Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 6 Solutions

1. In the following, we omit trivial equations of the form $\dfrac{d}{dt}V^i(t) = 0$.

   (a) $\dfrac{d}{dt}V^1(t) = V^1(t)\,\delta - (\mu^{12}_{y+t} + \mu^{13}_{x+t})(1 - V^1(t))$.

   (b) $\dfrac{d}{dt}V^1(t) = V^1(t)\,\delta - \mu^{12}_{y+t}\,(V^2(t) - V^1(t)) - \mu^{13}_{x+t}\,(V^3(t) - V^1(t))$
   $\dfrac{d}{dt}V^2(t) = V^2(t)\,\delta - \mu^{24}_{x+t}\,(1 - V^2(t))$
   $\dfrac{d}{dt}V^3(t) = V^3(t)\,\delta - \mu^{34}_{y+t}\,(1 - V^3(t))$.

   (c) $\dfrac{d}{dt}V^1(t) = V^1(t)\,\delta + \mu^{12}_{y+t}\,V^1(t) - \mu^{13}_{x+t}\,(1 - V^1(t))$.

   (d) $\dfrac{d}{dt}V^1(t) = V^1(t)\,\delta - \mu^{12}_{y+t}\,(V^2(t) - V^1(t)) + \mu^{13}_{x+t}\,V^1(t)$
   $\dfrac{d}{dt}V^2(t) = V^2(t)\,\delta - \mu^{24}_{x+t}\,(1 - V^2(t))$.

   (e) $\dfrac{d}{dt}V^1(t) = V^1(t)\,\delta - 1 + (\mu^{12}_{y+t} + \mu^{13}_{x+t})\,V^1(t)$.

   (f) $\dfrac{d}{dt}V^1(t) = V^1(t)\,\delta - 1 - \mu^{12}_{y+t}\,(V^2(t) - V^1(t)) - \mu^{13}_{x+t}\,(V^3(t) - V^1(t))$
   $\dfrac{d}{dt}V^2(t) = V^2(t)\,\delta - 1 + \mu^{24}_{x+t}\,V^2(t)$
   $\dfrac{d}{dt}V^3(t) = V^3(t)\,\delta - 1 + \mu^{34}_{y+t}\,V^3(t)$.

   (g) $\dfrac{d}{dt}V^1(t) = V^1(t)\,\delta + \mu^{12}_{y+t}\,V^2(t) - \mu^{13}_{x+t}\,(V^3(t) - V^1(t))$
   $\dfrac{d}{dt}V^3(t) = V^3(t)\,\delta - 1 + \mu^{34}_{y+t}\,V^3(t)$.

2. A spreadsheet to help with this exercise (tut6_q2.xls) can be downloaded from:

   `www.ma.hw.ac.uk/~andrea/f79AF`.

   The easiest approach is to program Thiele's equations with general annuity-type benefits b_i and assurance-type benefits b_{ij}, each defined in a separate cell, and then find the answers simply by setting each benefit to 0 or 1. The general equations are:

$$
\begin{aligned}
\frac{d}{dt}V^1(t) &= V^1(t)\,\delta - b_1 + \mu^{12}_{y+t}\,(b_{12} + V^2(t) - V^1(t)) - \mu^{13}_{x+t}\,(b_{13} + V^3(t) - V^1(t)) \\
\frac{d}{dt}V^2(t) &= V^2(t)\,\delta - b_2 + \mu^{24}_{x+t}\,(b_{24} - V^2(t)) \\
\frac{d}{dt}V^3(t) &= V^3(t)\,\delta - b_3 + \mu^{34}_{y+t}\,(b_{34} - V^3(t)).
\end{aligned}
$$

The answers (to 6 decimal places) are as follows:

(a) 0.215863

(b) 0.015415

(c) 0.130831

(d) 6.956499

(e) 7.829249

(f) 0.339573.

3. (a) ${}_{10}p_{30:40}$ is the probability that (30) and (40) both survive 10 years: ${}_{10}p_{30} \cdot {}_{10}p_{40} = 0.97853$.

   (b) $q_{30:40}$ is the probability that one or both of (30) and (40) die within one year: $1 - p_{30:40} = 0.001526$

   (c) $\mu_{40:50}$ multiplied by a small time element dt is interpreted as the probability that (40) or (50) or both die within time dt: $\mu_{40} + \mu_{50} = 0.003274$

   (d) ${}_{10}p_{[30]:[40]}$ is as for (a) but on a select basis: ${}_{10}p_{[30]} \cdot {}_{10}p_{[40]} = 0.97887$

   (e) $q_{[30]:[40]}$ is as for (b) but on a select basis: $1 - p_{[30]:[40]} = 0.001264$

   (f) $\mu_{[40]:[50]}$ is as for (c) but on a select basis: $\mu_{[40]} + \mu_{[50]} = 0.002293$

   (g) $\mu_{[40]+1:[60]+1}$ as for (f) but on select basis for (41) and (61) both with select duration 1: $\mu_{[40]+1:[60]+1} = 0.008129$.

   (h) ${}_{3|}q_{[30]+1:[40]+1}$ is the probability that one or both of (31) and (41), each with select duration of 1, will die within one year deferred for three years: ${}_{3|}q_{[30]+1:[40]+1} = 0.001976$

4. (a) The CDF of T_{max}, $P(T_{max} \leq t)$, denoted ${}_tq_{\overline{xy}}$, can be given as ${}_tq_{x\,t}q_y$. The density is therefore

$$
\begin{aligned}
f_{\overline{xy}}(t) &= \frac{d}{dt}{}_tq_{\overline{xy}} = \frac{d}{dt}{}_tq_{x\,t}q_y = \frac{d}{dt}\left(1 - {}_tp_x - {}_tp_y + {}_tp_{x\,t}p_y\right) \\
&= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{x\,t}p_y\left(\mu_{x+t} + \mu_{y+t}\right) \\
&= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t}
\end{aligned}
$$

(b) The density of T_{min} is ${}_tp_{xy}\mu_{x+t:y+t}$. Therefore its the expected value is given by $\mathrm{E}[T_{min}] = \int\limits_{t=0}^{\infty} t \cdot {}_tp_{xy}\mu_{x+t:y+t}dt$. Applying integration by parts, we let $u = t$ such that $\frac{du}{dt} = 1$ and we let $\frac{dv}{dt} = {}_tp_{xy}\mu_{x+t:y+t}$ such that $v = -{}_tp_{xy}$.

$$\mathrm{E}[T_{min}] = -t \cdot {}_tp_{xy}\Big|_{t=0}^{t=\infty} - \int\limits_{t=0}^{\infty} -{}_tp_{xy}dt = \int\limits_{t=0}^{\infty} {}_tp_{xy}dt.$$

(c)

$$\begin{aligned}
\mathrm{Cov}(T_{min}, T_{max}) &= \mathrm{E}[T_{min}T_{max}] - \mathrm{E}[T_{min}] \cdot \mathrm{E}[T_{max}] \\
&= \mathrm{E}[T_x] \cdot \mathrm{E}[T_y] - \mathring{e}_{xy}\left(\mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}\right) \\
&= \mathring{e}_x\mathring{e}_y - \mathring{e}_{xy}\left(\mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}\right) = \left(\mathring{e}_x - \mathring{e}_{xy}\right)\left(\mathring{e}_y - \mathring{e}_{xy}\right).
\end{aligned}$$

(d) i. The probability function of K_{min} is ${}_{t|}q_{xy}$ so that:

$$\begin{aligned}
\mathrm{E}[K_{min}] &= \sum_{k=0}^{k=\infty} k \cdot {}_{k|}q_{xy} = \sum_{k=0}^{k=\infty} k\left({}_kp_{xy} - {}_{k+1}p_{xy}\right) \\
&= 0\left({}_0p_{xy} - {}_1p_{xy}\right) + 1\left({}_1p_{xy} - {}_2p_{xy}\right) + 2\left({}_2p_{xy} - {}_3p_{xy}\right) + \cdots \\
&= {}_1p_{xy} + {}_2p_{xy} + {}_3p_{xy} + \cdots = \sum_{k=1}^{k=\infty} {}_kp_{xy}.
\end{aligned}$$

ii.

$$\begin{aligned}
\mathring{e}_{xy} &= \int\limits_{t=0}^{\infty} {}_tp_{xy}dt = \int\limits_{t=0}^{1} {}_tp_{xy}dt + \int\limits_{t=1}^{2} {}_tp_{xy}dt + \int\limits_{t=2}^{3} {}_tp_{xy}dt + \cdots \\
&\approx 0.5({}_0p_{xy} + {}_1p_{xy}) + 0.5({}_1p_{xy} + {}_2p_{xy}) + 0.5({}_2p_{xy} + {}_3p_{xy}) + \cdots \\
&= 0.5 + {}_1p_{xy} + {}_2p_{xy} + {}_2p_{xy} + \cdots = 0.5 + \sum_{k=1}^{k=\infty} {}_kp_{xy} = 0.5 + e_{xy}.
\end{aligned}$$

(e) The 'force of mortality' associated with T_{max} can be defined as

$$\mu_{\overline{x:y}}(t) = \frac{f_{\overline{xy}}(t)}{1 - F_{\overline{xy}}(t)} = \frac{{}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t}}{{}_tp_{\overline{xy}}}.$$

This way of defining a force is valid for any continuous random variable defining the time to a future event. For $t = 0$ we have $\mu_{\overline{x:y}}(0) = 0$. This may be surprising at first sight. However, considering the multiple-state model, for *both* lives to die in time dt requires two transitions, which is an event whose probability is $o(dt)$, hence:

$$\lim_{dt\to 0} \frac{\mathrm{P}[T_{max} \le dt | T_{max} > 0]}{dt} = \lim_{dt\to 0} \frac{o(dt)}{dt} = 0.$$

5. We have ${}_{10}p_x = \dfrac{l_{x+10:y}}{l_{x:y}} = 0.96$ and ${}_{10}p_y = \dfrac{l_{x:y+10}}{l_{x:y}} = 0.92$. Therefore the required probability is:

$$ {}_{10}p_x\left(1 - {}_{10}p_y\right) + {}_{10}p_y\left(1 - {}_{10}p_x\right) = 0.1136. $$

6. (a) This is the expected value of the random variable $\ddot{a}_{\overline{K_{min}+1}|}$. Therefore:

$$ \begin{aligned} \ddot{a}_{xy} &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} {}_{k|}q_{xy} = \sum_{k=0}^{\infty} \frac{1-v^{k+1}}{d}\left({}_{k}p_{xy} - {}_{k+1}p_{xy}\right) \\ &= \frac{1}{d}\sum_{k=0}^{\infty}\left({}_{k}p_{xy} - {}_{k+1}p_{xy} - v^{k+1}{}_{k}p_{xy} + v^{k+1}{}_{k+1}p_{xy}\right) \\ &= \frac{1}{d}\left(\sum_{k=0}^{\infty} {}_{k}p_{xy} - \sum_{k=0}^{\infty} {}_{k+1}p_{xy} - v\sum_{k=0}^{\infty} v^{k}{}_{k}p_{xy} + \sum_{k=0}^{\infty} v^{k+1}{}_{k+1}p_{xy}\right) \end{aligned} $$

But $\sum_{k=0}^{\infty} {}_{k+1}p_{xy} = \sum_{k=0}^{\infty} {}_{k}p_{xy} - 1$ and $\sum_{k=0}^{\infty} v^{k+1}{}_{k+1}p_{xy} = \sum_{k=0}^{\infty} v^{k}{}_{k}p_{xy} - 1$. Substituting gives

$$ \ddot{a}_{xy} = \frac{1}{d}\left((1-v)\sum_{k=0}^{\infty} v^{k}{}_{k}p_{xy}\right) = \sum_{k=0}^{\infty} v^{k}{}_{k}p_{xy} \quad \text{since} \quad d = 1 - v. $$

(b) This is the expected value of the random variable $\ddot{a}_{\overline{\min(K_{min}+1,n)}|}$.

$$ \begin{aligned} \ddot{a}_{xy:\overline{n}|} &= \sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_{xy} + {}_{n-1}p_{xy} \cdot \ddot{a}_{\overline{n}|} = \sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|}\left[{}_{k}p_{xy} - {}_{k+1}p_{xy}\right] + {}_{n-1}p_{xy} \cdot \ddot{a}_{\overline{n}|} \\ &= \sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|}\left[\left({}_{k}p_{x} + {}_{k}p_{y} - {}_{k}p_{\overline{xy}}\right) - \left({}_{k+1}p_{x} + {}_{k+1}p_{y} - {}_{k+1}p_{\overline{xy}}\right)\right] \\ &\qquad + \left({}_{n-1}p_{x} + {}_{n-1}p_{y} - {}_{n-1}p_{\overline{xy}}\right) \cdot \ddot{a}_{\overline{n}|} \\ &= \sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|}\left[\left({}_{k}p_{x} - {}_{k+1}p_{x}\right) + \left({}_{k}p_{y} - {}_{k+1}p_{y}\right) - \left({}_{k}p_{\overline{xy}} - {}_{k+1}p_{\overline{xy}}\right)\right] \\ &\qquad + \left({}_{n-1}p_{x} + {}_{n-1}p_{y} - {}_{n-1}p_{\overline{xy}}\right) \cdot \ddot{a}_{\overline{n}|} \\ &= \left(\sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_{x} + {}_{n-1}p_{x} \cdot \ddot{a}_{\overline{n}|}\right) + \left(\sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_{y} + {}_{n-1}p_{y} \cdot \ddot{a}_{\overline{n}|}\right) \\ &\qquad - \left(\sum_{k=0}^{n} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_{\overline{xy}} + {}_{n-1}p_{\overline{xy}} \cdot \ddot{a}_{\overline{n}|}\right) = \ddot{a}_{x:\overline{n}|} + \ddot{a}_{y:\overline{n}|} - \ddot{a}_{\overline{xy}:\overline{n}|} \end{aligned} $$

(c) $A_{\overline{xy}}$ is the expected value of the random variable $v^{K_{max}+1}$.

$$ \begin{aligned} A_{\overline{xy}} &= 1 - d\ddot{a}_{\overline{xy}} = 1 - d\left(\ddot{a}_{x} + \ddot{a}_{y} - \ddot{a}_{xy}\right) \\ &= (1 - d\ddot{a}_{x}) + (1 - d\ddot{a}_{y}) - (1 - d\ddot{a}_{xy}) = A_{x} + A_{y} - A_{xy}. \end{aligned} $$

7. The expected value of the random variable: $v^{K_{max}+1}$ is

$$A_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} \cdot {}_{k}|q_{\overline{xy}} \quad \text{where} \quad v = \frac{1}{1+i}.$$

The variance of $v^{K_{max}+1}$ is given by:

$$\begin{aligned}\text{Var}[v^{K_{max}+1}] &= \text{E}\left[\left(v^{K_{max}+1}\right)^2\right] - \left(\text{E}\left[v^{K_{max}+1}\right]\right)^2 \\ &= \sum_{k=0}^{\infty} \left(v^{k+1}\right)^2 \cdot {}_{k}|q_{\overline{xy}} - \left(A_{\overline{xy}}\right)^2 = \sum_{k=0}^{\infty} \left(v^2\right)^{k+1} \cdot {}_{k}|q_{\overline{xy}} - \left(A_{\overline{xy}}\right)^2.\end{aligned}$$

For a rate of interest j we define $V = 1/(1+j)$ and let $V = v^2$. This means that $j = i^2 + 2i$. Substituting in the above we get:

$$\text{Var}[v^{K_{max}+1}] = \sum_{k=0}^{\infty} V^{k+1} \cdot {}_{k}|q_{\overline{xy}} - \left(A_{\overline{xy}}\right)^2 = A^*_{\overline{xy}} - \left(A_{\overline{xy}}\right)^2$$

where the asterisk indicates rate of interest j.

8. (a) $\ddot{a}_{70:67} = 10.233$ (from tables).
   (b) $\ddot{a}^{(12)}_{70:67} \approx \ddot{a}_{70:67} - 0.458 = 9.775$.
   (c) $\ddot{a}_{70:67:\overline{10}|} = \ddot{a}_{70:67} - v^{10}\,{}_{10}p^m_{70} \cdot {}_{10}p^f_{67} . \ddot{a}_{80:77} = 7.458$.
   (d) $\ddot{a}^{(12)}_{70:67:\overline{10}|} = (\ddot{a}_{70:67} - 0.458) - v^{10}\,{}_{10}p^m_{70} \cdot {}_{10}p^f_{67} . (\ddot{a}_{80:77} - 0.458) = 7.204$.
   (e) $\ddot{a}_{\overline{70:67}} = \ddot{a}^m_{70} + \ddot{a}^f_{67} - \ddot{a}_{70:67} = 15.44$.
   (f) $\ddot{a}^{(12)}_{\overline{70:67}} = \ddot{a}_{\overline{70:67}} - 0.458 = 14.982$.

9. (a) $A_{\overline{xy}:\overline{n}|}$ is the EPV of an assurance of 1 payable at the end of the year in which the second of (x) and (y) dies, if that death occurs within n years.

$$A_{\overline{xy}:\overline{n}|} = \text{E}[v^{\min[K_{\max}+1,n]}] = \text{E}[1 - d\ddot{a}_{\overline{\min[K_{\max}+1,n]}|}] = 1 - d\text{E}[\ddot{a}_{\overline{\min[K_{\max}+1,n]}|}] = 1 - d\ddot{a}_{\overline{xy}:\overline{n}|}.$$

   (b) $\bar{A}_{xy:\overline{n}|}$ is the EPV of an assurance of 1 payable immediately upon the first death of (x) or (y), if that death occurs within n years.

$$\bar{A}_{xy:\overline{n}|} = \text{E}[v^{\min[T_{\min},n]}] = \text{E}[1 - \delta\bar{a}_{\overline{\min[T_{\min},n]}|}] = 1 - \delta\bar{a}_{xy:\overline{n}|}.$$