Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 5 Solutions

1. (a) Let $(DA)^{\,1}_{x+r:\overline{n-r}|}$ denote a decreasing term assurance with initial sum assured of $n-r$. Define ${}_tV\left((DA)^1_{x:\overline{n}|}\right)$ as the policy value just before the payment of the t^{th} premium and $P\left((DA)^1_{x:\overline{n}|}\right)$ as the net premium.

The policy value at time $t+1$, ${}_tV\left((DA)^1_{x:\overline{n}|}\right)$, together with the net premium paid at time $t+1$, $P\left((DA)^1_{x:\overline{n}|}\right)$, when accumulated at the rate of interest i should produce a value sufficient to provide a sum assured of $n-t$ at the end of year to the proportion q_{x+t} expected to die during the year and also set up a reserve of ${}_{t+1}V\left((DA)^1_{x:\overline{n}|}\right)$ for the proportion p_{x+t} expected to survive. Therefore

$$\left[{}_tV\left((DA)^1_{x:\overline{n}|}\right)+P\left((DA)^1_{x:\overline{n}|}\right)\right](1+i)=q_{x+t}\cdot(n-t)+p_{x+t}\cdot{}_{t+1}V\left((DA)^1_{x:\overline{n}|}\right)$$

Proof:

$$\begin{aligned}{}_tV\left((DA)^1_{x:\overline{n}|}\right)+P\left((DA)^1_{x:\overline{n}|}\right) &= (DA)^{\,1}_{x+t:\overline{n-t}|}-P\left((DA)^1_{x:\overline{n}|}\right)\cdot\ddot{a}_{x+t:\overline{n-t}|}+P\left((DA)^1_{x:\overline{n}|}\right)\\ &= (DA)^{\,1}_{x+t:\overline{n-t}|}-P\left((DA)^1_{x:\overline{n}|}\right)\left(\ddot{a}_{x+t:\overline{n-t}|}-1\right). \qquad (1)\end{aligned}$$

We note that

$$(DA)^{\,1}_{x+t:\overline{n-t}|}=q_{x+t}\cdot v\cdot(n-t)+p_{x+t}\cdot v\cdot(DA)^{\,1}_{x+t+1:\overline{n-t-1}|},$$

and

$$\ddot{a}_{x+t:\overline{n-t}|}-1=a_{x+t:\overline{n-t-1}|}=v\cdot p_{x+t}\cdot\ddot{a}_{x+t+1:\overline{n-t-1}|}.$$

Substituting these expressions for $(DA)^{\,1}_{x+t:\overline{n-t}|}$ and $\ddot{a}_{x+t:\overline{n-t}|}-1$ into equation (1), we can write

$$\begin{aligned}\left[{}_tV\left((DA)^1_{x:\overline{n}|}\right)+P\left((DA)^1_{x:\overline{n}|}\right)\right](1+i) &= q_{x+t}\cdot(n-t)+\\ &\quad p_{x+t}\cdot\left((DA)^{\,1}_{x+t+1:\overline{n-t-1}|}-P\left((DA)^1_{x:\overline{n}|}\right)\cdot\ddot{a}_{x+t+1:\overline{n-t-1}|}\right)\\ &= q_{x+t}\cdot(n-t)+p_{x+t}\cdot{}_{t+1}V\left((DA)^{\,1}_{x:\overline{n}|}\right)\end{aligned}$$

as required.

(b) The reserve can be found directly, or by using the relationship in (a) twice from the boundary condition ${}_nV\left((DA)^1_{x:\overline{n|}}\right) = 0$. Either way we need to find the net premium,

$$P\left((DA)^1_{x:\overline{n|}}\right) = \frac{(DA)^1_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}.$$

We have $x = 30$, $n = 30$, mortality is according to the A1967–70 ultimate table and interest is 4% p.a.

$$\begin{aligned}
\frac{(DA)^1_{30:\overline{30|}}}{\ddot{a}_{30:\overline{30|}}} &= \frac{31A^1_{30:\overline{30|}} - (IA)^1_{30:\overline{30|}}}{\ddot{a}_{30:\overline{30|}}} = \frac{31(M_{30} - M_{60}) - (R_{30} - R_{60} - 30M_{60})}{N_{30} - N_{60}} \\
&= \frac{30M_{30} - R_{31} + R_{61}}{N_{30} - N_{60}} = \frac{30(1981.9552) + 21167.520 - 75245.722}{219735.21 - 35841.261} \\
&= 0.029258.
\end{aligned}$$

Using the recursive relationship for the first time with $q_{59} = 0.01299373$, we have

$$\left({}_{29}V\left((DA)^1_{30:\overline{30|}}\right) + 0.029258\right)(1.04) = 1 \times 0.01299373 + 0$$

such that ${}_{29}V\left((DA)^1_{30:\overline{30|}}\right) = -0.016764$. Using the recursive relation for the second time with $q_{58} = 0.01168566$, we have

$$\left({}_{28}V\left((DA)^1_{30:\overline{30|}}\right) + 0.029258\right)(1.04) = 2\times 0.01168566 + (1 - 0.01168566)(-0.016764)$$

such that ${}_{28}V\left((DA)^1_{30:\overline{30|}}\right) = -0.022716$.

(c) The problem is that the reserve is negative. A level premium has been used to pay for a decreasing risk. This could cause problems if the policyholder lapses the policy since it will leave the insurer with a loss. One way to deal with this problem is to charge a higher level premium payable for a term shorter than the term of the policy so that negative reserves do not arise.

2. (a)

$${}_{6.5}V_{35:\overline{25|}} \cdot (1+i)^{0.5} = {}_{0.5}q_{41.5} + {}_{0.5}p_{41.5} \cdot {}_{7}V_{35:\overline{25|}}.$$

But ${}_{7}V_{35:\overline{25|}} = A_{42:\overline{18|}} - P_{35:\overline{25|}} \cdot \ddot{a}_{42:\overline{18|}} = 0.50121 - 0.02393(12.969) = 0.19081$.

Assuming a constant force of mortality between exact ages 41 and 42, then $({}_{0.5}p_{41.5})^2 = p_{41}$ giving ${}_{0.5}p_{41.5} = 0.99949$. Therefore

$${}_{6.5}V_{35:\overline{25|}} = \frac{(1 - 0.99949) + 0.99949(0.19081)}{(1.04)^{0.5}} = 0.18751.$$

(b)

$$\left({}_{16}V_{35:\overset{1}{\overline{25|}}} + P_{35:\overset{1}{\overline{25|}}}\right)(1+i)^{0.25} = {}_{0.25}q_{51} \times 0 + {}_{0.25}p_{51} \cdot {}_{16.25}V_{35:\overset{1}{\overline{25|}}} \quad \text{and}$$

$$\begin{aligned} {}_{16}V_{35:\overset{1}{\overline{25|}}} &= A_{51:\overset{1}{\overline{9|}}} - P_{35:\overset{1}{\overline{25|}}} \cdot \ddot{a}_{51:\overline{9|}} = A_{51:\overset{1}{\overline{9|}}} - \frac{A_{35:\overset{1}{\overline{25|}}}}{\ddot{a}_{35:\overline{25|}}} \cdot \ddot{a}_{51:\overline{9|}} \\ &= \frac{D_{60}}{D_{51}} - \frac{\frac{D_{60}}{D_{35}}}{16.027}(7.625) = 0.50604. \end{aligned}$$

Assuming a constant force of mortality between exact ages 51 and 52, $({}_{0.25}p_{51})^4 = p_{51}$ giving ${}_{0.25}p_{51} = 0.9993$. Therefore

$${}_{16.25}V_{35:\overset{1}{\overline{25|}}} = \frac{(0.50604 + 0.02197)(1.04)^{0.25}}{0.9993} = 0.53358.$$

(c) We note that the premiums are paid quarterly and the sum assured is paid at the end of year of death or on survival to maturity.

$$\left({}_{16.75}V^{(4)}_{35:\overline{25|}} + 0.25 \times P^{(4)}_{35:\overline{25|}}\right)(1+i)^{0.25} = {}_{0.25}q_{51.75} + {}_{0.25}p_{51.75} \cdot {}_{17}V^{(4)}_{35:\overline{25|}}.$$

But

$$P^{(4)}_{35:\overline{25|}} = \frac{A_{35:\overline{25|}}}{\ddot{a}^{(4)}_{35:\overline{25|}}} = \frac{0.38359}{16.027 - \frac{3}{8}(1 - \frac{D_{60}}{D_{35}})} = 0.0244.$$

$${}_{17}V^{(4)}_{35:\overline{25|}} = A_{52:\overline{8|}} - (0.0244)\ddot{a}^{(4)}_{52:\overline{8|}} = 0.73424 - (0.0244)(6.91 - \frac{3}{8}(1 - \frac{D_{60}}{D_{52}})) = 0.56836$$

Assuming a constant force of mortality between exact ages 51 and 52, then ${}_{0.25}p_{51.75} = 0.9993$, giving

$${}_{16.75}V^{(4)}_{35:\overline{25|}} = \frac{(1 - 0.9993) + 0.9993(0.56836)}{(1.04)^{0.25}} - 0.25(0.0244) = 0.55701.$$

Also

$$\left({}_{16.5}V^{(4)}_{35:\overline{25|}} + 0.25 \times P^{(4)}_{35:\overline{25|}}\right)(1+i)^{0.25} = {}_{0.25}q_{51.5} \cdot v^{0.25} + {}_{0.25}p_{51.5} \cdot {}_{16.75}V^{(4)}_{35:\overline{25|}}$$

such that

$${}_{16.5}V^{(4)}_{35:\overline{25|}} = \frac{(1 - 0.9993)(1.04)^{0.25} + 0.9993(0.55701)}{(1.04)^{0.25}} - 0.25(0.0244) = 0.54579.$$

3. (a) Thiele's equations are:

$$\begin{aligned}
\frac{d}{dt}{}_t\bar{V}_x^{(0)} &= {}_t\bar{V}_x^{(0)} \cdot \delta + \bar{P} - \mu_{x+t}^{01}({}_t\bar{V}_x^{(1)} - {}_t\bar{V}_x^{(0)}) - \mu_{x+t}^{02}(100 - {}_t\bar{V}_x^{(0)}) \\
\frac{d}{dt}{}_t\bar{V}_x^{(1)} &= {}_t\bar{V}_x^{(1)} \cdot \delta - 1 - \mu_{x+t}^{10}({}_t\bar{V}_x^{(0)} - {}_t\bar{V}_x^{(1)}) - \mu_{x+t}^{12}(100 - {}_t\bar{V}_x^{(1)}) \\
\frac{d}{dt}{}_t\bar{V}_x^{(2)} &= 0.
\end{aligned}$$

   (b) In this case you cannot solve Thiele's equations forwards, because ${}_t\bar{V}_x^{(1)}$ is not known at $t = 0$. However you know that ${}_n\bar{V}_x^{(j)} = 0$ for all three states, so you would solve the equations backwards from there.

4. We have

$$({}_tV_{x:\overline{n}|} + 179.3)(1.04) = q_{x+t}(1,000 + {}_{t+1}V_{x:\overline{n}|}) + p_{x+t} \cdot {}_{t+1}V_{x:\overline{n}|} = 1,000q_{x+t} + {}_{t+1}V_{x:\overline{n}|}.$$

Solving backwards ${}_4V_{40:\overline{5}|} = 784.48$, ${}_3V_{40:\overline{5}|} = 577.00$ and ${}_2V_{40:\overline{5}|} = 377.27$.