Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 8 Solutions

1. (a) Considering a time period $t+dt$, then the probability that a life who is healthy at age x will be dead by age $x+t+dt$ is

$$\begin{aligned}
{}_{t+dt}p_x^{13} &= {}_tp_x^{11}\cdot \text{P[dies between ages } x+t \text{ and } x+t+dt \text{ from healthy} \mid \text{healthy at } x+t]\\
&\quad +{}_tp_x^{12}\cdot \text{P[dies between ages } x+t \text{ and } x+t+dt \text{ from sick} \mid \text{sick at } x+t]\\
&\quad +{}_tp_x^{13}\cdot \text{P[stays dead between ages } x+t \text{ and } x+t+dt \mid \text{dead at } x+t]\\
&= {}_tp_x^{11}\left(\mu_{x+t}\cdot dt + o(dt)\right) + {}_tp_x^{12}\left(\nu_{x+t}\cdot dt + o(dt)\right) + {}_tp_x^{13}(1)
\end{aligned}$$

On rearranging, dividing by dt and taking the limit as dt approaches 0, we get

$$\frac{d}{dt}{}_tp_x^{13} = {}_tp_x^{11}\cdot\mu_{x+t} + {}_tp_x^{12}\cdot\nu_{x+t}.$$

To evaluate ${}_tp_x^{13}$ numerically we choose a small stepsize s and using Euler's method we have

$$\begin{aligned}
{}_sp_x^{13} &\approx {}_0p_x^{13} + s\left[\frac{d}{dt}{}_tp_x^{13}|_{t=0}\right]\\
&= 0 + s\left[{}_0p_x^{11}\cdot\mu_x + {}_0p_x^{12}\cdot\nu_x.\right]\\
&\quad 0 + s\left[1\cdot\mu_x + 0\cdot\nu_x.\right] = \mu_x\cdot s
\end{aligned}$$

For the next iteration we use ${}_sp_x^{13} = \mu_x\cdot s$ as the boundary point and have

$$\begin{aligned}
{}_{2s}p_x^{13} &\approx {}_sp_x^{13} + s\left[\frac{d}{dt}{}_tp_x^{13}|_{t=s}\right]\\
&= \mu_x\cdot s + s\left[{}_sp_x^{11}\cdot\mu_{x+s} + {}_sp_x^{12}\cdot\nu_{x+s}.\right]\\
&= {}_sp_x^{11}(\mu_{x+s}\cdot s) + {}_sp_x^{12}(\nu_{x+s}\cdot s) + \mu_x\cdot s
\end{aligned}$$

where ${}_sp_x^{11}$ and ${}_sp_x^{12}$ are evaluated in the same way as described in the lectures. These steps are repeated until the required occupancy probability is derived.

(b) (i)We can derive

$$\frac{d}{dt}{}_tp_x^{11} = -{}_tp_x^{11}(\sigma_{x+t}+\mu_{x+t}) + {}_tp_x^{12}\cdot\rho_{x+t} = -{}_tp_x^{11}(2\mu) + {}_tp_x^{12}\mu. \qquad (1)$$

and

$$\frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{11} \cdot \sigma_{x+t} - {}_tp_x^{12} \cdot (\rho_{x+t} + \nu_{x+t}) = {}_tp_x^{11} \cdot \mu - {}_tp_x^{12}(2\mu). \qquad (2)$$

Differentiating equation 1 w.r.t. t and substituting equation 2 we get

$$\frac{d^2}{dt^2}{}_tp_x^{11} = -2\mu\frac{d}{dt}{}_tp_x^{11} + \mu\frac{d}{dt}{}_tp_x^{12} = -2\mu\frac{d}{dt}{}_tp_x^{11} + \mu\left({}_tp_x^{11} \cdot \mu - {}_tp_x^{12}(2\mu)\right).$$

But from equation 1

$${}_tp_x^{12} = \frac{1}{\mu}\left(\frac{d}{dt}{}_tp_x^{11} + {}_tp_x^{11}(2\mu)\right)$$

such that

$$\frac{d^2}{dt^2}{}_tp_x^{11} = -2\mu\frac{d}{dt}{}_tp_x^{11} + \mu^2{}_tp_x^{11} - 2\mu\left(\frac{d}{dt}{}_tp_x^{11} + {}_tp_x^{11}(2\mu)\right)$$

which gives the second order differential equation

$$\frac{d^2}{dt^2}{}_tp_x^{11} + 4\mu\frac{d}{dt}{}_tp_x^{11} + 3\mu^2{}_tp_x^{11} = 0. \qquad (3)$$

We note that given ${}_tp_x^{11} = 0.5\left(e^{-\mu t} + e^{-3\mu t}\right)$ then

$$\frac{d}{dt}{}_tp_x^{11} = -0.5\mu\left(e^{-\mu t} + 3e^{-3\mu t}\right) \quad \text{and} \quad \frac{d^2}{dt^2}{}_tp_x^{11} = 0.5\mu^2\left(e^{-\mu t} + 9e^{-3\mu t}\right).$$

Substituting these three equations into the L.H.S of equation 3 we get 0 which shows that ${}_tp_x^{11} = 0.5\left(e^{-\mu t} + e^{-3\mu t}\right)$ satisfies the differential equation.

(ii) For $0 \leq d \leq t$

$$\begin{aligned} {}_{d,t}p_x^{12} &= \int_{s=t-d}^{t} {}_sp_x^{11}\,\mu\,{}_{t-s}p_{x+s}^{\overline{22}}ds \\ &= \int_{s=t-d}^{t} 0.5(e^{-\mu s} + e^{-3\mu s})\,\mu\,e^{-2\mu(t-s)}ds \end{aligned}$$

(Note that ${}_up_y^{\overline{22}} = e^{-2\mu u}$)

$$\begin{aligned} &= 0.5e^{-2\mu t}\int_{s=t-d}^{t} \mu(e^{\mu s} + e^{-\mu s})ds \\ &= 0.5e^{-2\mu t}\left[e^{\mu s} - e^{-\mu s}\right]_{t-d}^{t} \\ &= 0.5e^{-2\mu t}\left[e^{\mu t} - e^{-\mu t} - e^{\mu(t-d)} + e^{-\mu(t-d)}\right] \\ &= 0.5\left[e^{-\mu t} - e^{-3\mu t} - e^{-\mu(t+d)} + e^{-\mu(3t-d)}\right] \end{aligned}$$

(iii)Note that

$$
\begin{aligned}
{}_tp_{30}^{12} = {}_{t,t}p_{30}^{12} &= 0.5\left[e^{-\mu t} - e^{-3\mu t} - e^{-2\mu t} + e^{-2\mu t}\right] \\
&= 0.5\left[e^{-\mu t} - e^{-3\mu t}\right]
\end{aligned}
$$

Hence for $0 \leq d \leq t$

$$
{}_tp_{30}^{12} - {}_{d,t}p_{30}^{12} = 0.5\left(e^{-\mu(t+d)} - e^{-\mu(3t-d)}\right)
$$

(iv)The expected present value is

$$
\begin{aligned}
&= 10,000\int_{t=0}^{30} v^t\left({}_tp_{30}^{12} - {}_{0.25,t}p_{30}^{12}\right)dt \\
&= 10,000\int_{t=0}^{30} 0.5e^{-0.05t}\left(e^{-0.01(t+0.25)} - e^{-0.01(3t-0.25)}\right)dt \\
&= 5,000\int_{t=0}^{30}\left(e^{-0.06t-0.0025} - e^{-0.08t+0.0025}\right)dt \\
&= 5,000\left[-\frac{1}{0.06}e^{-0.06t-0.0025} + \frac{1}{0.08}e^{-0.08t+0.0025}\right]_0^{30} \\
&= 5,000\,(-2.7481 + 1.1368 + 16.6251 - 12.5313) \\
&= £12,413
\end{aligned}
$$

2. This is the probability that a life aged x who is currently sick, and has been sick for duration z, will be healthy at age $x+t$. For this to happen, the life must recover from the current sickness between ages $x+u$ and $x+u+du$, where du is small and $x \leq x+u \leq x+t$, (probability ${}_u\overline{p}_{x,z}^{22}.\rho_{x+u,z+u}.du$) and, from being healthy at age $x+u+du$, must be healthy again at age $x+t$ (probability ${}_{t-u}p_{x+u}^{11}$). The probability of this happening is:

$$
{}_u\overline{p}_{x,z}^{22}.\rho_{x+u,z+u}.du.{}_{t-u}p_{x+u}^{11}
$$

and the required probability is the sum (*i.e.* integral) of these probabilities over all possible values of u.

3. Consider ${}_{t+dt}p_x^{13}$, where $dt > 0$, and condition on the state at age $x+t$. The life may be:

   dead (probability ${}_tp_x^{13}$)

   healthy (probability ${}_tp_x^{11}$), in which case the life must die before age $x+t+dt$ (probability $\mu_{x+t}.dt + o(dt)$)

   sick, in which case the life must have fallen sick for the last time between ages $x+u$ and $x+u+du$, for some u between 0 and t, (probability ${}_up_x^{11}.\sigma_{x+u}.du$), remained sick until age $x+t$ (probability ${}_{t-u}\overline{p}_{x+u}^{22}$) and then died before age $x+t+dt$ (probability $\nu_{x+t,t-u}.dt$)

Combining these probabilities, we have:

$$ {}_{t+dt}p_x^{13} = {}_tp_x^{13} + {}_tp_x^{11}\,\mu_{x+t}.dt + \int_{u=0}^{t} {}_up_x^{11}.\sigma_{x+u}\cdot{}_{t-u}p_{x+u}^{\overline{22}}\cdot(\nu_{x+t,t-u}.dt)\,du + o(dt) $$

Rearranging, dividing by dt and letting dt decrease to 0 gives the required differential equation.

To calculate numerical values for ${}_tp_x^{13}$, we choose a small stepsize and using Eulers method and the boundary condition ${}_0p_x^{13} = 0$, we get a value for ${}_sp_x^{13}$. We do a second iteration using ${}_sp_x^{13}$ as the new boundary condition to get a value ${}_{2s}p_x^{13}$. The iterations are repeated for as many times as required to get the occupancy probability.

4. Consider ${}_{t+dt}p_{x,z}^{23}$. Using the same argument as in the answer to Question 2, we have:

$$ {}_{t+dt}p_{x,z}^{23} = {}_tp_{x,z}^{23} + {}_tp_{x,z}^{21}.\mu_{x+t}.dt + {}_tp_{x,z}^{\overline{22}}.\nu_{x+t,z+t}.dt + \int_{u=0}^{t} {}_up_{x,z}^{21}.\sigma_{x+u}\cdot{}_{t-u}p_{x+u}^{\overline{22}}\cdot(\nu_{x+t,t-u}.dt)\,du + o(dt) $$

Rearranging, dividing by dt and letting dt decrease to 0 gives the following differential equation:

$$ \frac{d}{dt}{}_tp_{x,z}^{23} = {}_tp_{x,z}^{21}.\mu_{x+t} + {}_tp_{x,z}^{\overline{22}}.\nu_{x+t,z+t} + \int_{u=0}^{t} {}_up_{x,z}^{21}.\sigma_{x+u}\cdot{}_{t-u}p_{x+u}^{\overline{22}}.\nu_{x+t,t-u}\,du $$