Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 8 Solutions

1. **Off-period** is the period which must be spent without sickness claims for later sickness to be considered separate from earlier spells for purpose of calculating sickness benefit. This is necessary if the sickness benefit falls after a certain period of sickness.

   Waiting period is the period after joining a sickness benefit scheme during which no sickness benefit is allowed.

   Deferred period is the period of sickness which must elapse before sickness benefit is payable.

2. (a) Consider time $t + dt$, and we have:

$$\begin{aligned}
{}_{t+dt}p_x^{13} &= {}_tp_x^{11}\,\mathrm{P}[\text{dies from able } x+t \to x+t+dt \mid \text{able at } x+t] \\
&\quad + {}_tp_x^{12}\,\mathrm{P}[\text{dies from ill } x+t \to x+t+dt \mid \text{ill at } x+t] \\
&\quad + {}_tp_x^{13}\,\mathrm{P}[\text{stays dead } x+t \to x+t+dt \mid \text{dead at } x+t] \\
&= {}_tp_x^{11}\,(\mu_{x+t}\,dt + o(dt)) + {}_tp_x^{12}\,(\nu_{x+t}\,dt + o(dt)) + {}_tp_x^{13}.
\end{aligned}$$

On rearranging, dividing by dt and taking the limit as $dt \to 0$, we get:

$$\frac{d}{dt}{}_tp_x^{13} = {}_tp_x^{11}\,\mu_{x+t} + {}_tp_x^{12}\,\nu_{x+t}.$$

To evaluate ${}_tp_x^{13}$ numerically we use the boundary conditions ${}_0p_x^{11} = 1$, ${}_0p_x^{12} = 0$ and ${}_0p_x^{13} = 0$, choose a small stepsize s and using Euler's method we have:

$$\begin{aligned}
{}_sp_x^{13} &\approx {}_0p_x^{13} + s\left[\left.\frac{d}{dt}{}_tp_x^{13}\right|_{t=0}\right] \\
&= 0 + s\left[{}_0p_x^{11}\,\mu_x + {}_0p_x^{12}\,\nu_x\right] \\
&= \mu_x\,s
\end{aligned}$$

Then we solve the related Kolmogorov equations for ${}_sp_x^{11}$ and ${}_sp_x^{12}$ (not shown here) and use these three values as the starting point for the next iteration, so:

$$
\begin{aligned}
{}_{2s}p_x^{13} &\approx {}_sp_x^{13} + s\left[\left.\frac{d}{dt}{}_tp_x^{13}\right|_{t=s}\right] \\
&= \mu_x\, s + s\left[{}_sp_x^{11}\,\mu_{x+s} + {}_sp_x^{12}\,\nu_{x+s}\right] \\
&= {}_sp_x^{11}\,(\mu_{x+s}\, s) + {}_sp_x^{12}\,(\nu_{x+s}\, s) + \mu_x\, s.
\end{aligned}
$$

These steps are repeated until the required occupancy probability is derived.

(b) (1) We can derive:

$$\frac{d}{dt}{}_tp_x^{11} = -{}_tp_x^{11}(\sigma_{x+t} + \mu_{x+t}) + {}_tp_x^{12}\,\rho_{x+t} = -{}_tp_x^{11}(2\mu) + {}_tp_x^{12}\mu \tag{1}$$

and:

$$\frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{11}\,\sigma_{x+t} - {}_tp_x^{12}\,(\rho_{x+t} + \nu_{x+t}) = {}_tp_x^{11}\,\mu - {}_tp_x^{12}(2\mu). \tag{2}$$

Differentiating Equation (1) w.r.t. t and substituting Equation (2) we get:

$$\frac{d^2}{dt^2}{}_tp_x^{11} = -2\mu\frac{d}{dt}{}_tp_x^{11} + \mu\frac{d}{dt}{}_tp_x^{12} = -2\mu\frac{d}{dt}{}_tp_x^{11} + \mu\left({}_tp_x^{11}\,\mu - {}_tp_x^{12}(2\mu)\right)$$

But from Equation (1):

$${}_tp_x^{12} = \frac{1}{\mu}\left(\frac{d}{dt}{}_tp_x^{11} + {}_tp_x^{11}(2\mu)\right)$$

so substututing this into the last equation gives:

$$\frac{d^2}{dt^2}{}_tp_x^{11} = -2\mu\frac{d}{dt}{}_tp_x^{11} + \mu^2{}_tp_x^{11} - 2\mu\left(\frac{d}{dt}{}_tp_x^{11} + {}_tp_x^{11}(2\mu)\right)$$

which leads to the second order differential equation:

$$\frac{d^2}{dt^2}{}_tp_x^{11} + 4\mu\frac{d}{dt}{}_tp_x^{11} + 3\mu^2{}_tp_x^{11} = 0. \tag{3}$$

We note that given ${}_tp_x^{11} = 0.5\,(e^{-\mu t} + e^{-3\mu t})$ then:

$$\frac{d}{dt}{}_tp_x^{11} = -0.5\mu\left(e^{-\mu t} + 3e^{-3\mu t}\right) \quad \text{and} \quad \frac{d^2}{dt^2}{}_tp_x^{11} = 0.5\mu^2\left(e^{-\mu t} + 9e^{-3\mu t}\right).$$

Substituting these three equations into the left-hand side of Equation (3) we get 0 which shows that:

$${}_tp_x^{11} = 0.5\left(e^{-\mu t} + e^{-3\mu t}\right)$$

satisfies the differential equation.

(2) For $0 \le d \le t$:

$$
\begin{aligned}
{}_{d,t}p_x^{12} &= \int_{s=t-d}^{t} {}_sp_x^{11}\,\mu\,{}_{t-s}p_{x+s}^{\overline{22}}ds \\
&= \int_{s=t-d}^{t} 0.5(e^{-\mu s}+e^{-3\mu s})\,\mu\,e^{-2\mu(t-s)}ds.
\end{aligned}
$$

Note that ${}_up_y^{\overline{22}} = e^{-2\mu u}$ so:

$$
\begin{aligned}
{}_{d,t}p_x^{12} &= 0.5\,e^{-2\mu t}\int_{s=t-d}^{t} \mu(e^{\mu s}+e^{-\mu s})ds \\
&= 0.5\,e^{-2\mu t}\left[e^{\mu s}-e^{-\mu s}\right]_{t-d}^{t} \\
&= 0.5\,e^{-2\mu t}\left[e^{\mu t}-e^{-\mu t}-e^{\mu(t-d)}+e^{-\mu(t-d)}\right] \\
&= 0.5\,\left[e^{-\mu t}-e^{-3\mu t}-e^{-\mu(t+d)}+e^{-\mu(3t-d)}\right].
\end{aligned}
$$

(3) Note that:

$$
\begin{aligned}
{}_tp_{30}^{12} = {}_{t,t}p_{30}^{12} &= 0.5\left[e^{-\mu t}-e^{-3\mu t}-e^{-2\mu t}+e^{-2\mu t}\right] \\
&= 0.5\,\left[e^{-\mu t}-e^{-3\mu t}\right].
\end{aligned}
$$

Hence for $0 \le d \le t$:

$$
{}_tp_{30}^{12} - {}_{d,t}p_{30}^{12} = 0.5\,\left(e^{-\mu(t+d)}-e^{-\mu(3t-d)}\right).
$$

(4) The expected present value is:

$$
\begin{aligned}
&= 10,000\int_0^{30} v^t\left({}_tp_{30}^{12} - {}_{0.25,t}p_{30}^{12}\right)dt \\
&= 10,000\int_0^{30} 0.5e^{-0.05t}\left(e^{-0.01(t+0.25)}-e^{-0.01(3t-0.25)}\right)dt \\
&= 5,000\int_0^{30} \left(e^{-0.06t-0.0025}-e^{-0.08t+0.0025}\right)dt \\
&= 5,000\left[-\frac{1}{0.06}e^{-0.06t-0.0025}+\frac{1}{0.08}e^{-0.08t+0.0025}\right]_0^{30} \\
&= 5,000\,(-2.7481+1.1368+16.6251-12.5313) \\
&= £12,413.
\end{aligned}
$$

3. This is the probability that a life aged x who is currently sick, and has been sick for duration z, will be healthy at age $x+t$. For this to happen, the life must recover

from the current sickness between ages $x+u$ and $x+u+du$, where du is small and $x \le x+u \le x+t$, (probability ${}_u p^{\overline{22}}_{x,z}\, \rho_{x+u,z+u}\, du$) and, from being healthy at age $x+u+du$, must be healthy again at age $x+t$ (probability ${}_{t-u}p^{11}_{x+u}$, up to terms that are $o(du)$). Hence the probability of the required life history, with the first recovery being at time u, is:

$$ {}_u p^{\overline{22}}_{x,z}\, \rho_{x+u,z+u}\, du\, {}_{t-u}p^{11}_{x+u} $$

and the required probability is the sum (*i.e.* integral) of these probabilities over all possible values of u.

4. Consider ${}_{t+dt}p^{13}_x$, where $dt > 0$, and condition on the state occupied at age $x+t$. The life may be:

   - dead (probability ${}_t p^{13}_x$);
   - healthy (probability ${}_t p^{11}_x$), in which case the life must die before age $x+t+dt$ (probability $\mu_{x+t}\, dt + o(dt)$); or
   - sick, in which case the life must have fallen sick for the last time between ages $x+u$ and $x+u+du$, for some u between 0 and t (probability ${}_u p^{11}_x\, \sigma_{x+u}\, du$), remained sick until age $x+t$ (probability ${}_{t-u}p^{\overline{22}}_{x+u}$) and then died before age $x+t+dt$ (probability $\nu_{x+t,t-u}\, dt$).

   Combining these probabilities, we have:

$$ {}_{t+dt}p^{13}_x = {}_t p^{13}_x + {}_t p^{11}_x\, \mu_{x+t}.dt + \int_{u=0}^{t} {}_u p^{11}_x .\sigma_{x+u}.{}_{t-u}p^{\overline{22}}_{x+u}.(\nu_{x+t,t-u}.dt)\, du + o(dt). $$

   Rearranging, dividing by dt and letting $dt \to 0$ gives the required integro-differential equation.

   To calculate numerical values of ${}_t p^{13}_x$, we choose a small stepsize s and Euler's method. From the boundary condition ${}_0 p^{13}_x = 0$, we get an approximate value for ${}_s p^{13}_x$. We carry out the second step using ${}_s p^{13}_x$ as the new starting value to get an approximate value of ${}_{2s}p^{13}_x$. The iterations are repeated for as many times as required to get the occupancy probability.

5. Consider ${}_{t+dt}p^{23}_{x,z}$. Using the same argument as in the answer to Question 3, and noting that if the life is ill at age $x+t$ we must treat separately the possibilities that they never left the ill state, or left and returned to the ill state, we have:

$$ \begin{aligned} {}_{t+dt}p^{23}_{x,z} &= {}_t p^{23}_{x,z} + {}_t p^{21}_{x,z}\, \mu_{x+t}\, dt + {}_t p^{\overline{22}}_{x,z}\, \nu_{x+t,z+t}\, dt \\ &\quad + \int_{u=0}^{t} {}_u p^{21}_{x,z}\, \sigma_{x+u}\, {}_{t-u}p^{\overline{22}}_{x+u}\, (\nu_{x+t,t-u}\, dt)\, du + o(dt). \end{aligned} $$

Rearranging, dividing by dt and letting $dt \to 0$ gives the following integro-differential equation:

$$\frac{d}{dt} {}_tp^{23}_{x,z} = {}_tp^{21}_{x,z}\, \mu_{x+t} + {}_tp^{\overline{22}}_{x,z}\, \nu_{x+t,z+t} + \int_{u=0}^{t} {}_up^{21}_{x,z}\, \sigma_{x+u}\, {}_{t-u}p^{\overline{22}}_{x+u}\, \nu_{x+t,t-u}\, du.$$