \ddot{a}_{x|y}$.

Example: Calculate the annual premium (payable while both lives are alive) for the following contract for a man aged 70 and woman aged 64.

Benefits:

- SA of $10,000 payable immediately on first death; and
- a reversionary annuity of $5,000 payable
  continuously to the survivor for the remainder of their lifetime.

Basis:

- Renewal expenses 10% of all premiums.
- a(55) Ultimate, male/female as appropriate
- Interest at 8%.

Solution: Let P = Annual Premium.

EPV of premiums less expenses

$$\begin{aligned} &= 0.9\, P\, \ddot{a}^{m\ \ f}_{70:64} \\ &= 0.9\, P\ (1 + 5.576) \\ &= 5.9184\, P. \end{aligned}$$

EPV of assurance

$$\begin{aligned} &= 10,000 \bar{A}^{m\ \ f}_{70:64} \\ &= 10,000 \left(1 - \delta\, \bar{a}^{m\ \ f}_{70:64}\right) \\ &\approx 10,000 \left(1 - 0.076961 (a^{m\ \ f}_{70:64} + 0.5)\right) \\ &= 10,000\, (1 - 0.076961 (5.576 + 0.5)) \\ &= 5,323.85. \end{aligned}$$

EPV of reversionary annuity

$$
\begin{aligned}
&= 5,000\left(\bar{a}^{m\ \ f}_{70|64} + \bar{a}^{f\ \ m}_{64|70}\right) \\
&= 5,000\left(a^{m\ \ f}_{70|64} + a^{f\ \ m}_{64|70}\right) \\
&= 5,000\left(a^{f}_{64} - a^{m\ \ f}_{70:64} + a^{m}_{70} - a^{f\ \ m}_{64:70}\right) \\
&= 5,000\,(8.574 + 6.268 - 2 \times 5.576) \\
&= 18,450.0.
\end{aligned}
$$

Now set: EPV$\left[\text{Income}\right] = $ EPV$\left[\text{Outgo}\right]$

$$
\begin{aligned}
\Longrightarrow \quad 5.9184\,P &= 5,323.85 + 18,450.0 \\
\Longrightarrow \quad P &= 4,016.94.
\end{aligned}
$$

Contingent assurances

Consider a contingent assurance with sum assured \$1 payable immediately on the death of (x), if (y) is still then alive.

The probability that life (x) dies within t years with life (y) being alive when life (x) dies is denoted ${}_tq^{\,1}_{xy}$ and:

$$
\begin{aligned}
{}_tq^{\,1}_{xy} &= \mathsf{P}[(x) \text{ dies within } t \text{ years } \textbf{and} \text{ before } (y)] \\
&= \mathsf{P}[T_x < t \text{ and } T_x < T_y].
\end{aligned}
$$

This is an example of a **contingent probability,** so called because they are associated with the death of a life contingent on the survival or death of another life.

Now let $f_x(r)$ be the density of T_x and $f_y(r)$ be the density of T_y, then:

$$
{}_tq^{\,1}_{xy} = \int_{r=0}^{t} \int_{s=r}^{\infty} f_x(r) f_y(s) ds\, dr
$$

$$= \int_{r=0}^{t} f_x(r) \left[\int_{s=r}^{\infty} f_y(s) ds \right] dr.$$

Since:

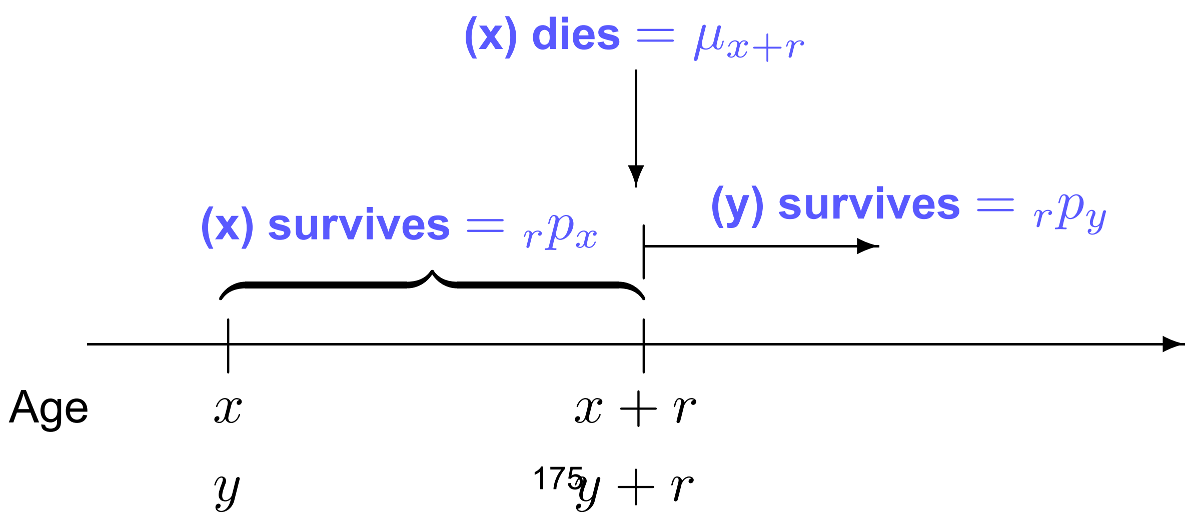
$$\int_{s=r}^{\infty} f_y(s) ds = {}_r p_y$$

then:

$${}_t q_{xy}^{\;1} = \int_{r=0}^{t} {}_r p_x \, \mu_{x+r} \, {}_r p_y \, dr.$$

Illustration: (x) survives to time r and then dies and (y) survives beyond r, giving:

${}_r p_x \, \mu_{x+r} \, {}_r p_y$



Note that since one of (x) and (y) has to die first:

$$ {}_tq_{\overset{1}{x}y} + {}_tq_{x\overset{1}{y}} = {}_tq_{xy}. $$

We define ${}_tq_{\overset{2}{x}y}$ as the probability that (x) **dies within** t **years but after** (y) **has died.**

Therefore:

$$ {}_tq_x = {}_tq_{\overset{1}{x}y} + {}_tq_{\overset{2}{x}y} $$

from which:

$$ {}_tq_{\overset{2}{x}y} = \int_0^t {}_rp_x \, \mu_{x+r} \, \left(1 - {}_rp_y\right) \, dr. $$

With these probabilities we can now evaluate the EPV $\bar{A}_{\overset{1}{x}y}$.

$$ \bar{A}_{\overset{1}{x}y} = \begin{cases} v^{T_x} & \text{if } T_x < T_y \\ 0 & \text{if } T_x \geq T_y. \end{cases} $$

¿From this it can be shown that:

$$\bar{A}^{1}_{xy} = \int_{0}^{\infty} v^{r} \, {}_{r}p_{x} \, \mu_{x+r} \, {}_{r}p_{y} \, dr.$$

Similarly, we can derive an expression for the EPV of a benefit of $1 payable immediately on the death of (x) if it is after that of (y), denoted $\bar{A}^{2}_{xy}$:

$$\bar{A}^{2}_{xy} = \int_{0}^{\infty} v^{r} \, {}_{r}p_{x} \, \mu_{x+r} \, \left(1 - {}_{r}p_{y}\right) \, dr.$$

We see, what is intuitively clear, that:

$$\bar{A}_{x} = \bar{A}^{1}_{xy} + \bar{A}^{2}_{xy}.$$

Other important relationships:

(a) $\bar{A}_{xy} = \bar{A}^{1}_{xy} + \bar{A}_{xy}^{\;\;1}$ and $\bar{A}_{\overline{xy}} = \bar{A}^{2}_{xy} + \bar{A}_{xy}^{\;\;2}$.

(b) $A_{xy}^{1} = \sum_{t=0}^{\infty} v^{t+1} {}_t p_x {}_t p_y \cdot q_{x+t:y+t}^{\quad 1}$.

(c) $A_{xy} = A_{xy}^{1} + A_{xy}^{\ \ 1}$ and $A_{\overline{xy}} = A_{xy}^{2} + A_{xy}^{\ \ 2}$.

(d) $A_{xy}^{1} \approx (1+i)^{-\frac{1}{2}} \bar{A}_{xy}^{1}$.

If the two lives are the same age, and the same mortality table is applies to both, then:

$$
\begin{aligned}
\bar{A}_{xx}^{1} &= \frac{1}{2}\bar{A}_{xx} \\
A_{xx}^{1} &= \frac{1}{2}A_{xx} \\
\bar{A}_{xx}^{2} &= \frac{1}{2}\bar{A}_{\overline{xx}} \\
A_{xx}^{2} &= \frac{1}{2}A_{\overline{xx}}.
\end{aligned}
$$

6.5 Examples

Example 1

Find the expected present value of an annuity of $10,000 p.a. payable annually in advance to a man aged 65, reducing to $5,000 p.a. continuing in payment to his wife, now aged 61, if she survives him.

Basis:

- Mortality — a(55) ultimate, male/female as appropriate
- Interest: 8% p.a.
- Ignore the possibility of divorce.

Solution

EPV

$$\begin{aligned} &= 10,000\, \ddot{a}^{m}_{65} + 5,000\, \ddot{a}^{m\ \ f}_{65|61} \\ &= 10,000\, (a^{m}_{65} + 1) + 5,000\, a^{m\ \ f}_{65|61} \end{aligned}$$

$$
\begin{aligned}
&= 10,000\ (7.4+1) + 5,000\ \left(a^{f}_{61} - a^{m\ f}_{65:61}\right) \\
&= 10,000\ (8.4) + 5,000\ (9.124 - 6.619) \\
&= 96,525.
\end{aligned}
$$

Since:

$$
\begin{aligned}
a^{m\ f}_{65:61} &\approx \frac{1}{4}\left(a^{m\ f}_{66:62} + a^{m\ f}_{66:60} + a^{m\ f}_{64:62} + a^{m\ f}_{64:60}\right) \\
&= \frac{1}{4}\ (6.392 + 6.519 + 6.709 + 6.854) \\
&= 6.619.
\end{aligned}
$$

Example 2

(a) Express ${}_nq^{\ 2}_{xy}$ in terms of single life probabilities and contingent probabilities referring to the first death.

(b) Suppose $\mu_x = \frac{1}{80-x}$ for $0 \le x \le 80$, evaluate ${}_{20}q^{\;\;2}_{40:50}$.

Solution

(a) ${}_{t}q^{\;\;2}_{xy}$

$$\begin{aligned} &= \int_{r=0}^{t} {}_{r}p_{y}\,\mu_{y+r}\,\left(1 - {}_{r}p_{x}\right)\,dr \\ &= \int_{r=0}^{t} {}_{r}p_{y}\,\mu_{y+r}\,dr - \int_{r=0}^{t} {}_{r}p_{y}\,\mu_{y+r}\,{}_{r}p_{x}\,dr \\ &= {}_{t}q_{y} - {}_{t}q^{\;\;1}_{xy}. \end{aligned}$$

(b) ${}_{n}q_{x} = 1 - {}_{n}p_{x}$

$$= 1 - \exp\left\{-\int_{0}^{n} \mu_{x+t}\,dt\right\}$$

$$
\begin{aligned}
&= 1 - \exp\left\{ -\int_0^n \frac{1}{80 - x - t}\, dt \right\} \\
&= 1 - \exp\left\{ [\log(80 - x - t)]_0^n \right\} \\
&= 1 - \exp\left\{ \log\left(\frac{80 - x - n}{80 - x} \right) \right\} \\
&= \frac{n}{80 - x}.
\end{aligned}
$$

and we have ${}_nq_{xy}^{\ \ 1}$

$$
\begin{aligned}
&= \int_0^n {}_tp_x\, {}_tp_y\, \mu_{y+t}\, dt \\
&= \frac{1}{(80 - x)(80 - y)} \int_0^n (80 - x - t)\, dt \\
&= \frac{n\,(80 - x) - \frac{1}{2} n^2}{(80 - x)(80 - y)}
\end{aligned}
$$

$$
\begin{aligned}
\therefore {}_{20}q_{40:\overset{2}{50}} &= {}_{20}q_{50} - {}_{20}q_{40:\overset{1}{50}} \\
&= \frac{20}{30} - \frac{20 \times 40 - \frac{1}{2}\,20^2}{40 \times 30} \\
&= \frac{1}{6}.
\end{aligned}
$$

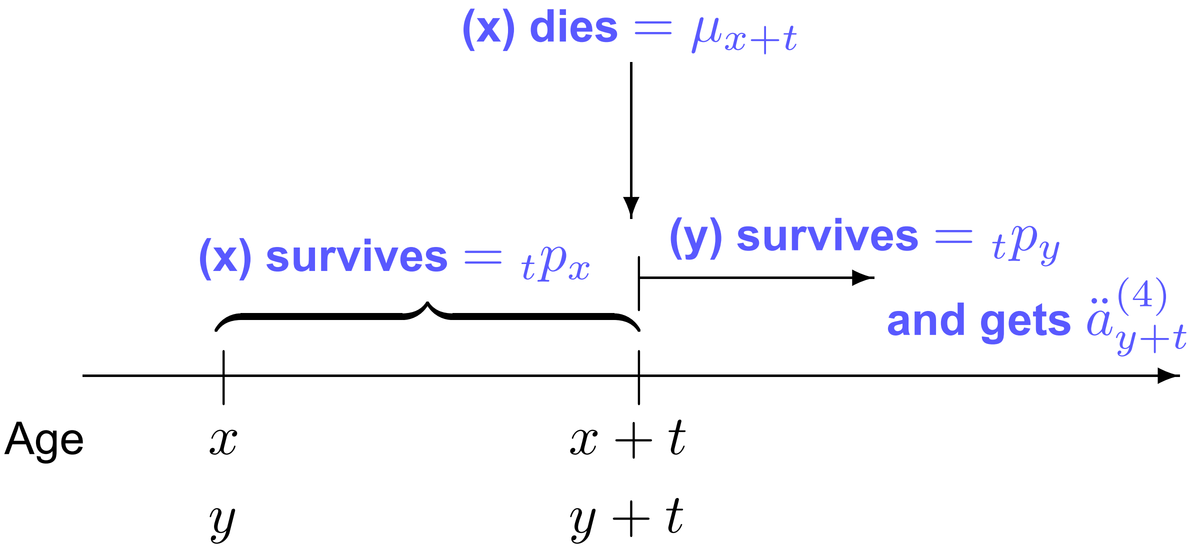
Example 3

Consider a reversionary annuity of \$1 p.a. payable quarterly in advance during the lifetime of (y) following the death of (x). Show that the expected present value of this benefit is approximately equal to:

$$a_{x|y} + \frac{1}{8}\bar{A}^{\,1}_{xy}.$$

Solution

Hint: Fix a time t to represent the death of (x), as below:



Hence:

$$
\begin{aligned}
\text{EPV} &= \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t} \, {}_tp_y \, \ddot{a}^{(4)}_{y+t} \, dt \\
&\approx \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t} \, {}_tp_y \left(\bar{a}_{y+t} + \frac{1}{8} \right) dt \\
&= \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t} \, {}_tp_y \, \bar{a}_{y+t} \, dt
\end{aligned}
$$

$$+ \frac{1}{8} \int_0^{\infty} v^t \, {}_tp_x \, \mu_{x+t} \, {}_tp_y \, dt$$

$$= \quad \bar{a}_{x|y} + \frac{1}{8} \bar{A}^{1}_{xy} \approx a_{x|y} + \frac{1}{8} \bar{A}^{1}_{xy}.$$