Parameter risk in time-series mortality forecasts

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- Projecting mortality rates is an essential part of valuing liabilities in life-insurance portfolios and pension schemes.
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- In particular, ARIMA processes and random-walks with drift are often used to generate scenarios for the period effects.
- While random walk models are the most widely used models, projections based on ARIMA models can look very different.

Motivation

- Parameters for those models need to be estimated and, therefore, parameter risk becomes an issue.
- We consider parameter risk from the point of view of an insurer using stochastic models for regulatory risk reporting.
- Decomposing overall risk into undiversifiable trend risk (parameter uncertainty) and diversifiable volatility.

Questions for this Presentation

- How should time series be projected for mortality forecasts?
- What is the importance of different sources of uncertainty?
- Is goodness of fit a reliable criterion for choosing forecasting models?
- What impact does parameter instability have on projected mortality rates and solvency capital requirements?
- How do central projections compare to the CMI model and how can we set the long-term rate in the CMI model?

The Lee-Carter model

$$D_{x,t} \quad \sim \quad \text{Poisson}\left(\mu_{x,t} E^c_{x,t}\right)$$

For each calendar year y and age x we observe

$D_{x,t}$: Number of deaths,

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Model for the force of mortality μ:

$$\log \mu_{x,t} = \alpha_x + \beta_x \kappa_t$$

with age effects α_x and β_x, and period effect κ_t.

The Lee-Carter model

Future liabilities in year $t+h$ will depend on the number of deaths $D_{x,t+h}$.

$$
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Sources of Uncertainty (given information up to current calendar year t):

- Poisson noise in year $t+h$
- uncertainty about estimated age parameters α_x, β_x

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- **uncertainty about future values κ_{t+h} of period effect κ**

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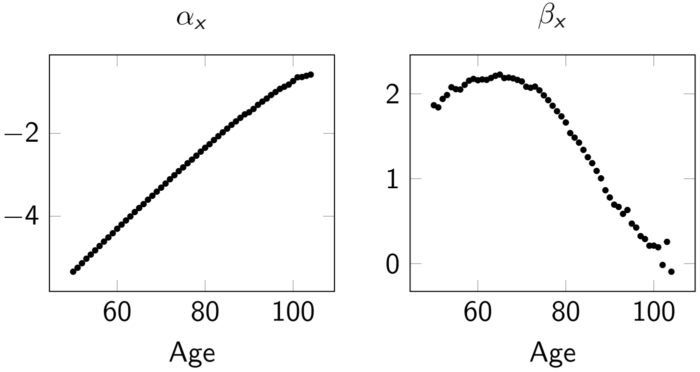
We will study the distribution of $\hat{\kappa}_t(h)$ and compare it to κ_{t+h} for an example data set (England & Wales).

The Lee-Carter model

Parameter estimates for Lee-Carter model fitted to mortality data for males in England & Wales aged 50–104 over the period 1971–2013.



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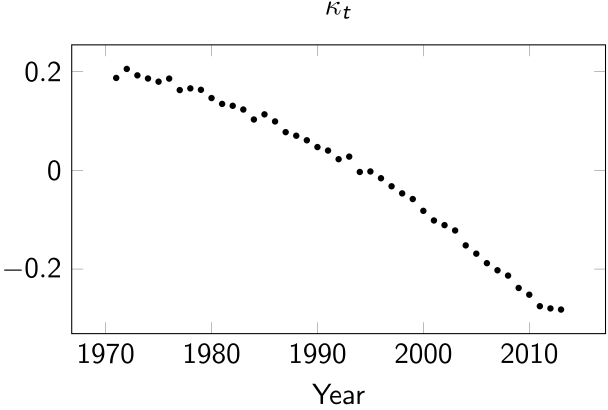

Period Effect as Random Walk

Model for the period effect κ ($\epsilon_t \sim N(0, \sigma_\epsilon^2)$ i.i.d.):

$$\kappa_{t+1} = \kappa_t + \mu_0 + \epsilon_{t+1}$$

And the realised value h years ahead is given by

$$\kappa_{t+h} = \kappa_t + h\mu_0 + \sum_{j=1}^{h} \epsilon_{t+j}$$

What is unknown at time t?

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- drift parameter μ_0: Replaced with an estimate, $\hat{\mu}_0$ to obtain a forecast estimator h years ahead as

$$\hat{\kappa}_t(h) = \kappa_t + h\hat{\mu}_0 \tag{1}$$

Period Effect as Random Walk

We will use the standard estimator

$$\hat{\mu}_0 = \frac{1}{t-1}\sum_{i=2}^{t}(\kappa_i - \kappa_{i-1}) = \frac{\kappa_t - \kappa_1}{t-1} \tag{2}$$

with variance:

$$\text{Var}(\hat{\mu}_0) = \text{Var}\left(\frac{\kappa_t - \kappa_1}{t-1}\right) = \frac{\sigma_\epsilon^2}{t-1} \tag{3}$$

For our data we obtain $\hat{\mu}_0 = -0.011176$.

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Our estimate of σ_ϵ^2 is the appropriate sample variance, σ_ϵ^2: which gives $\hat{\sigma}_\epsilon^2 = 0.00011$ ($\hat{\sigma}_\epsilon = 0.010512$).

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Projection error:

$$\mathrm{E}\left[\left(\hat{\kappa}_t(h) - \kappa_{t+h}\right)^2\right] = \underbrace{\frac{h}{t-1} h\sigma_\epsilon^2}_{\substack{\text{parameter}\\\text{uncertainty}}} + \underbrace{h\sigma_\epsilon^2}_{\text{volatility}} \tag{4}$$

where the parameter uncertainty is the variance of $h\hat{\mu}_0$, i.e. $h^2\mathrm{Var}(\hat{\mu}_0)$.

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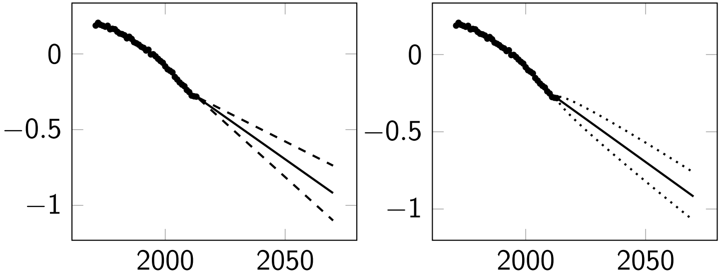
$$\textbf{Parameter uncertainty (variance of } h\hat{\mu}_0\textbf{)} = \frac{h}{t-1}\ \textbf{Volatility}$$

Period Effect as Random Walk

(i) Uncertainty about $\hat{\mu}$. (ii) Uncertainty from volatility.



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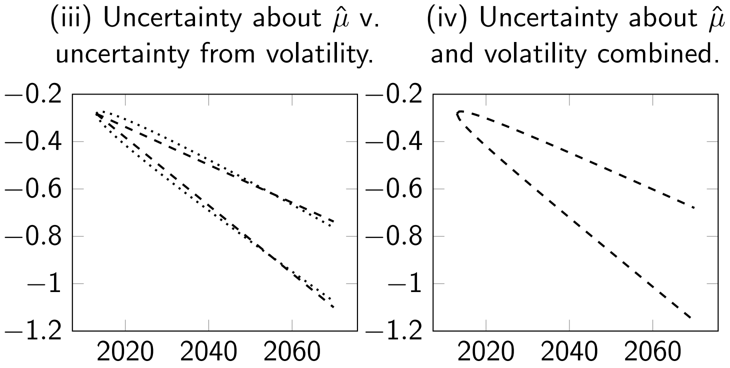

Period Effect as Random Walk

Predicted κ values $\hat{\kappa}_t(h)$ from RW model

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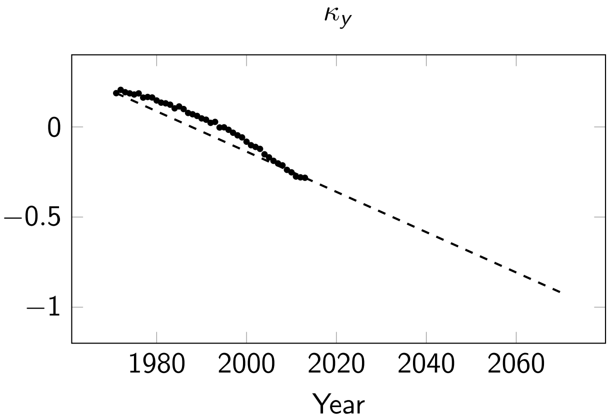
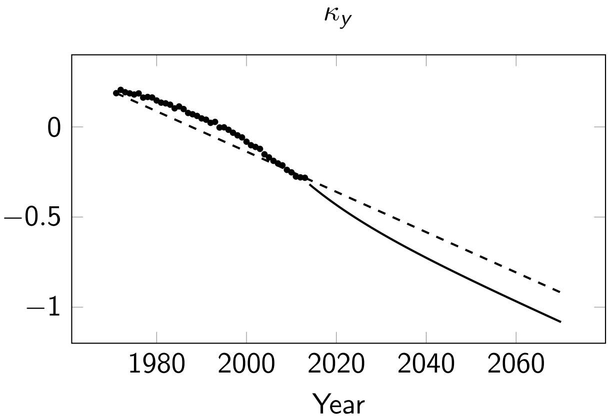


Period Effect as Random Walk

Predicted κ values $\hat{\kappa}_t(h)$ from RW model and ARIMA(1,1,2) model.



Period Effect as ARIMA process

The structure of an ARIMA(p,1,q) process is just like the structure of a RW:

$$\begin{aligned}\kappa_{t+1} &= \kappa_t + \mu + X^0_{t+1}\\ \kappa_{t+h} &= \kappa_t + h\mu + \sum_{i=1}^{h} X^0_{t+i}\end{aligned}$$

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But with a different noise process:

$$X^0_t = ar_1 X^0_{t-1} + \ldots + ar_p X^0_{t-p} + ma_1 \varepsilon_{t-1} + \ldots + ma_q \varepsilon_{t-q} + \varepsilon_t$$

where ε_t are i.i.d. normal.
In particular, an ARIMA(0,1,0) process is a random walk.

Period Effect as ARIMA process

$$\kappa_{t+h} \;=\; \kappa_t + h\mu + \sum_{i=1}^{h} X^0_{t+i}$$

We define h-step ahead projections for κ as in the previous section, that is:

$$\hat{\kappa}_t(h) \;=\; \kappa_t + \sum_{i=1}^{h} \hat{X}^0_t(i) + h\hat{\mu}$$

Period Effect as ARIMA process

h-step ahead projections for κ:

$$\hat{\kappa}_t(h) \quad = \quad \kappa_t + \sum_{i=1}^{h} \hat{X}_t^0(i) + h\hat{\mu}$$

For ARIMA(1,1,2) we obtain (all future noise terms ε are set to zero)

$$\begin{aligned} i = 1: & \qquad \hat{X}_t^0(1) = \hat{ar}_1 X_t^0 + \hat{ma}_1 \varepsilon_t + \hat{ma}_2 \varepsilon_{t-1} \\ i = 2: & \qquad \hat{X}_t^0(2) = \hat{ar}_1 \hat{X}_t^0(1) + \hat{ma}_2 \varepsilon_t \\ i > 2: & \qquad \hat{X}_t^0(i) = \hat{ar}_1^{i-2} \hat{X}_t^0(2) \end{aligned}$$

Period Effect as ARIMA process

As for the Random Walk, we estimate μ with

$$\hat{\mu} = \frac{1}{t-1}\sum_{i=2}^{t}(\kappa_i - \kappa_{i-1}) = \frac{\kappa_t - \kappa_1}{t-1} \tag{5}$$

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Variance of $\hat{\mu}$:

$$\mathrm{Var}\left(\hat{\mu}\right) = \frac{\mathrm{Var}(X^0)}{t} + \frac{2}{t} \sum_{k=1}^{t-1} \gamma(k) \left[1 - \frac{k}{t}\right] \tag{6}$$

where $\gamma(k) = \mathrm{Cov}(X_t^0, X_{t+k}^0)$ is the auto-covariance function of X^0.

Period Effect as ARIMA process

AICc (Akaike information criterion) values for various ARIMA(p, 1, q) models.

	q			
p	0	1	2	3
0	-260.16	-259.54	-260.81	-262.78
1	-260.22	-257.88	-269.83	-267.14
2	-258.10	-261.00	-267.14	-264.58
3	-258.95	-262.60	-264.17	-261.29

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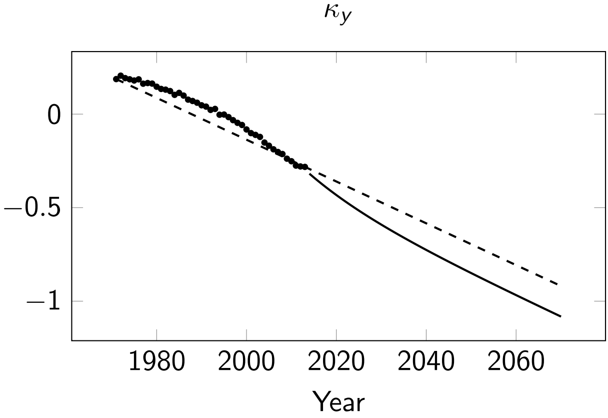
Period Effect as ARIMA process

Parameter estimates for ARIMA(1,1,2)

Parameter	Estimate	Standard error
ar_1	0.935	0.060
ma_1	-1.577	0.173
ma_2	0.815	0.149
σ_ϵ^2	0.000068	n/a
$\hat{\mu}$	-0.011	0.002

Period Effect as ARIMA process

κ values with RW and ARIMA(1,1,2) forecasts, $\hat{\kappa}_t(h)$.



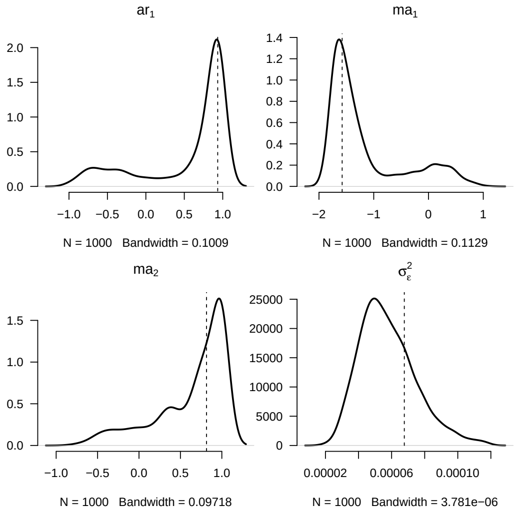
Period Effect as ARIMA process

A Bootstrap method (Pascual et al. 2004)[1] is applied to study uncertainty about the Time series parameters.
Idea:

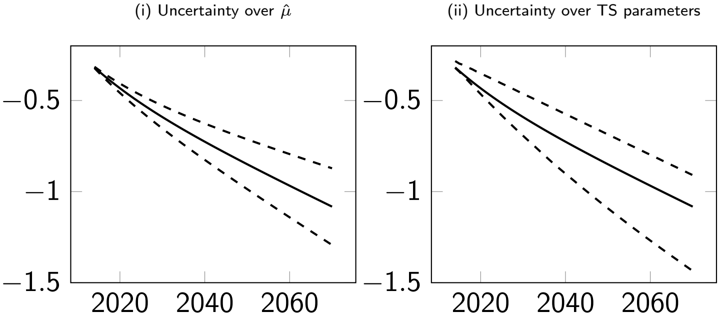
- Simulate the past many times using estimated parameters and
- then re-estimate parameters for each simulated scenario.
- This gives an empirical distribution for the estimated parameters.

[1] L. Pascual, J. Romo and E. Ruiz (2004): Bootstrap Predictive Inference for ARIMA Processes, Journal of Time Series Analysis 25(4)

Period Effect as ARIMA process



Forecast κ values from ARIMA(1,1,2)



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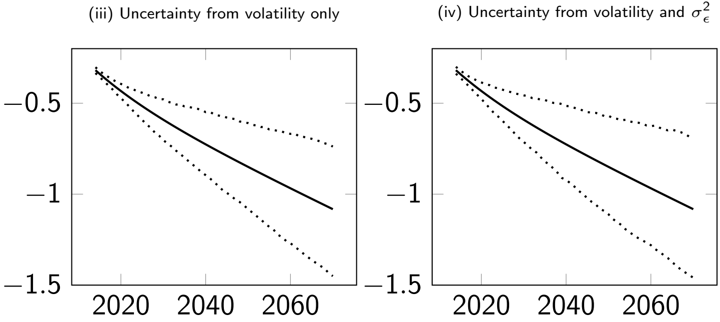


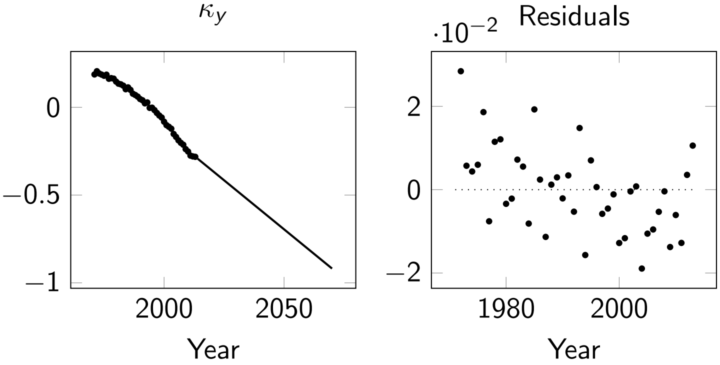
Alternative ARIMA(1,1,0) process

Long term central projections for ARIMA(p,1,q) processes depend on AR terms and drift, but not on MA terms.
But estimated parameter values and goodness of fit depend on both, AR and MA terms.
Estimated values for ARIMA(1,1,0):

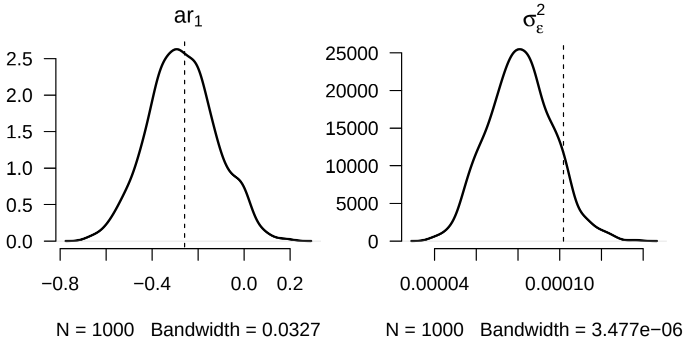
	ARIMA(1,1,0)		ARIMA(1,1,2)	
Parameter	Estimate	Std. error	Estimate	Std. error
ar_1	-0.259	0.166	0.935	0.060
σ_ϵ^2	0.000102		0.000068	
$\hat{\mu}$	-0.011	0.002		

Impact on Central projections

κ values with ARIMA(1,1,0) forecast and residuals from the ARIMA(1,1,0) fit



Period Effect as ARIMA process



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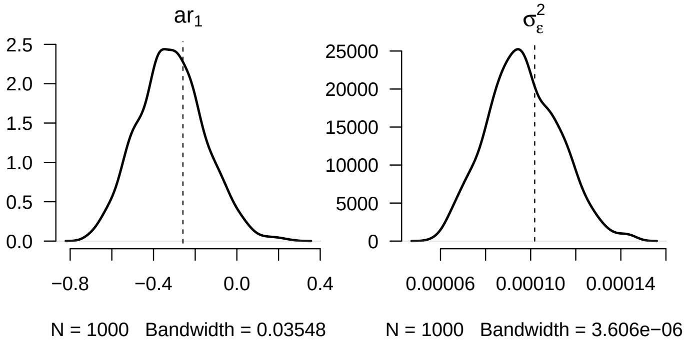
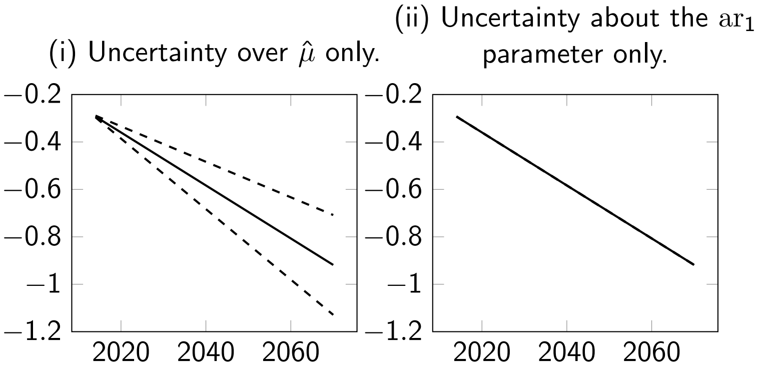


Period Effect as ARIMA process

Figure: κ values with forecast from ARIMA(1,1,0) model with 95% bounds for various kinds of uncertainty.



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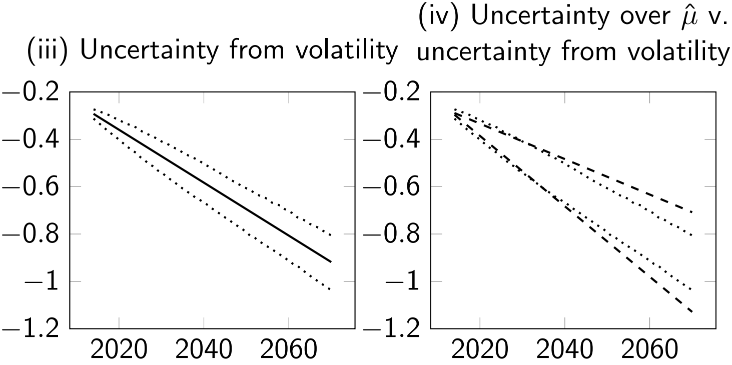


Period Effect as ARIMA process

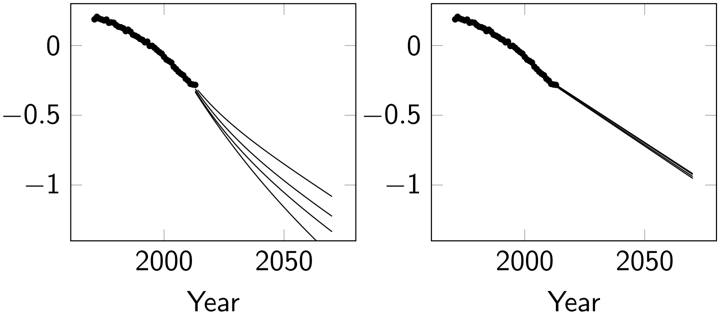
The importance of ar_1 can be seen when central projections are considered, i.e. where we set future error terms ε to zero.

h	RW and ARIMA(1,1,0)	ARIMA(1,1,2)
1	$X_t^0(1) = ar_1 X_t^0$	$X_t^0(1) = ar_1 X_t^0 + ma_1 \varepsilon_t + ma_2 \varepsilon_{t-1}$
2	$X_t^0(2) = ar_1^2 X_t^0$	$X_t^0(2) = ar_1 X_t^0(1) + ma_2 \varepsilon_t$
> 2	$X_t^0(h) = ar_1^{h-2} X_t^0(2)$	

Random Walk: $ar_1 = 0$
ARIMA(1,1,0): $ar_1 = -0.259$
ARIMA(1,1,2): $ar_1 = 0.935$

Period Effect as ARIMA process

Figure: Sensitivity of central projections when up to three years are removed from the end of the sample. Left panel: ARIMA(1,1,2) model. Right panel: ARIMA(1,1,0) model.



Capital Requirements

Model	Volatility	Para. uncert.	$\bar{a}^{50\%}_{\overline{70:35}\rceil}$	$\bar{a}^{99.5\%}_{\overline{70:35}\rceil}$	VaR99.5 capital	CTE99 capital
RW	Yes	No	12.50	12.70	1.62%	1.70%
	No	Yes	12.50	12.54	0.33%	0.34%
	Yes	Yes	12.49	12.72	1.79%	1.96%
(1,1,0)	Yes	No	12.51	12.68	1.36%	1.40%
	No	Yes	12.51	12.55	0.31%	0.34%
	Yes	Yes	12.51	12.69	1.43%	1.58%
(1,1,2)	Yes	No	12.59	12.87	2.25%	2.32%
	No	Yes	12.53	12.61	0.63%	0.64%
	Yes	Yes	12.53	12.87	2.70%	2.79%

interest rate: 2.5% p.a.

Capital Requirements

- Both Value-at-risk and CTE calculations are driven by the variability of mortality experience over a one-year horizon and how the model fit responds to this.
- Volatility makes the largest contribution: short time horizon for projections (1 year)
- The best fitting model, ARIMA(1,1,2), leads to
  - highest capital requirements
  - highest extra requirements for parameter uncertainty

Comparing Time Series Models

Random Walk

- standard model, most commonly used, very simple statistical methods required
- least flexible, few estimated parameters and strong assumptions (mortality improvements over time are i.i.d.)

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ARIMA(1,1,2)

- best fitting model
- great uncertainty about parameters and projected rates leading to high capital requirements

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- least flexible, few estimated parameters and strong assumptions (mortality improvements over time are i.i.d.)

ARIMA(1,1,2)

- best fitting model
- great uncertainty about parameters and projected rates leading to high capital requirements

ARIMA(1,1,0)

- similar goodness of fit as RW
- lowest capital requirements
- weaker assumptions than RW allowing for structure in error terms

Conclusions

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- Choosing a model requires actuarial judgement taking objectives into account

Questions?