Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 2 Solutions

1. (a) The recursive relationship between pure endowment policy values is:

$$(V(t)+P)(1+i) = p_{x+t}\,V(t+1).$$

To prove this, write:

$$(V(t)+P) = A_{x+t:\overset{\,1}{\overline{n-t}|}} - P\,(\ddot{a}_{x+t:\overline{n-t}|} - 1)$$

and note that:

$$A_{x+t:\overset{\,1}{\overline{n-t}|}} = q_{x+t}\,v \times 0 + p_{x+t}\,v\,A_{x+t+1:\overset{\,1}{\overline{n-t-1}|}}$$

and:

$$\ddot{a}_{x+t:\overline{n-t}|} = 1 + p_{x+t}\,v\,\ddot{a}_{x+t+1:\overline{n-t-1}|}$$

so:

$$\begin{aligned}(A_{x+t:\overset{\,1}{\overline{n-t}|}} - P\,(\ddot{a}_{x+t:\overline{n-t}|} - 1))(1+i) &= p_{x+t}A_{x+t+1:\overset{\,1}{\overline{n-t-1}|}} - P\,p_{x+t}\,\ddot{a}_{x+t+1:\overline{n-t-1}|} \\ &= p_{x+t}\,V(t+1).\end{aligned}$$

(b) The policy value for a level annuity-due payable for life is just $\ddot{a}_{x+t}$. The recursive relationship between policy values is therefore:

$$(\ddot{a}_{x+t} - 1)(1+i) = p_{x+t}\cdot\ddot{a}_{x+t+1}.$$

The statement stands as its own proof.

2. An Excel workbook showing how this example may be laid out can be accessed through the module webpage.

3. (a) Let $(DA)^{\,1}_{x+r:\overline{n-r}|}$ denote the EPV of a decreasing term assurance with initial sum assured of $n-r$. Define $V(t)$ as the policy value just before the payment of the t^{th} premium and P as the net premium. The policy value at time t, together with the net premium paid at time t, when accumulated at the rate of interest i should produce a value sufficient to provide a sum assured of $n-t$ at the end

of year to the proportion q_{x+t} expected to die during the year, and also to set up a reserve of $V(t+1)$ for the proportion p_{x+t} expected to survive. Therefore:

$$(V(t)+P)(1+i) = q_{x+t}\,(n-t) + p_{x+t}\,V(t+1).$$

Proof:

$$\begin{aligned} V(t)+P &= (DA)^{\,1}_{x+t:\overline{n-t}|} - P\,\ddot{a}_{x+t:\overline{n-t}|} + P \\ &= (DA)^{\,1}_{x+t:\overline{n-t}|} - P\,(\ddot{a}_{x+t:\overline{n-t}|} - 1)\,. \end{aligned} \tag{1}$$

We note that

$$(DA)^{\,1}_{x+t:\overline{n-t}|} = q_{x+t}\,v\,(n-t) + p_{x+t}\,v\,\,(DA)^{\,1}_{x+t+1:\overline{n-t-1}|}\,,$$

and

$$\ddot{a}_{x+t:\overline{n-t}|} - 1 = a_{x+t:\overline{n-t-1}|} = v\,p_{x+t}\,\ddot{a}_{x+t+1:\overline{n-t-1}|}.$$

Substituting these into equation (1), we can write:

$$\begin{aligned} (V(t)+P)(1+i) &= q_{x+t}\,(n-t) + p_{x+t}\,\left((DA)^{\,1}_{x+t+1:\overline{n-t-1}|} - P\,\ddot{a}_{x+t+1:\overline{n-t-1}|}\right) \\ &= q_{x+t}\,(n-t) + p_{x+t}\,V(t+1) \end{aligned}$$

as required.

(b) The reserve can be found directly, or by using the relationship in (a) twice from the boundary condition $V(n)=0$. Either way we need to find the net premium,

$$P = \frac{(DA)^{\,1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

We have $x=30$, $n=30$, mortality is according to the A1967–70 ultimate table and interest is 4% p.a.

$$\begin{aligned} \frac{(DA)^{1}_{30:\overline{30}|}}{\ddot{a}_{30:\overline{30}|}} &= \frac{31A^{1}_{30:\overline{30}|} - (IA)^{1}_{30:\overline{30}|}}{\ddot{a}_{30:\overline{30}|}} = \frac{31(M_{30}-M_{60}) - (R_{30}-R_{60}-30M_{60})}{N_{30}-N_{60}} \\ &= \frac{30M_{30} - R_{31} + R_{61}}{N_{30}-N_{60}} = \frac{30(1981.9552) + 21167.520 - 75245.722}{219735.21 - 35841.261} \\ &= 0.029258. \end{aligned}$$

Using the recursive relationship for the first time with $q_{59} = 0.01299373$, we have

$$(V(29) + 0.029258)(1.04) = 1 \times 0.01299373 + 0$$

so that $V(29) = -0.016764$. Using the recursive relation for the second time with $q_{58} = 0.01168566$, we have

$$(V(28) + 0.029258)(1.04) = 2 \times 0.01168566 + (1 - 0.01168566)(-0.016764)$$

so that $V(28) = -0.022716$.

(c) The problem is that the reserve is negative. A level premium has been used to pay for a decreasing risk. This could cause problems if the policyholder lapses the policy since it will leave the insurer with a loss. One way to deal with this problem is to charge a higher level premium payable for a term shorter than the term of the policy so that negative reserves do not arise.

4. (a)

$$ {}_{6.5}V_{35:\overline{25}|} \cdot (1+i)^{0.5} = {}_{0.5}q_{41.5} + {}_{0.5}p_{41.5} \cdot {}_{7}V_{35:\overline{25}|}. $$

But ${}_{7}V_{35:\overline{25}|} = A_{42:\overline{18}|} - P_{35:\overline{25}|} \cdot \ddot{a}_{42:\overline{18}|} = 0.50121 - 0.02393(12.969) = 0.19081.$

Assuming a constant force of mortality between exact ages 41 and 42, then $({}_{0.5}p_{41.5})^2 = p_{41}$ giving ${}_{0.5}p_{41.5} = 0.99949$. Therefore

$$ {}_{6.5}V_{35:\overline{25}|} = \frac{(1-0.99949) + 0.99949(0.19081)}{(1.04)^{0.5}} = 0.18751. $$

(b)

$$\left({}_{16}V_{35:\overset{1}{\overline{25}|}} + P_{35:\overset{1}{\overline{25}|}}\right)(1+i)^{0.25} = {}_{0.25}q_{51} \times 0 + {}_{0.25}p_{51} \cdot {}_{16.25}V_{35:\overset{1}{\overline{25}|}} \quad \text{and}$$

$$\begin{aligned} {}_{16}V_{35:\overset{1}{\overline{25}|}} &= A_{51:\overset{1}{\overline{9}|}} - P_{35:\overset{1}{\overline{25}|}} \cdot \ddot{a}_{51:\overline{9}|} = A_{51:\overset{1}{\overline{9}|}} - \frac{A_{35:\overset{1}{\overline{25}|}}}{\ddot{a}_{35:\overline{25}|}} \cdot \ddot{a}_{51:\overline{9}|} \\ &= \frac{D_{60}}{D_{51}} - \frac{\frac{D_{60}}{D_{35}}}{16.027}(7.625) = 0.50604. \end{aligned}$$

Assuming a constant force of mortality between exact ages 51 and 52, $({}_{0.25}p_{51})^4 = p_{51}$ giving ${}_{0.25}p_{51} = 0.9993$. Therefore

$${}_{16.25}V_{35:\overset{1}{\overline{25}|}} = \frac{(0.50604 + 0.02197)(1.04)^{0.25}}{0.9993} = 0.53358.$$

(c) We note that the premiums are paid quarterly and the sum assured is paid at the end of year of death or on survival to maturity.

$$\left({}_{16.75}V^{(4)}_{35:\overline{25}|} + 0.25 \times P^{(4)}_{35:\overline{25}|}\right)(1+i)^{0.25} = {}_{0.25}q_{51.75} + {}_{0.25}p_{51.75} \cdot {}_{17}V^{(4)}_{35:\overline{25}|}.$$

But

$$P^{(4)}_{35:\overline{25}|} = \frac{A_{35:\overline{25}|}}{\ddot{a}^{(4)}_{35:\overline{25}|}} = \frac{0.38359}{16.027 - \frac{3}{8}(1 - \frac{D_{60}}{D_{35}})} = 0.0244.$$

$${}_{17}V^{(4)}_{35:\overline{25}|} = A_{52:\overline{8}|} - (0.0244)\ddot{a}^{(4)}_{52:\overline{8}|} = 0.73424 - (0.0244)(6.91 - \frac{3}{8}(1 - \frac{D_{60}}{D_{52}})) = 0.56836$$

Assuming a constant force of mortality between exact ages 51 and 52, then ${}_{0.25}p_{51.75} = 0.9993$, giving

$${}_{16.75}V^{(4)}_{35:\overline{25}|} = \frac{(1 - 0.9993) + 0.9993(0.56836)}{(1.04)^{0.25}} - 0.25(0.0244) = 0.55701.$$

Also

$$\left({}_{16.5}V^{(4)}_{35:\overline{25}|} + 0.25 \times P^{(4)}_{35:\overline{25}|}\right)(1+i)^{0.25} = {}_{0.25}q_{51.5} \cdot v^{0.25} + {}_{0.25}p_{51.5} \cdot {}_{16.75}V^{(4)}_{35:\overline{25}|}$$

such that

$${}_{16.5}V^{(4)}_{35:\overline{25}|} = \frac{(1 - 0.9993)(1.04)^{0.25} + 0.9993(0.55701)}{(1.04)^{0.25}} - 0.25(0.0244) = 0.54579.$$

5. We have:

$$(V(t)+179.3)(1.04) = q_{x+t}(1,000+V(t+1))+p_{x+t}\,V(t+1) = 1,000\,q_{x+t}+V(t+1).$$

Solving backwards $V(4) = 784.48$, $V(3) = 577.00$ and $V(2) = 377.27$.

6. (a) Thiele's equation is:

$$\frac{d}{dt}V(t) = V(t)\,\delta - 1 + \mu_{x+t}\,V(t).$$

Intuitive reasoning: during time dt, interest of $V(t)\,dt$ is earned and annuity benefit of dt is paid out. If the annuitant dies during time dt, which happens with probability $\mu_{x+t}\,dt$ there is no sum assured to pay but the life office keeps the reserve $V(t)$.

For proof, note that the policy value is $V(t) = \bar{a}_{x+t}$ so that $\bar{A}_{x+t} = 1 - \delta\,V(t)$ and Thiele's equation above is thus equivalent to Question 2(b) of Tutorial 1.

The appropriate boundary condition is $V(n) = 0$.

(b) Thiele's equation is:

$$\frac{d}{dt}V(t) = V(t)\,\delta + \mu_{x+t}\,V(t).$$

Intuitive reasoning: during time dt, interest of $V(t)\,dt$ is earned and no benefit is paid out. If the annuitant dies during time dt, which happens with probability $\mu_{x+t}\,dt$ there is no sum assured to pay but the life office keeps the reserve $V(t)$.

For proof, note that the policy value is $V(t) = A_{x+t:\overline{n}|}^{\;\;1} = e^{-\delta(n-t)}\;{}_{n-t}p_{x+t}$. Hence:

$$\begin{aligned}
\frac{d}{dt}V(t) &= \delta\,e^{-\delta(n-t)}\;{}_{n-t}p_{x+t} + e^{-\delta(n-t)}\,\frac{d}{dt}\left({}_{n-t}p_{x+t}\right)\\
&= \delta\,V(t) + e^{-\delta(n-t)}\,\frac{d}{dt}\left(\frac{{}_np_x}{{}_tp_x}\right)\\
&= V(t)\,\delta + e^{-\delta(n-t)}\;{}_np_x\,\frac{-1}{{}_tp_x^2}\left(-{}_tp_x\,\mu_{x+t}\right)\\
&= V(t)\,\delta + e^{-\delta(n-t)}\left(\frac{{}_np_x}{{}_tp_x}\right)\mu_{x+t}\\
&= V(t)\,\delta + \mu_{x+t}\,V(t).
\end{aligned}$$

The appropriate boundary condition is $V(n) = 1$.

(c) Thiele's equation is of the form in (a) above during payment of the annuity, and of the form in (b) above during deferral.

The appropriate boundary condition is $V(\omega - x) = 0$, where ω is the limiting age of the life table.

7. We have:

$$\begin{aligned}\frac{d}{dt}V(t) &= V(t)\,\delta + P - \mu_{x+t}\,(10,000 + V(t) - V(t)) \\ &= V(t)\,\delta + P - 10,000\,\mu_{x+t}.\end{aligned}$$

The appropriate boundry condition is $V(n) = 10,000$.