Factor-Based GLM Model with Socio-Economic Inputs

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Content

Data Overview

Multi-population mortality data for England:

- Death and exposure counts at granular level - D_{itx} and E_{itx} available at each individual LSOA i, in year t at age x.
- Overall age range is 0-90 and for years 2001-2016. We concentrate on the 'pensionable' age 65-89.
- Set of socio-economic factors available at LSOA-level, i.e. indices and numerical measures related to income, education, housing affordability, etc.
- Categorical variables indicating certain features of each individual LSOA, i.e. which region is it located in, which urban-rural class does it belong to, etc.

Data Overview (cont.)

Actual-vs-Expected Death Ratio (A-E Ratio):

$$D_i = \sum_{tx} D_{itx} \quad \text{actual deaths at LSOA i}$$

$$\hat{D}_i^0 = \sum_{tx} m_{tx}^b E_{itx} \quad \text{'expected' deaths at LSOA i}$$

$$m_{tx}^b = \frac{\sum_i D_{itx}}{\sum_i E_{itx}} \quad \text{national-level death rate}$$

$$y_i = \frac{D_i}{\hat{D}_i^0} \quad \text{A-E Ratio of LSOA i}$$

It measures the relative mortality at individual LSOAs to national level. A-E ratio helps on lots of analysis as it is a scalar LSOA-level mortality measure, which makes works more straight forward.

Data Overview (cont.)

Considerations:

- Individual LSOAs have limited exposure size and are likely to make modelling volatile.
- Deaths and exposures can be combined over all LSOAs. National-level mortality is modelled by the combined data.
- LSOA-level mortalities are modelled via top-down approach starting with national mortality.
- Factors at LSOA-level are used as predictors (input variables) to model the LSOA-specific mortality shift from the national level.
- It limits the volatilities from small death/exposure sizes at individual LSOAs.

Data Overview (cont.)

Some of the LSOA-level socio-economic factors are listed as example. Most of them are dated in 2011 and gender neutral.

Predictive variables / Risk factors	**Code**
IMD income score	inc
IMD income-old score	incOld
IMD employment score	emp
IMD education score	edu
IMD crime score	cri
IMD barriers to housing and services score	bHS
IMD wider barriers score	widerB
Average number of bedrooms	numb
Prportion of population above age 65 with no qualification	pNoqual65
Ratio of UK-born population	rUK
Proportion of population living in care home with nursing	pCHwithN
Proportion of population living in care home without nursing	pCHnoN
Proportion of population working 49+ hours weekly	work49h

Data Overview (cont.)

- Most of the variables studied have been standardized to be $N(0,1)$ distributed.
- Deprivation scores are constructed from quite different indicators and some other factors are taken as different measurements, they are at quite different scales from each other.
- Standardization transforms different factors to consistent scale, which avoids the modelling from being significantly distorted.
- Standardization follows uniform and normal transformation step-by-step. Starting from the original factors F:

For certain variable F_i at LSOA $i = 1, 2, 3, \ldots, N$:

$$U_i = \frac{rank(F_i)}{N+1}$$

$$X_i = \Phi^{-1}(U_i) \quad \Phi \text{ is the probability function of N(0,1)}$$

Content

Model Outlook

- m_{itx} is death rate at LSOA i;
- m^b_{tx} is national level death rate ("base mortality");
- RR_{itx} is the relative risk factor of LSOA i.

The model has its overall structure as:

$$\log m_{itx} = \log m^b_{tx} + RR_{itx}$$

The national-level base mortality m^b_{tx} can be evaluated by different methods

$$m^b_{tx} = \begin{cases} \text{crude national death rates} \\ \text{OR} \\ \kappa^1_t + \kappa^2_t(x - \bar{x}) \quad \text{or other mortality models} \end{cases}$$

Relative risk factors RR_{itx}:
Evaluated as linear combination of risk factors at LSOA-level as a typical GLM does.

Model Outlook (cont.)

A simple model structure is:

$$\log m_{itx} = \underbrace{\kappa_t^1 + \kappa_t^2(x - \bar{x})}_{m_{tx}^b} + \underbrace{\sum_{j=1}^{p} \beta_j X_{ji}}_{RR_{itx}}$$

- κ's capture the overall period effect of mortality at national level.
- β's are coefficients of the socio-economic risk factors mapping their effects onto mortality of individual LSOAs.
- The national-level mortality is modelled by the CBD structure, while the predictors shift individual LSOAs up and down from national.
- This structure assumes all LSOAs follow the same age pattern as national-level mortalities do.
- This structure assumes factors X have constant effects onto mortality over time.

Model Outlook (cont.)

Now age interaction and time dependecy are added to capture more complex patterns.

$$\log m_{itx} = \underbrace{\kappa_t^1 + \kappa_t^2(x - \bar{x})}_{m_{tx}^b} + \underbrace{\sum_{j=1}^{p} \Big(\beta_{j0,t} + \beta_{j1,t}(x - \bar{x}) \Big) X_{ji}}_{RR_{itx}}$$

- For each individual predictor j, there are intercept β_{j0} and age interaction β_{j1}.
- Coefficients vary over time, which capture timely movements of mortality pattern over the socio-economic factors.
- The estimated β's indicate for how much do their corresponding factors contribute to inequality of mortality over different LSOAs.

Model Outlook (cont.)

The GLM coefficients are estimated by the Maximum Liklihood Estimation (MLE) approach with the deaths and exposures data, D_{itx} and E_{itx}. Likelihood functions are constructed based on Poisson assumption for the number of deaths (death frequency).

$$D_{itx} \sim Poisson(\hat{D}_{itx})$$
$$\hat{D}_{itx} = E_{itx}\hat{m}_{itx}$$

The estimated death rate $\hat{m}_{itx}$ is modelled by the GLM-type models with deaths, exposures and socio-economic factors as input data. The Poisson likelihood function is therefore constructed as:

$$L(\theta|D, E) = \prod_{itx} \frac{e^{-\hat{D}_{itx}} \hat{D}_{itx}^{D_{itx}}}{D_{itx}!}$$

Content

Parameter Estimation and Model Selection Criteria

- The main model selection criteria is BIC, defined as $k \log N - 2 \log \hat{L}$, with number of unknown parameters k and sample size N (smaller BIC is better following the definition).
- BIC penalizes overfitting if too many redundant variables are included.
- Many different combinations of variables are tested and those giving most preferable BICs are highlighted for further study.

List of combinations with most preferable BIC:

$\underline{X}$	**LL**	**BIC**
incOld+emp+pNoqual65+widerb	$-6,569,457$	$13,141,537$
incOld+pNoqual65+numb+rUK	$-6,569,646$	$13,141,914$
incOld+pNoqual65+numb+widerb	**-6,569,364**	**13,141,351**
incOld+emp+pNoqual65+numb+rUK	$-6,569,263$	$13,141,674$

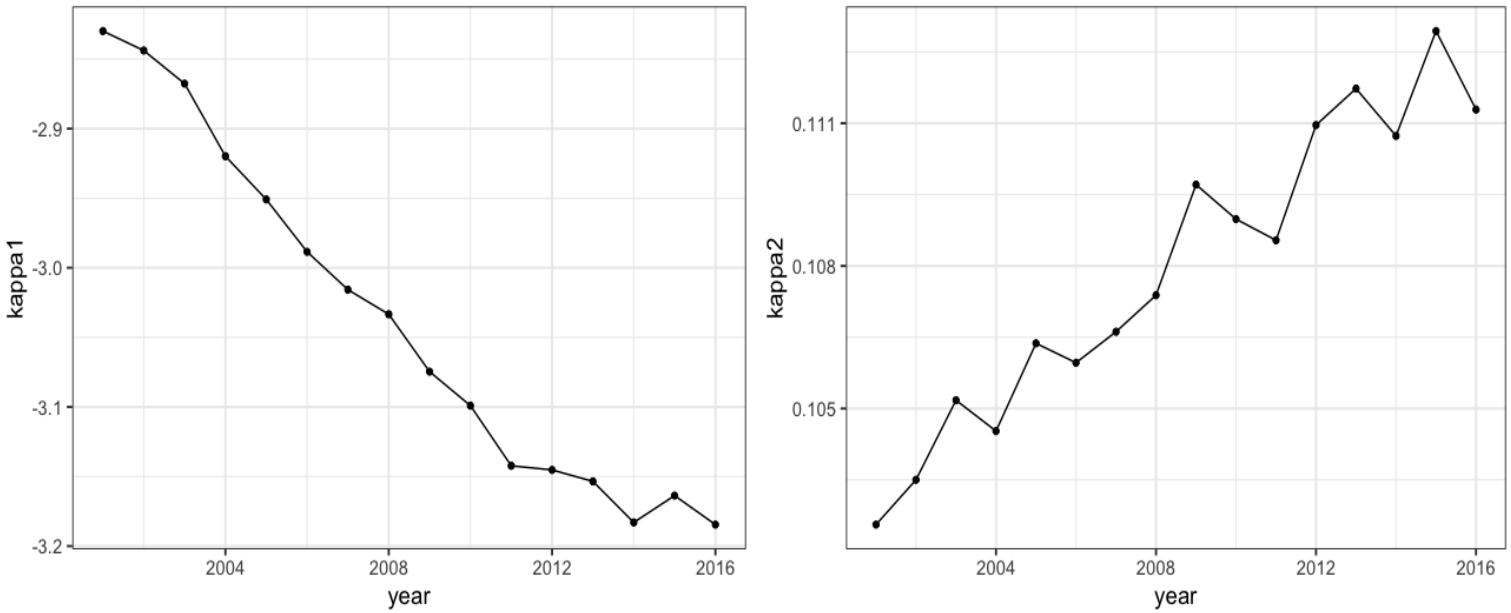
Parameter Estimation and Model Selection Criteria (cont.)

The estimated parameters under the BIC-optimized profile: incOld+pNoqual65+numb+widerb

$$\begin{aligned}\log m_{itx} = \kappa_t^1 + \kappa_t^2(x-\bar{x}) &+ \big(\beta_{10,t} + \beta_{11,t}(x-\bar{x})\big)X_1 \\ &+ \big(\beta_{20,t} + \beta_{21,t}(x-\bar{x})\big)X_2 \\ &+ \big(\beta_{30,t} + \beta_{31,t}(x-\bar{x})\big)X_3 \\ &+ \big(\beta_{40,t} + \beta_{41,t}(x-\bar{x})\big)X_4\end{aligned}$$

Parameter Estimation and Model Selection Criteria (cont.)

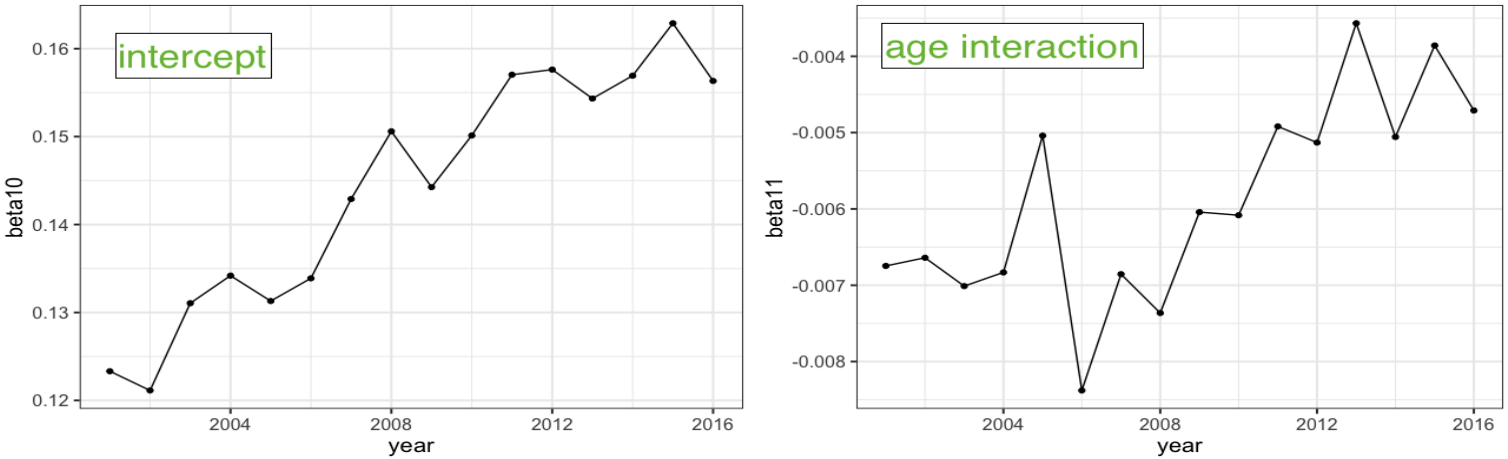
The common κ's at national level:



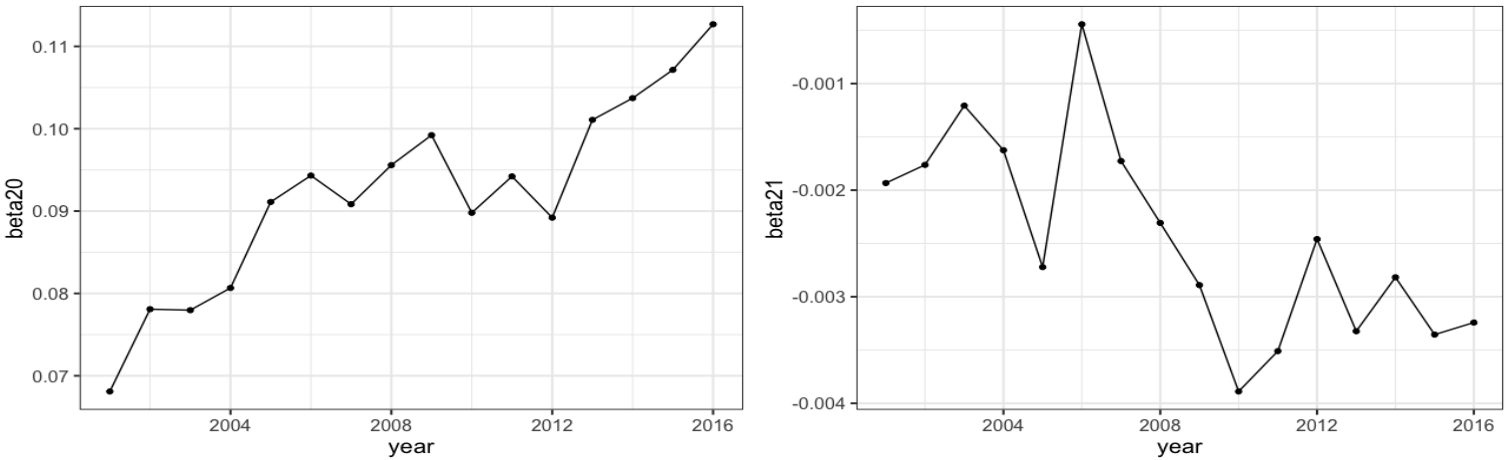
- κ^1 shows the declining mortality at national level over 2001-2016, with improvement slown down slightly around 2011.
- κ^2 being positive shows there has been bigger improvement at relatively young ages (below the mean age).

Parameter Estimation and Model Selection Criteria (cont.)

IMD income-old score (*incOld*):

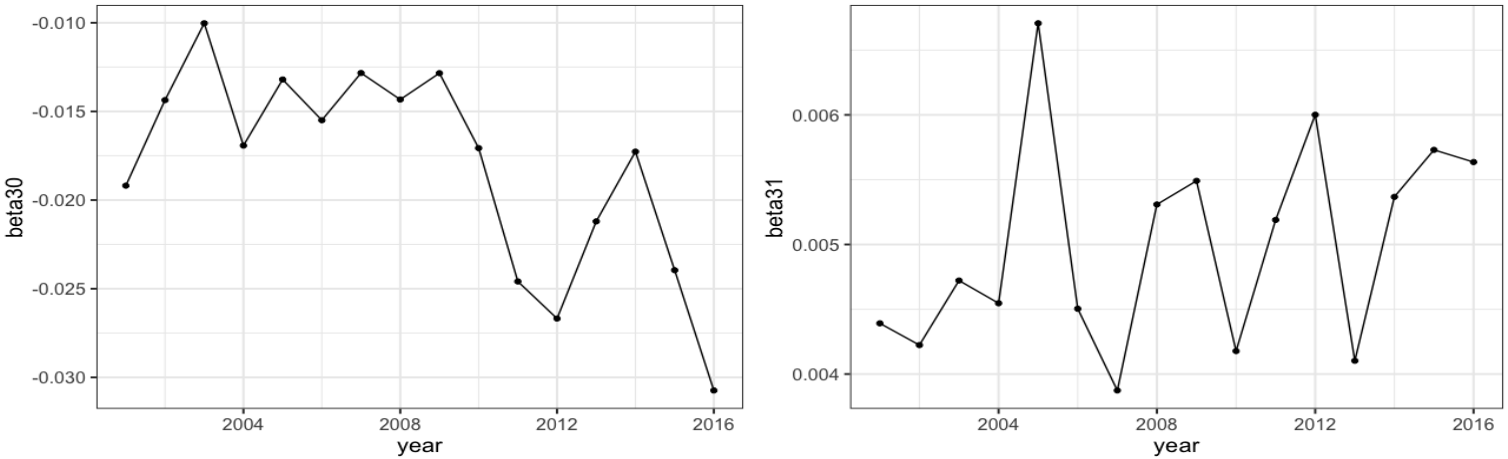


Proportion of no-qualification population above 65 (*pNoqual*65):

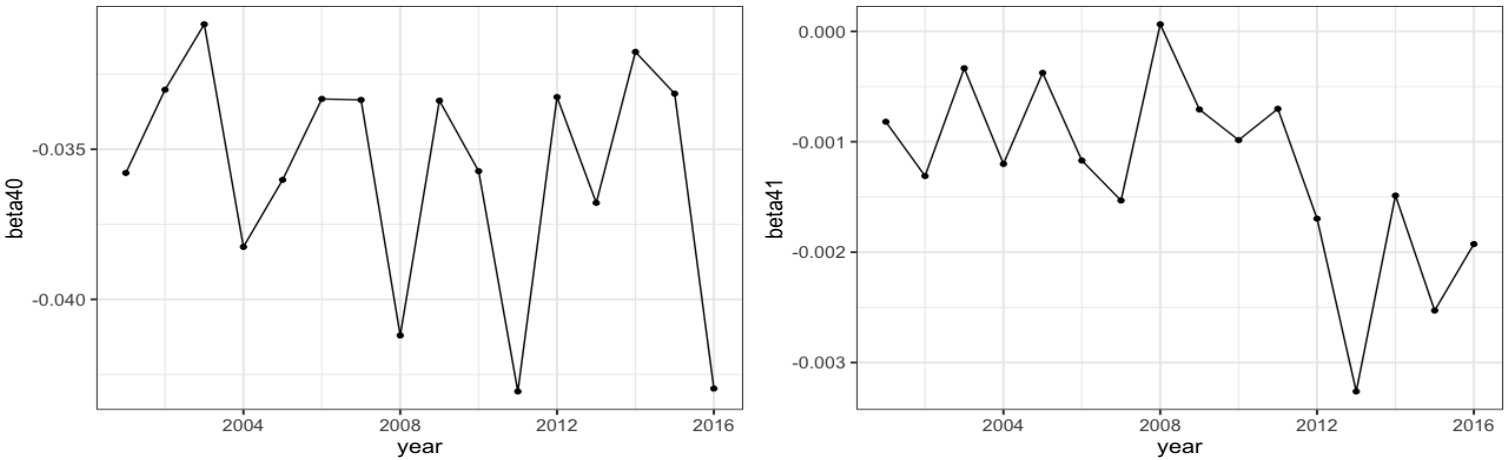


Parameter Estimation and Model Selection Criteria (cont.)

Average number of bedrooms (*numb*):



IMD wider barriers score (*widerb*):



Parameter Estimation and Model Selection Criteria (cont.)

Conclusions from estimated coefficients

The 4 important variables are:

- incOld is IMD income-old deprivation score indicating proportion of population receiving relevant type of benefit due to income shortage.
- pNoqual65 is proportion of population over age 65 with no degree qualification.
- numb is the average number of bedrooms per household.
- widerb is the IMD wider-barriers deprivation score indicating some housing related facts: overcrowding, affordability, etc.

Parameter Estimation and Model Selection Criteria (cont.)

- β_{10} and β_{20} have increasing trend indicating increasing contribution from *incOld* and *pNoqual*65 to inequality of mortality across different LSOAs over 2001-2016.
- β_{30} and β_{40} do not have clear pattern and their fluctuations are more likely to be from sampling variation.
- LSOAs with large value of *incOld* and *pNoqual*65 tend to have higher mortality.
- *widerb* has negative sign over all years, which is counterintuitive. Few reasons drive its behavior: effects from London-based LSOAs, dependency between different risk factors (e.g. with incOld), etc.

Parameter Estimation and Model Selection Criteria (cont.)

Considerations on the estimated cofficients β:

- Except for β_{10} and β_{20}, the other coefficients do not have any signficant pattern over time.
- Their fluctuations over time are more likely to be from sampling noise rather than crude mortality pattern.
- They can be simplifed to constant coefficients for mitigating over-parameterization.
- BIC is improved to **13,140,030** from **13,141,351** after the simplification. The simplified model has formula of:

$$\begin{aligned}\log m_{itx} = \kappa_t^1 + \kappa_t^2(x-\bar{x}) &+ \left(\beta_{10,t} + \beta_{11}(x-\bar{x})\right) \times incOld \\ &+ \left(\beta_{20,t} + \beta_{21}(x-\bar{x})\right) \times pNoqual65 \\ &+ \left(\beta_{30} + \beta_{31}(x-\bar{x})\right) \times numb \\ &+ \left(\beta_{40} + \beta_{41}(x-\bar{x})\right) \times widerb\end{aligned}$$

Parameter Estimation and Model Selection Criteria (cont.)

There are still further works to do around the GLM model regarding to flexible time dependency, flexible age interaction and finding out the most appropriate variables to use in the GLM predictor.

We have too many variables that are potentially related to mortality. However we are not using all of them, otherwise the model will be overfitted and any interaction between variables can distort the modelling result.

As a part of GLM modelling, we work on variable selection by different techniques to pick up the most important variables from the whole universe.

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Variable Selection

1. Shrinkage

For variable selection purpose, LASSO is one of the commonly used machine learning approaches. It is based on optimization of penalized model fitting criteria, i.e. MLE or OLS.

LASSO Penalized Log-likelihood

$$l^L(\underline{\beta}) = l(\underline{\beta}) - \lambda \sum_{j=1}^{p} |\beta_j|$$

Shrink ratio of LASSO

$$sr^L(\lambda) = \frac{\sum_j |\hat{\beta}_j^{L,\lambda}|}{\sum_j |\hat{\beta}_j|}$$

As λ increasing from zero, the estimated coefficients $\hat{\beta}$ are shrinked and reduced towards zero.

Variable Selection (cont.)

Simplified GLM (no age interaction or time dependency) is used to calculate the Poisson MLE for LASSO, as the target is to select important variables instead of modelling accurately.

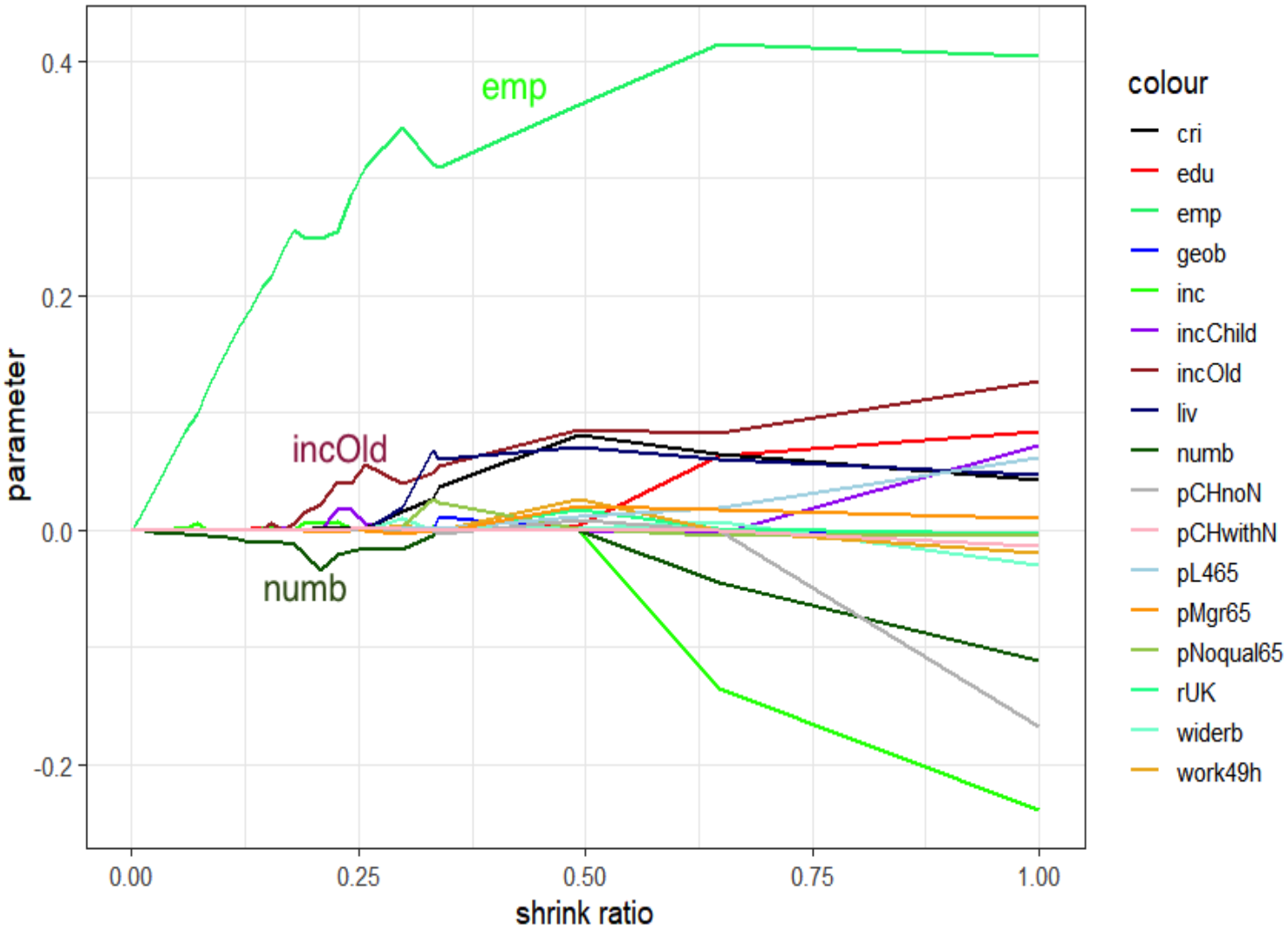
$$\log m_{itx} = \log m^b_{tx} + \beta_0 + \sum_{j=1}^{p} \beta_j X_{ji}$$

with m^b_{tx} representing the base mortality derived from crude observations taking all LSOAs into account, $m^b_{tx} = \frac{\sum_i D_{itx}}{\sum_i E_{itx}}$.

The simplifed model should be fitted over relatively narrow age buckets as there is no age interaction, i.e. 5-year or 10-year.

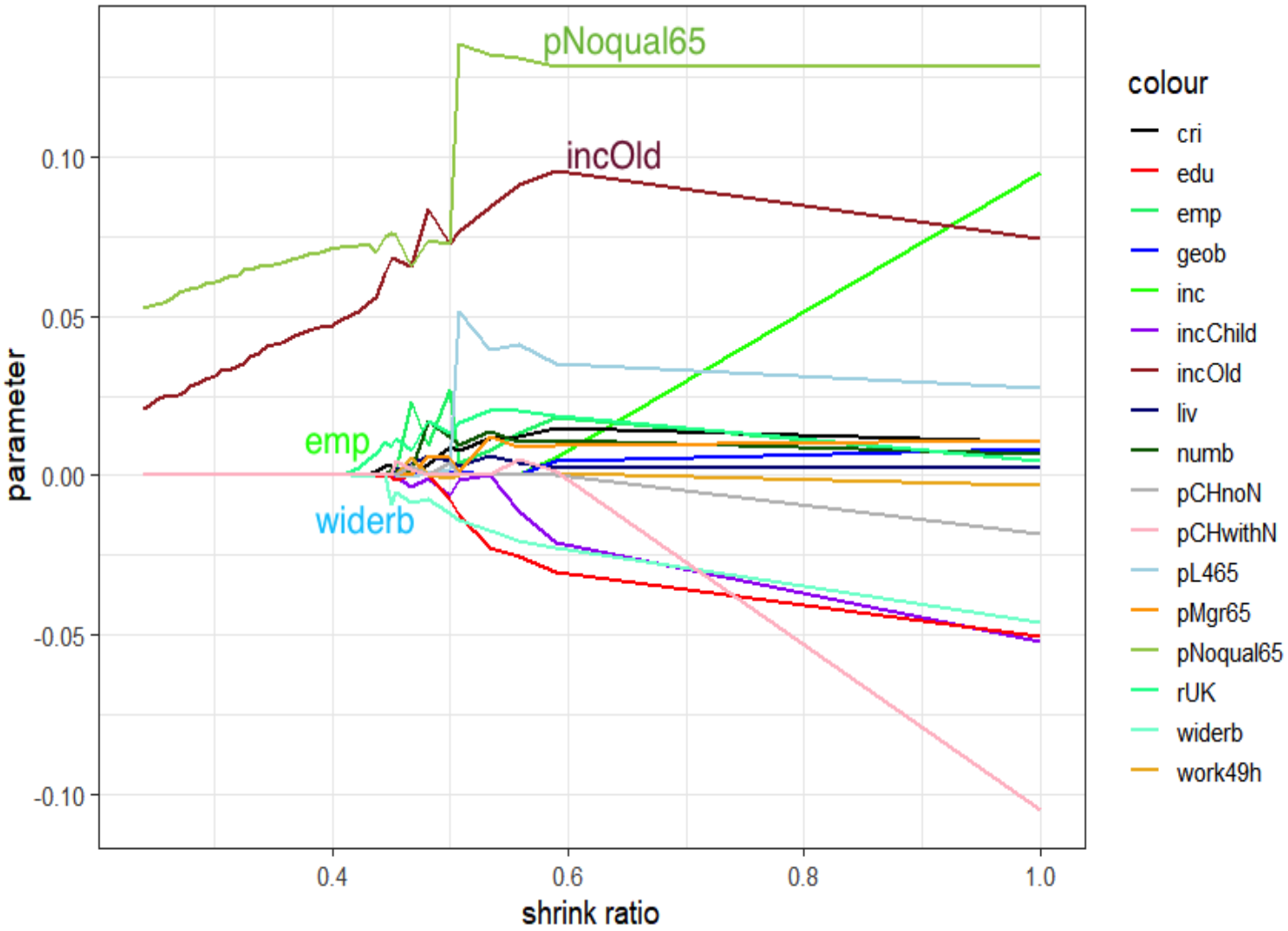
Variable Selection (cont.)

Age 41-45:



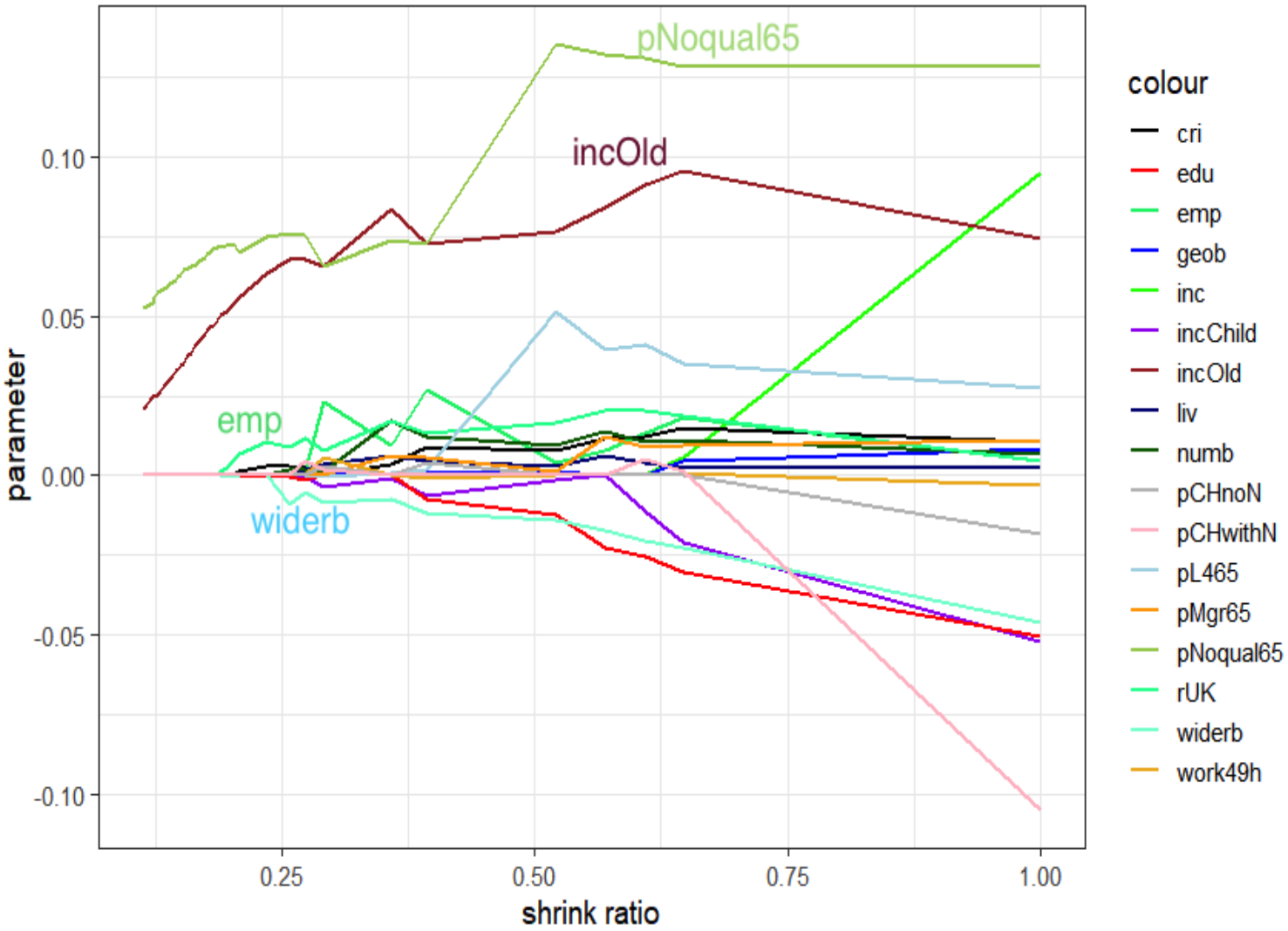
Variable Selection (cont.)

Age 61-65:



Variable Selection (cont.)

Age 81-85:



Variable Selection (cont.)

LASSO suggests the important variables over different age buckets to be:

- 41-45: emp, incOld, numb
- 61-65: pNoqual65, incOld, emp, widerb
- 81-85: pNoqual65, incOld, emp, widerb

Variables suggested to be important by LASSO have been broadly consistent with those having the most preferrable BIC of the GLM model.

Variable Selection (cont.)

2. Regression Tree and Random Forest (RF)

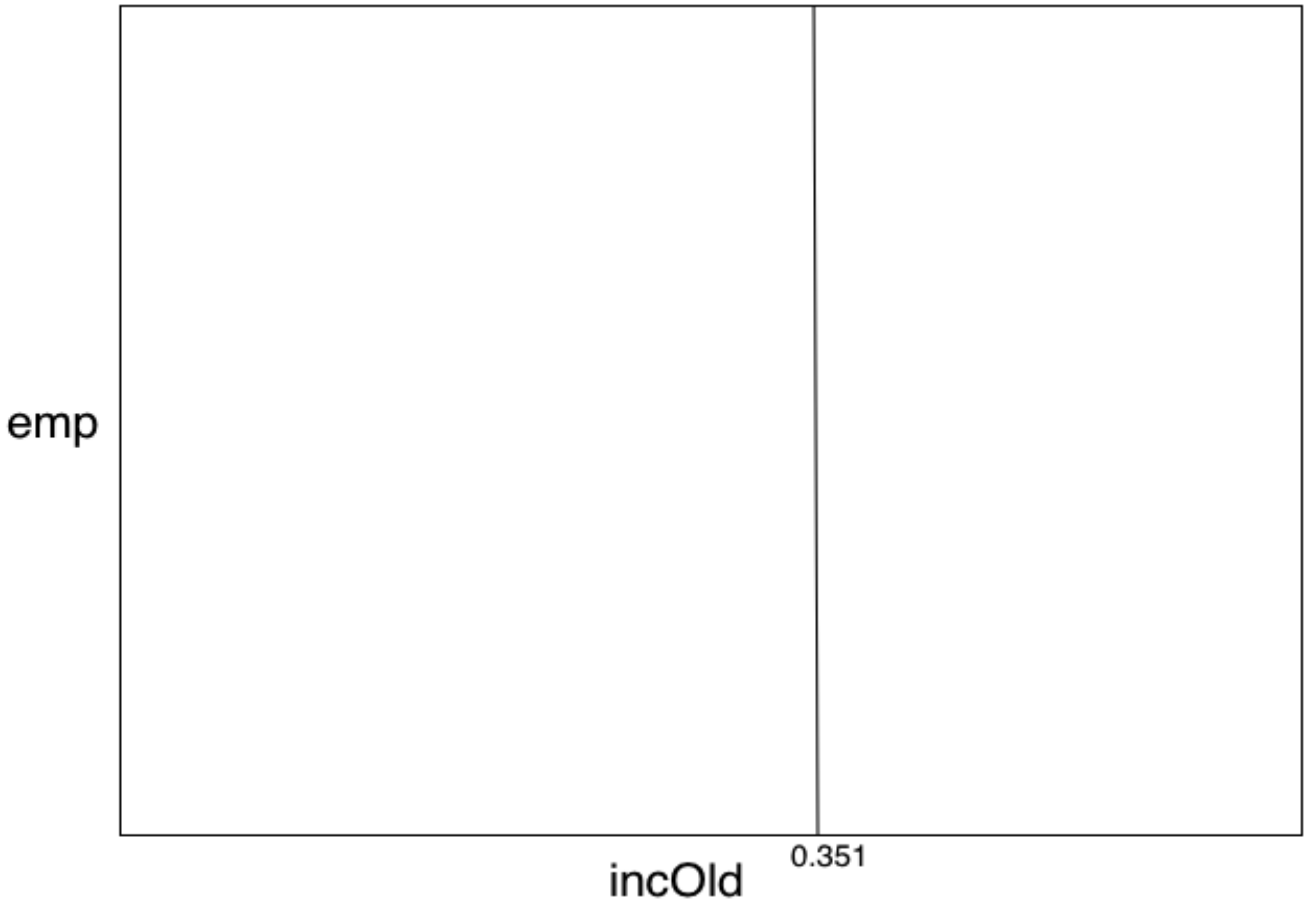
- Tree-based supervised learning methods (non-parametric methods, no closed-form formula) recognise complex patterns in labelled datasets.
- They can be alternative to BIC, LASSO, etc. for selecting the most important variables.
- They split the training dataset (LSOAs) by values of input variables $\mathbf{X}$ into different groups. Each split is made by only one variable.
- Input response variable y here is set by the A-E ratio over a certain age range to represent mortality level of individual LSOAs.
- All in all, the trees are all about the non-linear regression problem: $\hat{y} = f(X_1, X_2, \ldots, X_p)$, while f is a non-linear tree function.

Variable Selection (cont.)

- Each split in the tree is called a 'node', the two distinct groups under each node are called 'leaves'.
- The final groups splitted are called the 'terminal nodes'.
- Within each terminal node (group), the estimated $\hat{y}$ is calculated as the average of all observations in that node, i.e. $\hat{y}_R = \frac{\sum_{i \in R} y_i}{N_R}$ for all i in group R.
- Variable to split at each node is selected such that the overall mean-squared error (MSE) is reduced the most. Splitting stops when certain stopping criterion is achieved.

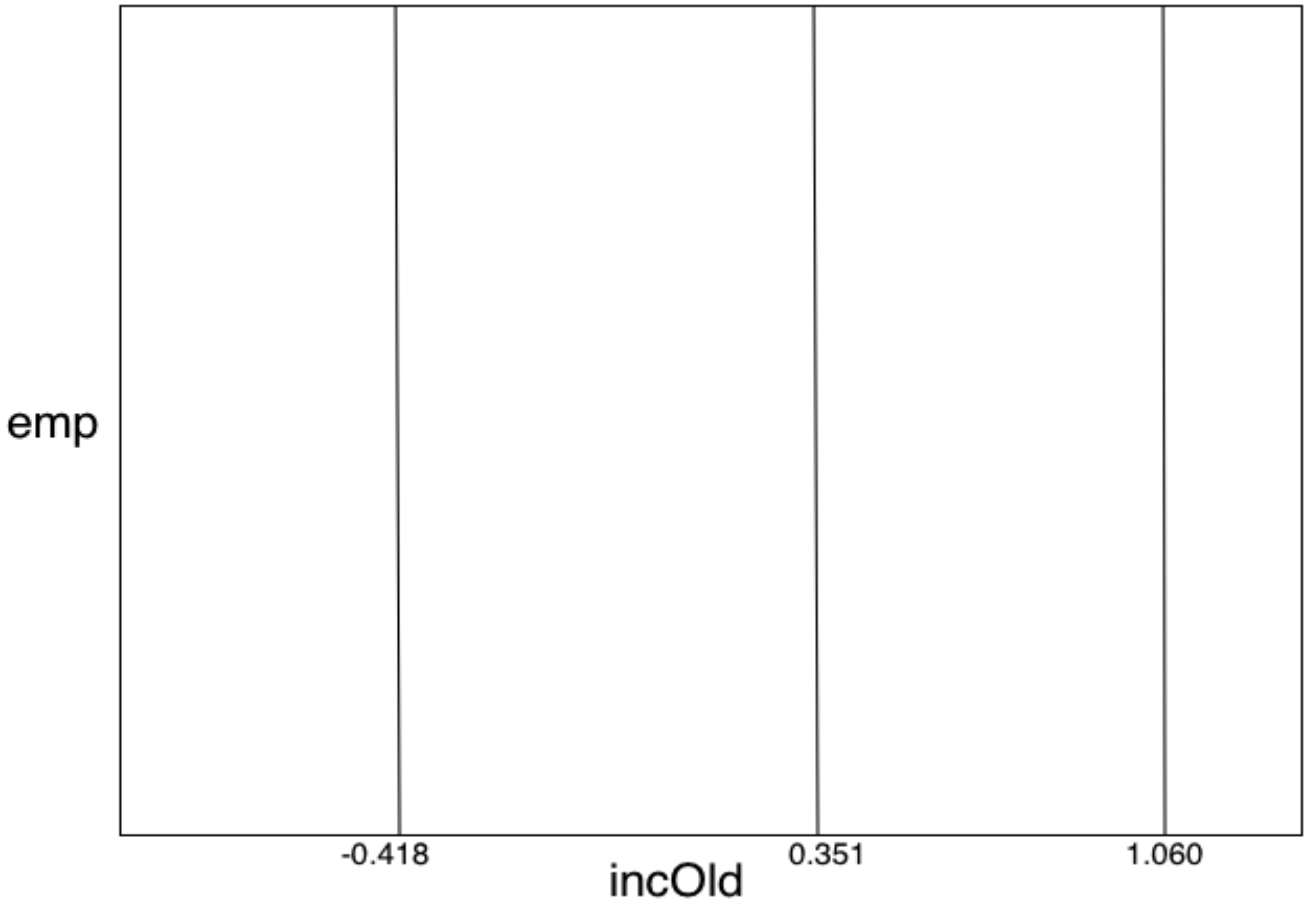
Variable Selection (cont.)

Single regression tree:
Splitting of nodes shown in the 2D plan of ($X_1 = incOld, X_2 = emp$):



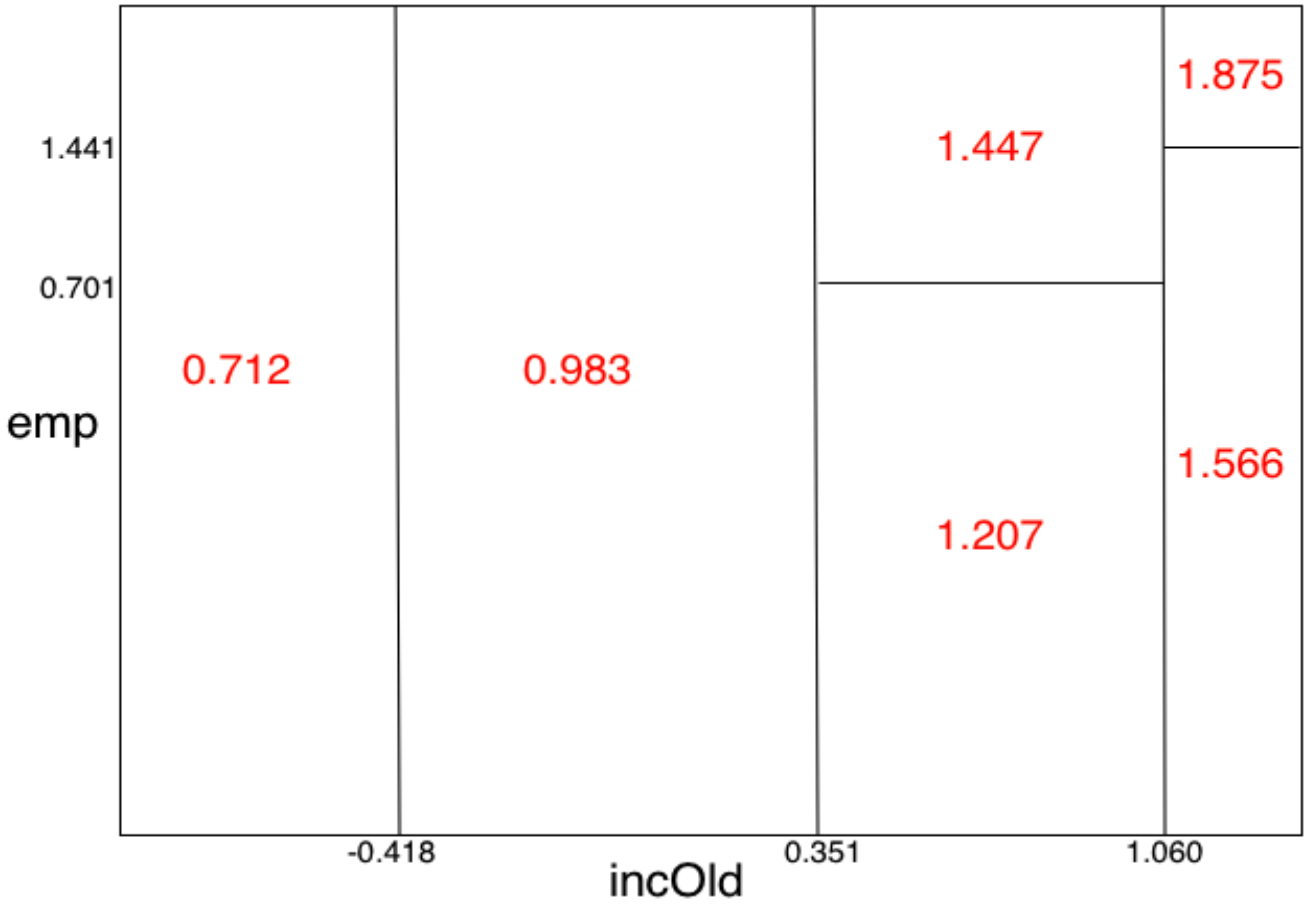
Variable Selection (cont.)

Single regression tree:
Splitting of nodes shown in the 2D plan of ($X_1 = incOld, X_2 = emp$):



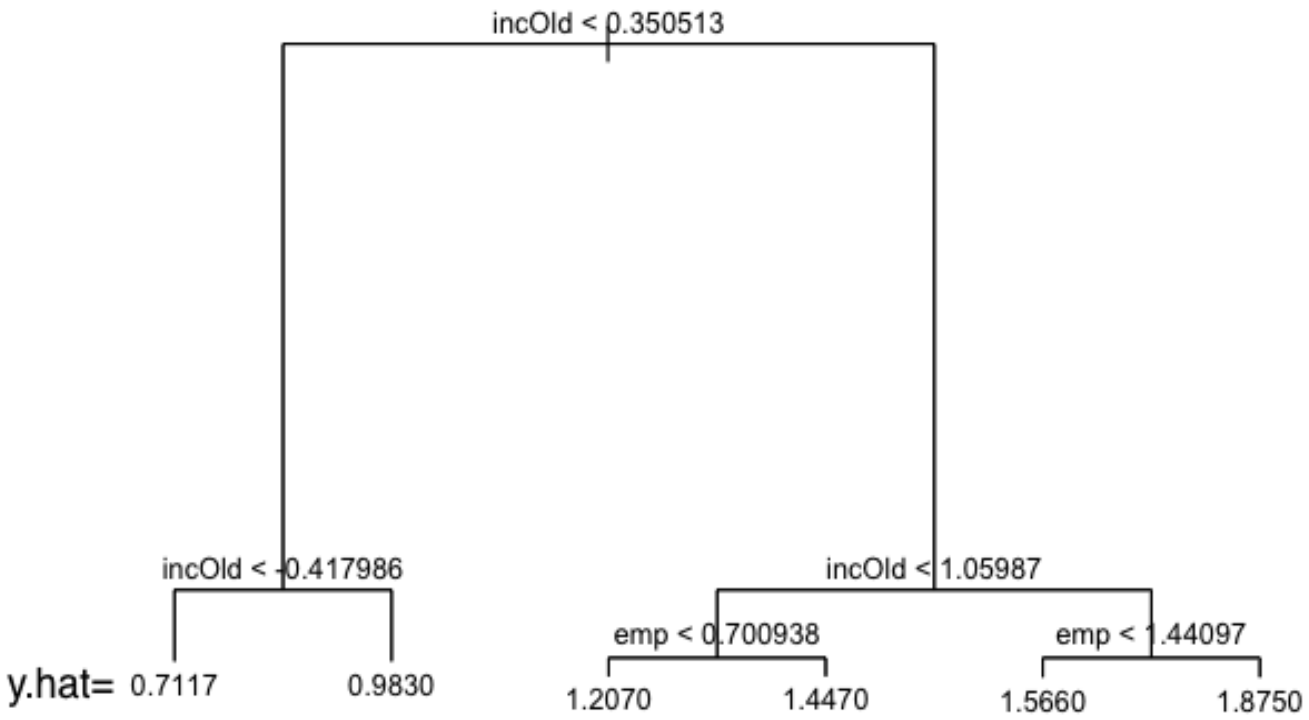
Variable Selection (cont.)

Single regression tree:
Splitting of nodes shown in the 2D plan of ($X_1 = incOld, X_2 = emp$):



Variable Selection (cont.)

Single regression tree:
y = A-E ratio of age 60-69; $\hat{y} = f(X_1 = incOld, X_2 = emp)$



Variable Selection (cont.)

- RF grows a set of trees, with each tree grown from a randomly selected subset of obserations and variables out of the training dataset.
- Output of RF is calculated as an average over outputs from all individual trees in the forest, $\hat{y} = \frac{\sum_{t=1}^{T} \hat{y}^t}{T}$
- RF can be used for regression (when y is numerical) or classification (when y is categorical) of new observations. We apply them here to investigate variable importance.
- Other tree-based regression tools are available and might lead to further improvements in accuracy, i.e. Gradient Boosting Machine (GBM).

Variable Selection (cont.)

Predictive variables

$$
\textit{LSOAs}\begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{14} & \cdots & X_{1,p-1} & X_{1p} \\
X_{21} & X_{22} & X_{23} & X_{24} & \cdots & X_{2,p-1} & X_{2p} \\
X_{31} & X_{32} & X_{33} & X_{34} & \cdots & X_{3,p-1} & X_{3p} \\
X_{41} & X_{42} & X_{43} & X_{44} & \cdots & X_{4,p-1} & X_{4p} \\
X_{51} & X_{52} & X_{53} & X_{54} & \cdots & X_{5,p-1} & X_{5p} \\
X_{61} & X_{62} & X_{63} & X_{64} & \cdots & X_{6,p-1} & X_{6p} \\
X_{71} & X_{72} & X_{73} & X_{74} & \cdots & X_{7,p-1} & X_{7p} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
X_{N-1,1} & X_{N-1,2} & X_{N-1,3} & X_{N-1,4} & \cdots & X_{N-1,p-1} & X_{N-1,p} \\
X_{N1} & X_{N2} & X_{N3} & X_{N4} & \cdots & X_{N,p-1} & X_{Np}
\end{pmatrix}
$$

Red: randomly selected LSOAs with variables used to grow a single tree.

Black: LSOAs not included in growing of the tree, which are called out-of-bag (OOB) samples.

Variable Selection (cont.)

RF produces two different measures of variable importance under two aspects of model criteria.

- Percentage increase in MSE: increase in out-of-bag MSE after a specific variable is randomly shuffled (after trees are grown), as a percentage of the original out-of-bag MSE. It is calculated for each individual tree and averaged over all trees (out-of-sample).
- Increase in node purity: total reduction of in-sample residual sum of squares (RSS) from all splits in a tree that is driven by a specific variable, averaged over all trees (in-sample).

Variable Selection (cont.)

RF with A-E ratio over age 60-69 as the response variable y, over the set of 27 explanatory variables in $\mathbf{X}$:

Variable	**%IncMSE**	**Variable**	**IncNodePurity**
pCHwithN	108.89	incOld	652.95
rUK	74.64	inc	456.64
pNoqual	59.48	emp	367.77
numb	53.76	incChild	273.15
work49h	48.23	pNoqual65	268.75
geob	47.52	numb	203.69
incOld	46.54	pMgr	172.00
liv	45.24	edu	145.74
bHS	45.18	pMgr65	135.03
pMgr65	43.57	adultSkill	113.10

Content

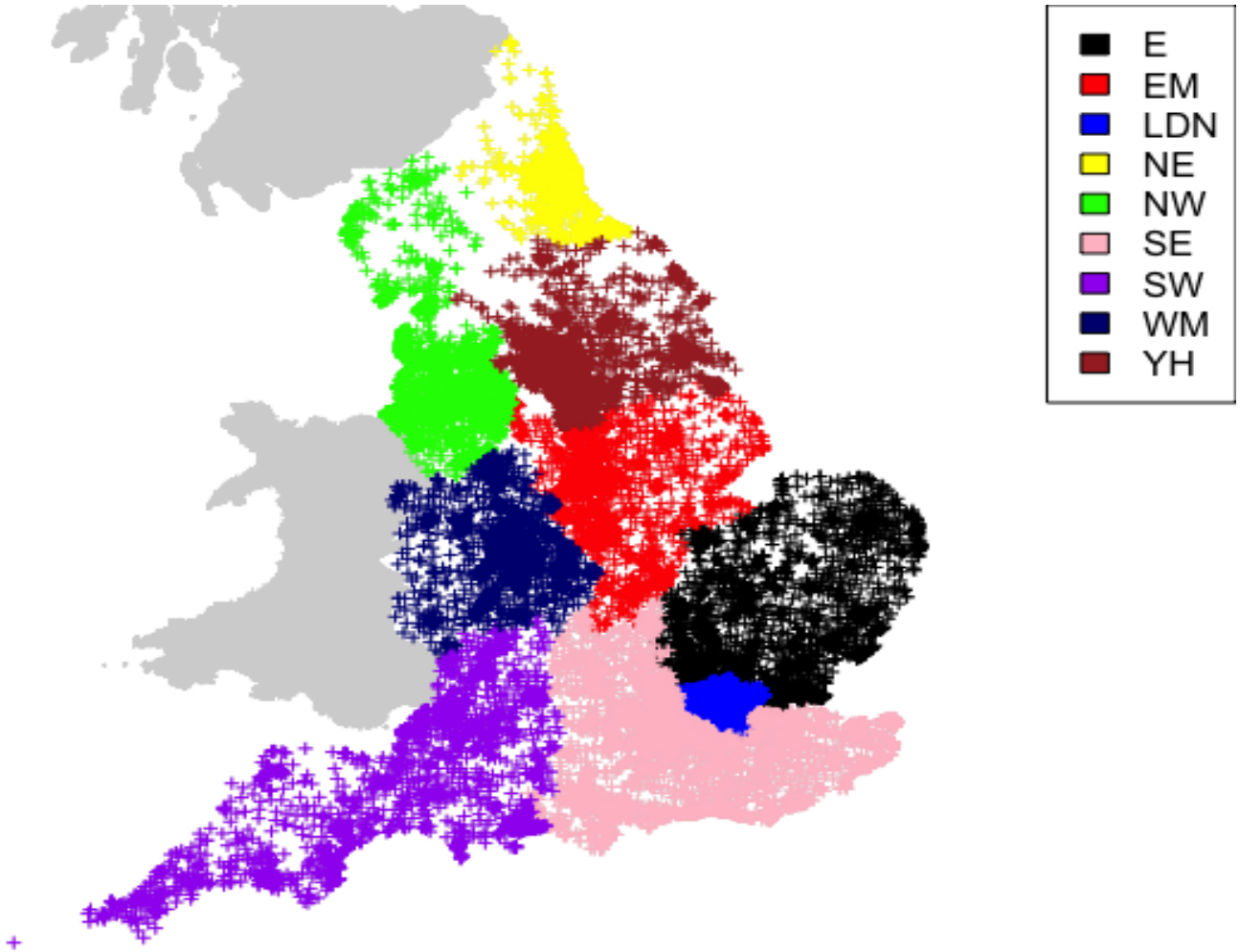
Region/Urban-Rural Pattern

Nine Geographical regions:

- East
- East Midlands
- London
- North East
- North West
- South East
- South West
- West Midlands
- Yorkshire and The Humber

Region/Urban-Rural Pattern (cont.)

Distribution of LSOAs over the 9 Regions:



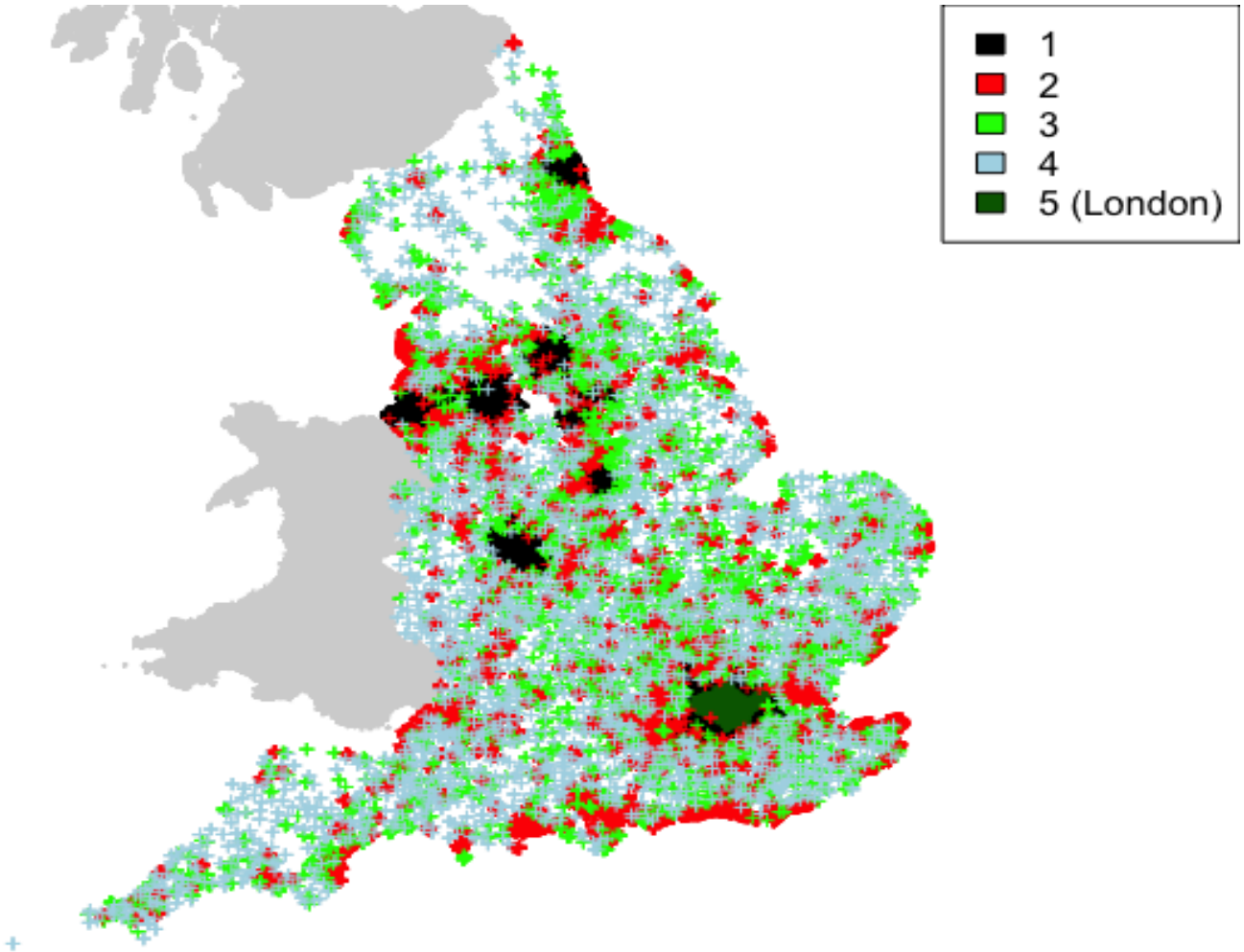
Region/Urban-Rural Pattern (cont.)

Five urban-rural classes:

- Class 1: Urban conurbation (except London)
- Class 2: Urban city and town
- Class 3: Rural town and fringe
- Class 4: Rural village and dispersed
- Class 5: Urban conurbation in London

Region/Urban-Rural Pattern (cont.)

Distribution of LSOAs over the urban-rural classes:



Region/Urban-Rural Pattern (cont.)

Region/UR specific GLM model

A distinct model for each individual region or urban-rural class:

$$\log m_{itx} = \kappa^1_{t,r(i)} + (x - \bar{x})\kappa^2_{t,r(i)} + \sum_j \left(\beta_{j0,t,r(i)} + \beta_{j1,t,r(i)}(x - \bar{x}) \right) X_{ji}$$

$r(i)$ is the corresponding Region or Urban-Rural class of LSOA i. It is equivalent to decompositing the national-level CBD model into a multi-population version with groups of LSOAs.

Region/Urban-Rural Pattern (cont.)

The GLM model is fitted with *incOld* only - over age 65-89 for LSOAs in different urban-rural classes separately.

Period-effect factor κ's:

Region/Urban-Rural Pattern (cont.)

Income-related coefficients β_{10} and β_{11}:

Region/Urban-Rural Pattern (cont.)

Conclusions of region/urban-rural analysis

- The London-based LSOAs have significantly different behaviour from most of the others. Allowing for London effect in the GLM could improve the model's performance.
- The 2011 slowdown of mortality improvement affects LSOAs in urban-rural class 1 to 3 more than 4 and 5.
- The LSOAs in class 4 (least urbanized) are most sensitive to age, reflected by the two age-interacted parameters κ^2.
- Regions with higher mortality have had least improvements over 2001-2016, which widens the gap of mortality over high and low-mortality areas.
- North West and North East are the two regions with highest mortality over 2001-2016.

Content

Key Takeaways

Highlights

- Income, employment, education and housing related factors have relatively strong predictive power to mortality.
- Income is one of the main contributors to the increasing inequality of mortality across different areas.
- There is significant heterogeneity of mortality over different regions and urban-rural classes in England.
- London-based LSOAs have significantly different behaviour from most of the others with respect to socio-economic factors.

Key Takeaways (cont.)

Future Researches

- More research on the selective age interaction and time dependency for GLM model - some variables might not need them.
- Based on the current findings, build up the GLM model with the most appropriate set of variables including socio economic factors and other factors such as spatial information (region/urban-rural) that gives the best BIC.

ANY QUESTIONS?