Heriot-Watt University

BSc in Actuarial Mathematics and Statistics

Life Insurance Mathematics I

Formula Sheet (November 12, 2001)

Note: this sheet will ***not*** be available during the exam.

$$
\begin{aligned}
T_{[x]+r} &= \text{complete future lifetime of a life currently aged} \\
&\qquad\quad x+r \text{ and selected } r \text{ years ago} \\
K_{[x]+r} &= \text{curtate future lifetime of a life currently aged} \\
&\qquad\quad x+r \text{ and selected } r \text{ years ago} \\
&= \lfloor T_{[x]+r} \rfloor, \quad \text{the integer part of } T_{[x]+r} \\
D_{[x]+r} &= l_{[x]+r} v^{x+r} \\
N_{[x]+r} &= \sum_{k=0}^{\infty} D_{[x]+r+k} \\
S_{[x]+r} &= \sum_{k=0}^{\infty} N_{[x]+r+k} = \sum_{k=0}^{\infty} (k+1) D_{[x]+r+k} \\
C_{[x]+r} &= d_{[x]+r} v^{x+r+1} \\
M_{[x]+r} &= \sum_{k=0}^{\infty} C_{[x]+r+k} \\
R_{[x]+r} &= \sum_{k=0}^{\infty} M_{[x]+r+k} = \sum_{k=0}^{\infty} (k+1) C_{[x]+r+k} \\
\bar{N}_{[x]+r} &= \int_{t=0}^{\infty} D_{[x]+r+t} \\
\bar{S}_{[x]+r} &= \sum_{k=0}^{\infty} \bar{N}_{[x]+r+k} \\
\bar{C}_{[x]+r} &= \int_0^1 {}_tp_{[x]+r} \mu_{[x]+r+t} v^{x+r+t} dt \approx d_{[x]+r} v^{x+r+\frac{1}{2}} \\
\bar{M}_{[x]+r} &= \sum_{k=0}^{\infty} \bar{C}_{[x]+r+k} \\
\bar{R}_{[x]+r} &= \sum_{k=0}^{\infty} \bar{M}_{[x]+r+k} = \sum_{k=0}^{\infty} (k+1) \bar{C}_{[x]+r+k}
\end{aligned}
$$

Note that $\bar{R}_{[x]+r}$ is **NOT** equal to $\int_0^\infty {}_tp_{[x]+r} \mu_{[x]+r+t} t \times v^{x+r+t} dt$.

$$
\begin{aligned}
\ddot{a}_{[x]+r:\overline{n}|} &= \sum_{k=0}^{n-1} {}_kp_{[x]+r}v^k = \frac{N_{[x]+r} - N_{[x]+r+n}}{D_{[x]+r}} \\
a_{[x]+r:\overline{n}|} &= \sum_{k=1}^{n} {}_kp_{[x]+r}v^k = \frac{N_{[x]+r+1} - N_{[x]+r+n+1}}{D_{[x]+r}} \\
A_{[x]+r:\overline{n}|} &= \sum_{k=0}^{n-1} k|q_{[x]+r}v^{k+1} + {}_np_{[x]+r}v^n = \frac{M_{[x]+r} - M_{[x]+r+n} + D_{[x]+r+n}}{D_{[x]+r}} \\
A^{\ 1}_{[x]+r:\overline{n}|} &= \sum_{k=0}^{n-1} k|q_{[x]+r}v^{k+1} = \frac{M_{[x]+r} - M_{[x]+r+n}}{D_{[x]+r}} \\
A_{[x]+r:\overset{1}{\overline{n}|}} &= {}_np_{[x]+r}v^n = \frac{D_{[x]+r+n}}{D_{[x]+r}} \\
\bar{A}^{\ 1}_{[x]+r:\overline{n}|} &= \int_0^n {}_tp_{[x]+r}\mu_{[x]+r+t}v^t dt \\
&\approx (1+i)^{1/2}A^{\ 1}_{[x]+r:\overline{n}|} \\
\bar{A}_{[x]+r:\overline{n}|} &= \int_0^n {}_tp_{[x]+r}\mu_{[x]+r+t}v^t dt + \ np_{[x]+r}v^n \\
&\approx (1+i)^{1/2}A^{\ 1}_{[x]+r:\overline{n}|} + \frac{D_{[x]+r+n}}{D_{[x]+r}} \\
(IA)^{\ 1}_{[x]+r:\overline{n}|} &= E\left[(K_{[x]+r}+1)v^{(K_{[x]+r}+1}I(K_{[x]+r} < n)\right] \quad \text{where } I(k<n) = 1 \text{ if } k<n \text{ and 0 otherwise} \\
&= \sum_{k=0}^{n-1} {}_k|q_{[x]+r}(k+1)v^{k+1} \\
&= \frac{R_{[x]+r} - R_{[x]+r+n} - nM_{[x]+r+n}}{D_{[x]+r}} \\
(IA)_{[x]+r:\overline{n}|} &= E\left[\min\{K_{[x]+r}+1, n\}v^{\min\{K_{[x]+r}+1,n\}}\right] \\
&= \sum_{k=0}^{n-1} {}_k|q_{[x]+r}(k+1)v^{k+1} + {}_np_{[x]+r}nv^n \\
&= \frac{R_{[x]+r} - R_{[x]+r+n} - nM_{[x]+r+n} + nD_{[x]+r+n}}{D_{[x]+r}} \\
(I\bar{A})^{\ 1}_{[x]+r:\overline{n}|} &= E\left[(K_{[x]+r}+1)v^{T_{[x]+r}}I(T_{[x]+r} < n)\right] \\
&= \sum_{k=0}^{n-1}(k+1)\int_k^{k+1} {}_tp_{[x]+r}\mu_{[x]+r+t}v^t dt \\
&= \frac{\bar{R}_{[x]+r} - \bar{R}_{[x]+r+n} - n\bar{M}_{[x]+r+n}}{D_{[x]+r}} \\
&\approx (1+i)^{1/2}(IA))^{\ 1}_{[x]+r:\overline{n}|}
\end{aligned}
$$

$$
(\bar{I}\bar{A})^{\ 1}_{[x]+r:\overline{n}|} = E\left[T_{[x]+r}v^{T_{[x]+r}}I(T_{[x]+r} < n)\right]
$$

$$
\begin{aligned}
&= \int_0^n {}_tp_{[x]+r}\mu_{[x]+r+t}t\ v^t dt \\
&\approx (I\bar{A})^{\ 1}_{[x]+r:\overline{n}|} - \frac{1}{2}\bar{A}^{\ 1}_{[x]+r:\overline{n}|} \\
&\approx (1+i)^{1/2}\left((IA)^{\ 1}_{[x]+r:\overline{n}|} - \frac{1}{2}A^{\ 1}_{[x]+r:\overline{n}|}\right)
\end{aligned}
$$

Note that $(\bar{I}\bar{A})^{\ 1}_{[x]+r:\overline{n}|}$ is **NOT** equal to $(\bar{R}_{[x]+r} - \bar{R}_{[x]+r+n} - n\bar{M}_{[x]+r+n})/D_{[x]+r}$ because of the earlier definition of $\bar{R}_{[x]+r}$ and the subsequent comment.

$$
\begin{aligned}
(I\ddot{a})_{[x]+r:\overline{n}|} &= E\left[(I\ddot{a})_{\overline{\min(K_{[x]+r}+1,n)}|}\right] \\
&= \sum_{k=0}^{n-1}(k+1)\ {}_kp_{[x]+r}v^k \\
&= \frac{S_{[x]+r} - S_{[x]+r+n} - nN_{[x]+r+n}}{D_{[x]+r}} \\
\ddot{a}^{(m)}_{[x]+r:\overline{n}|} &\approx \ddot{a}_{[x]+r:\overline{n}|} - \frac{(m-1)}{2m}\left(1 - \frac{D_{[x]+r+n}}{D_{[x]+r}}\right) \\
a^{(m)}_{[x]+r:\overline{n}|} &\approx a_{[x]+r:\overline{n}|} + \frac{(m-1)}{2m}\left(1 - \frac{D_{[x]+r+n}}{D_{[x]+r}}\right) \\
\bar{a}_{[x]+r:\overline{n}|} &\approx \ddot{a}_{[x]+r:\overline{n}|} - \frac{1}{2}\left(1 - \frac{D_{[x]+r+n}}{D_{[x]+r}}\right)
\end{aligned}
$$