

A FAMILY OF TERM-STRUCTURE MODELS WITH STOCHASTIC VOLATILITY

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PLAN

- MOTIVATION, DATA
- NEW MODEL
- EXAMPLES
  - long-term dynamics under P
  - forward-rate volatility term structure
  - caplet volatility smiles and skews

WHY DO WE NEED A NEW MODEL?

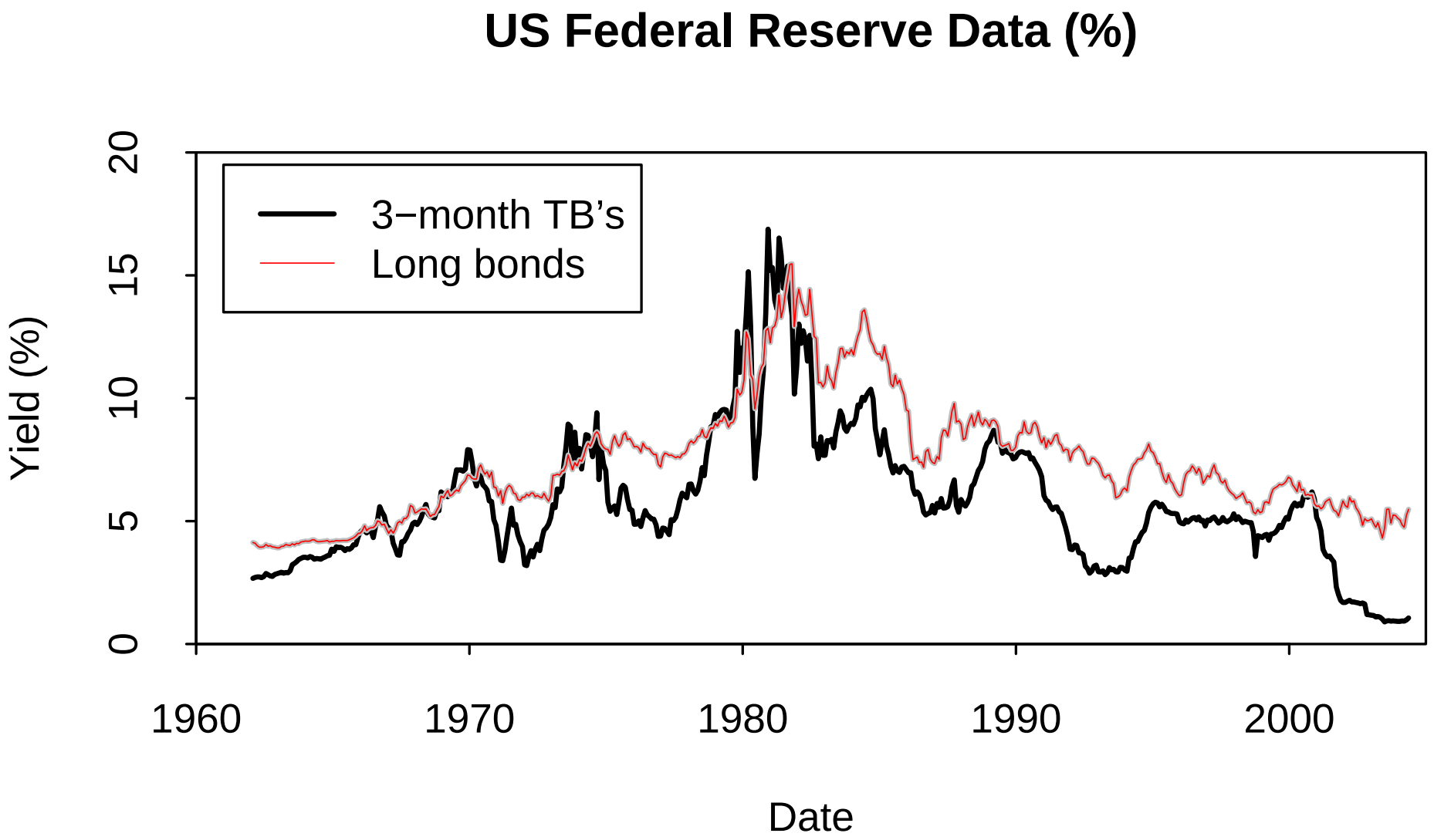
Problems of interest:

- Long-term risk assessment
- Long-term risk management
- Pricing of long-term interest-rate derivatives

Aim:

- To tackle these problems using a parsimonious and time-homogeneous interest-rate model.
- To complement the use of market models.

USA: short and long-term interest rates



Model requirements: Medium/long term:

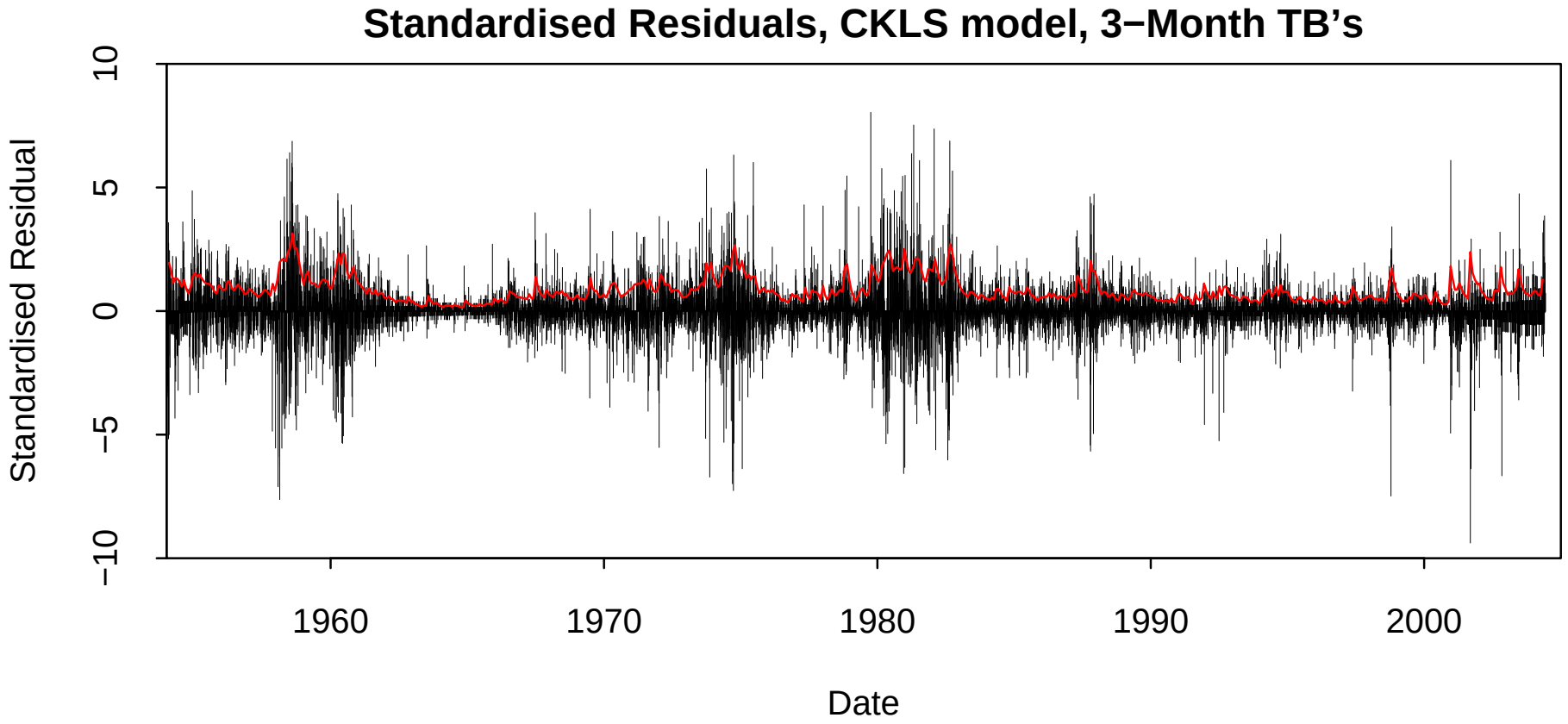
- arbitrage free
- positive interest rates
- mean reverting
- short/long rates not perfectly correlated
- Faithful to historical data
  - long periods of high/low rates
  - wide range of attainable rates

Very few existing models satisfy these requirements

US daily data, 3-month TB's, standardised residuals based on MLE

CKLS model: $\Delta Y_{t+1} = \alpha(\mu - Y_t) + \sigma Y_t^{\gamma} Z_{t+1}$

Standardised residuals should be *i.i.d.* $\sim N(0,1)$.



Model requirements:

Stochastic volatility

- important for
  - short-term dynamics
  - short/long term derivative pricing

    (including volatility smiles)
- not so important (???) for
  - modelling static, long-term bond portfolio

NEW FAMILY OF TERM-STRUCTURE MODELS

Underlying framework: Flesaker & Hughston; Rogers; Rutkowski

$$
\begin{aligned}
P(t,T) &= \text{Zero Coupon bond price at } t \text{ for \$ 1 at } T \\
&= \frac{D(t,T,X(t))}{D(t,t,X(t))} \\
X(t) &= \text{state variables, } \Big(X_1(t),\ldots,X_n(t)\Big)' \\
D(t,T,X(t)) &= \int_T^\infty M(t,s,X(t))ds
\end{aligned}
$$

For each s: $M(t,s,X(t)) > 0$ a $\hat{P}$-martingale for $0 \leq t \leq s$.

$\hat{P}$ is the Pricing Measure

Structure of $M(t,s,x)$ depends on the model for $X(t)$

Special case:

$$\begin{aligned} M(t,s,X(t)) &= e^{-\beta t + A(s-t) + B(s-t)'X(t)} \\ D(t,T,X(t)) &= \int_T^\infty M(t,s,X(t))ds \end{aligned}$$

What

- models for $X(t)$
- forms for $A(u)$, $B(u)$

result in $M(t,s,X(t))$ a $\hat{P}$-martingale?

Two previous solutions:

Solution A: Cairns (2004, Math.Finance), $n \geq 1$ factors

$X_1(t), \ldots, X_n(t)$ are correlated Ornstein-Uhlenbeck processes

$$\Rightarrow dX_i(t) = -\alpha_i X_i(t)dt + d\hat{Z}_i(t)$$

$\alpha_i =$ mean reversion rate

$\hat{Z}_1(t), \hat{Z}_2(t), \ldots$: correlated Brownian motions

Model designed for

- long-term risk management
- pricing long-term interest-rate derivatives

Solution B: Brody & Hughston (2002, Quant.Finance), $n = 1$

$X(t)$ a Cox-Ingersoll-Ross (CIR) process

$$dX(t) = \alpha(\mu - X(t))dt + \sigma\sqrt{X(t)}d\hat{Z}(t)$$

Solutions A, B reminiscent of Affine Term Structure models for the risk-free rate

(Vasicek, CIR, Duffie & Kan)

Consider $D(t,T,X(t)) = e^{-\beta t}\int_{T-t}^{\infty} e^{A(u)+B(u)'X(t)}du$

New solution C

$$M(t,s,X(t)) \;=\; e^{-\beta t+A(s-t)+B(s-t)'X(t)}$$

is a $\hat{P}$-martingale, for some β, $A(u)$, $B(u)$

$\Leftrightarrow$

$X(t)$ is an affine diffusion process.

$A(u),\ B_1(u),\ \ldots, B_n(u) =$ solutions of Riccatti Equations.

Modelling highlights

Constant volatility processes (Solution A, earlier slide)

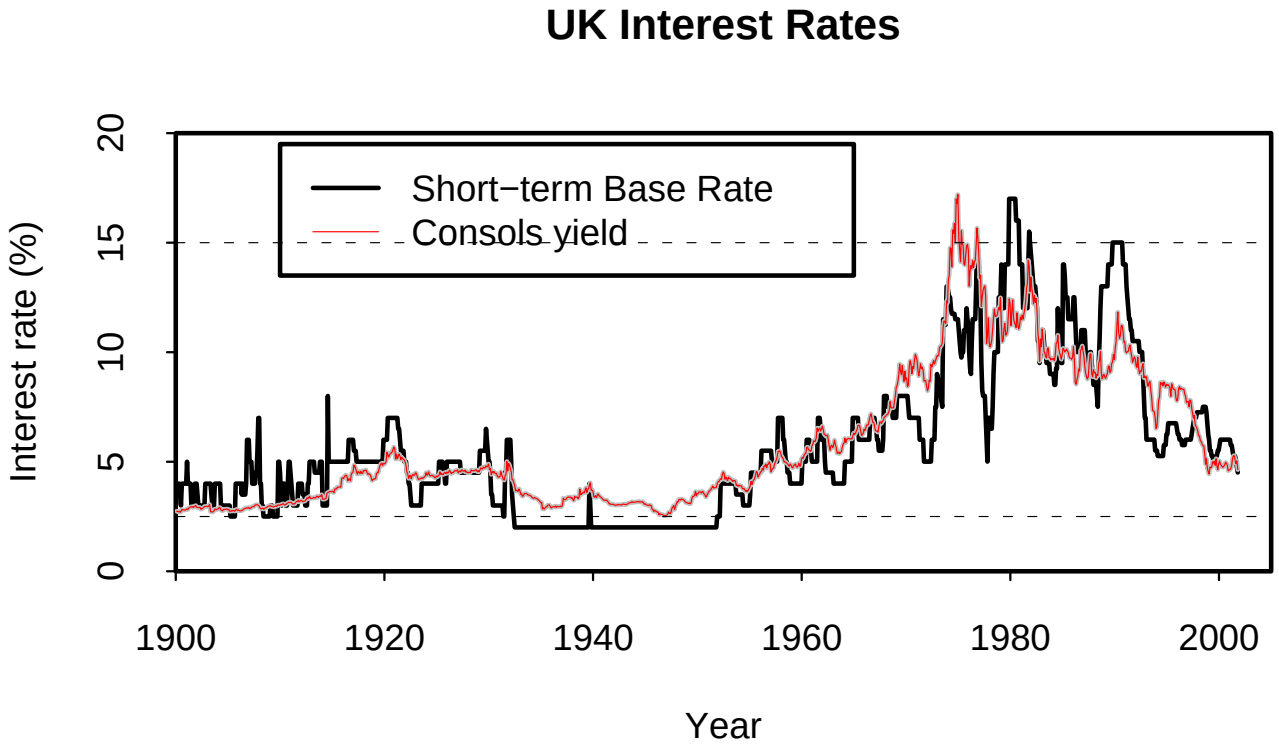
e.g. $dX_i(t) = -\alpha_i X_i(t)dt + d\hat{W}_i(t)$

$\Rightarrow$ key determinants of yield curve shape

Low α_i key to

- SUSTAINED periods of high or low interest rates
- SUBSTANTIAL VARIABILITY in long-bond yields consistent with historical data

Distribution of values for short and long rates

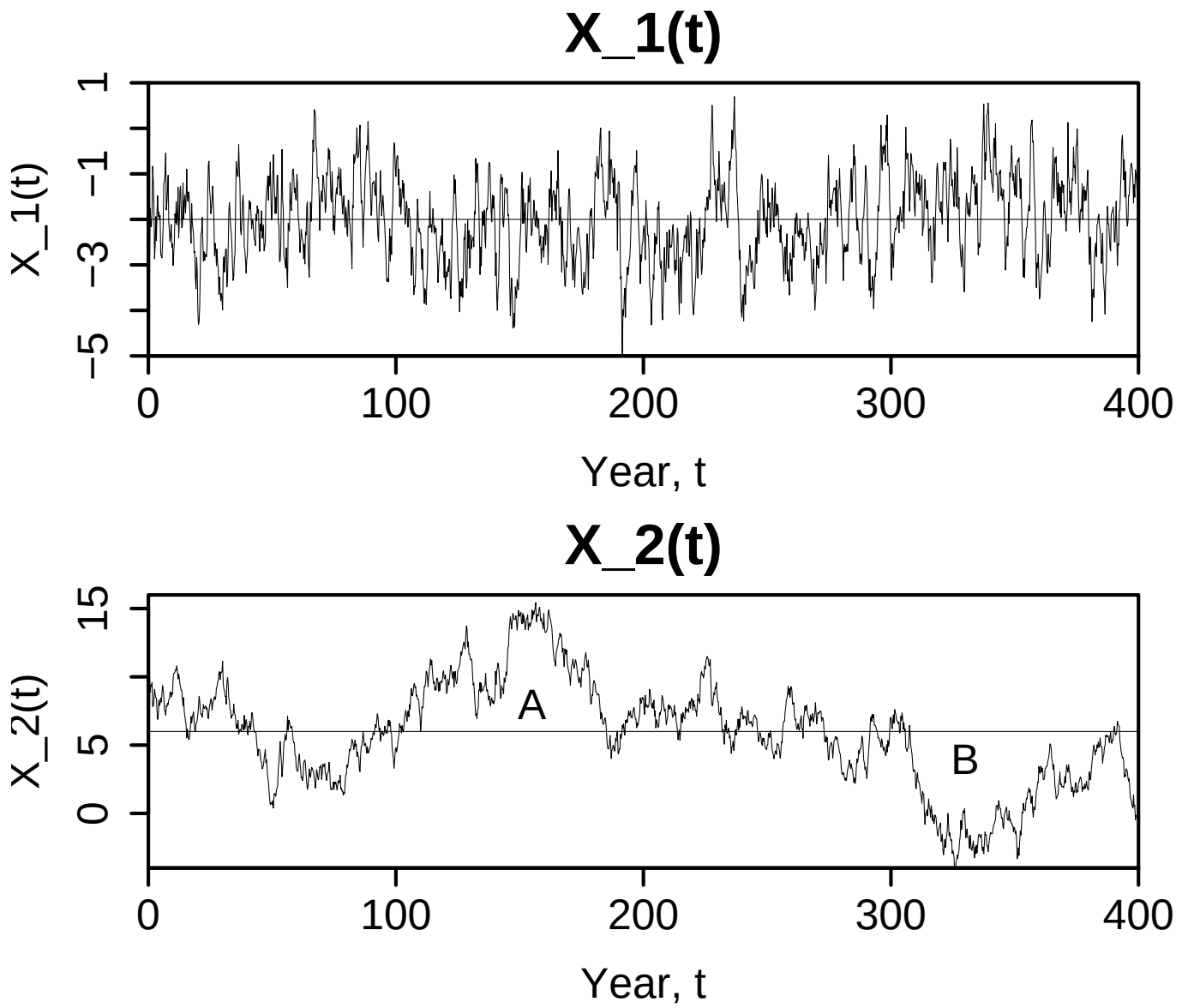


Desirable characteristic: model should produce a similar range

- Short rates: $< 2.5\%$ or $> 15\%$ with "reasonable probability"
- Long rates: $< 3\%$ or $> 15\%$ with "reasonable probability"

Simulation example: $n = 2$

- $\beta = 0.04$ long-term forward rate
- $\alpha_i = (0.6, 0.06)$ mean reversion rates for $X_i(t)$
- $\sigma_i = (0.6, 0.4)$ volatility parameter
- $\rho_{12} = -0.5$ local correlation between $X_1(t)$ and $X_2(t)$


X_1(t)
X_1(t)
1
−1
−3
−5
0
100
200
300
400
Year, t
X_2(t)
X_2(t)
15
5
0
A
B
0
100
200
300
400
Year, t

Simulation: comment

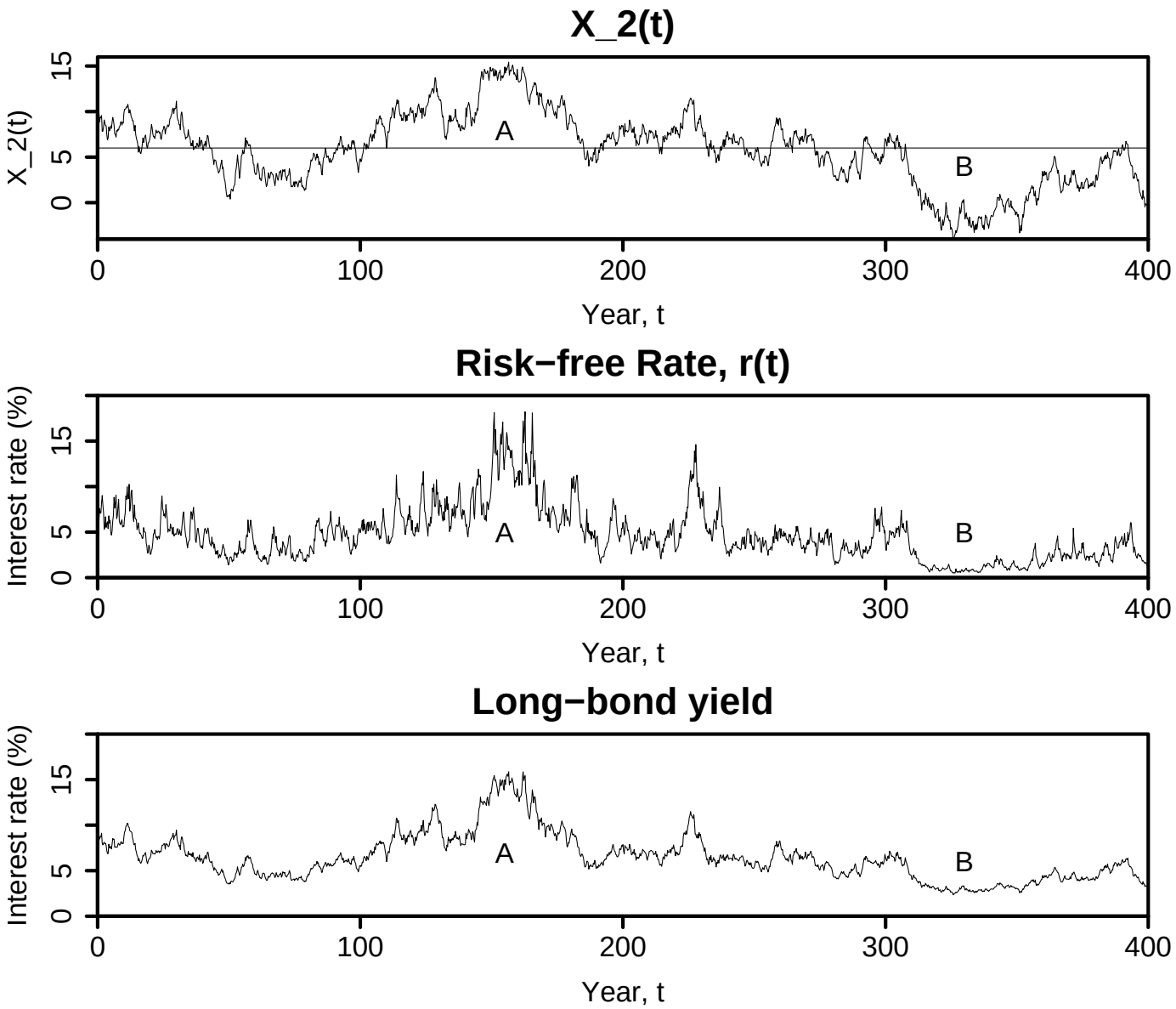
- Pricing measure:

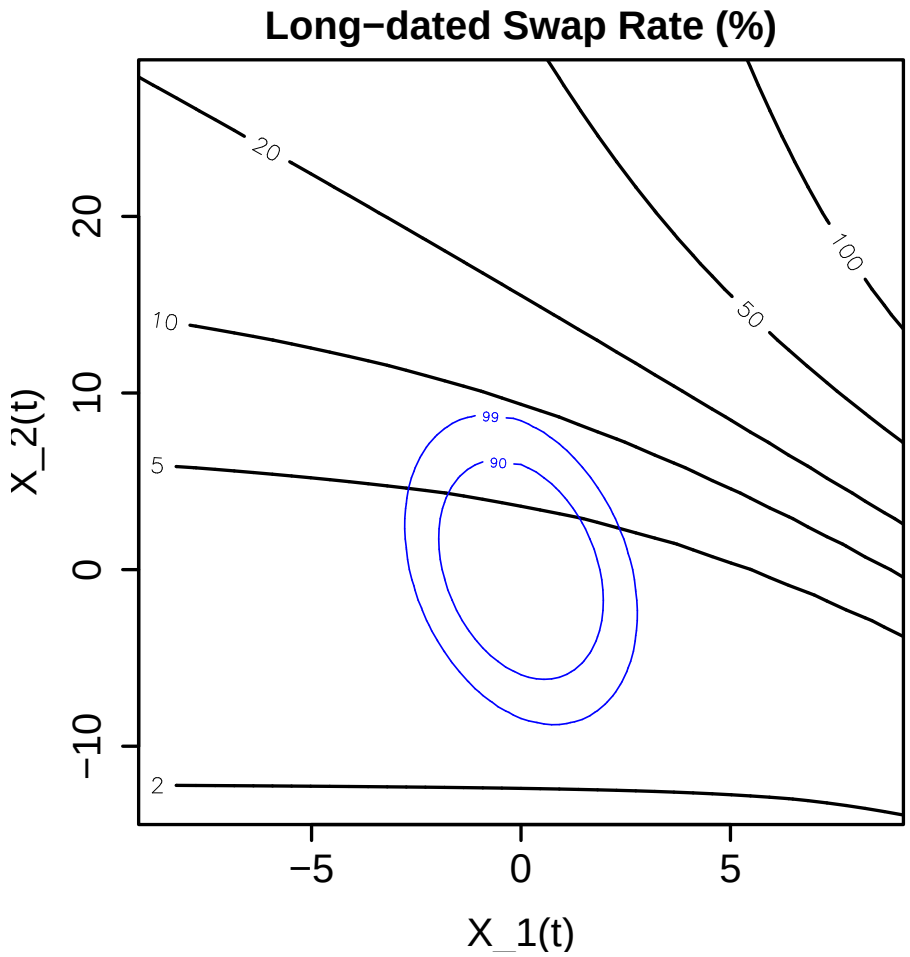
  $\hat{P} \Rightarrow X_1(t),\ X_2(t)$ mean reverting to 0

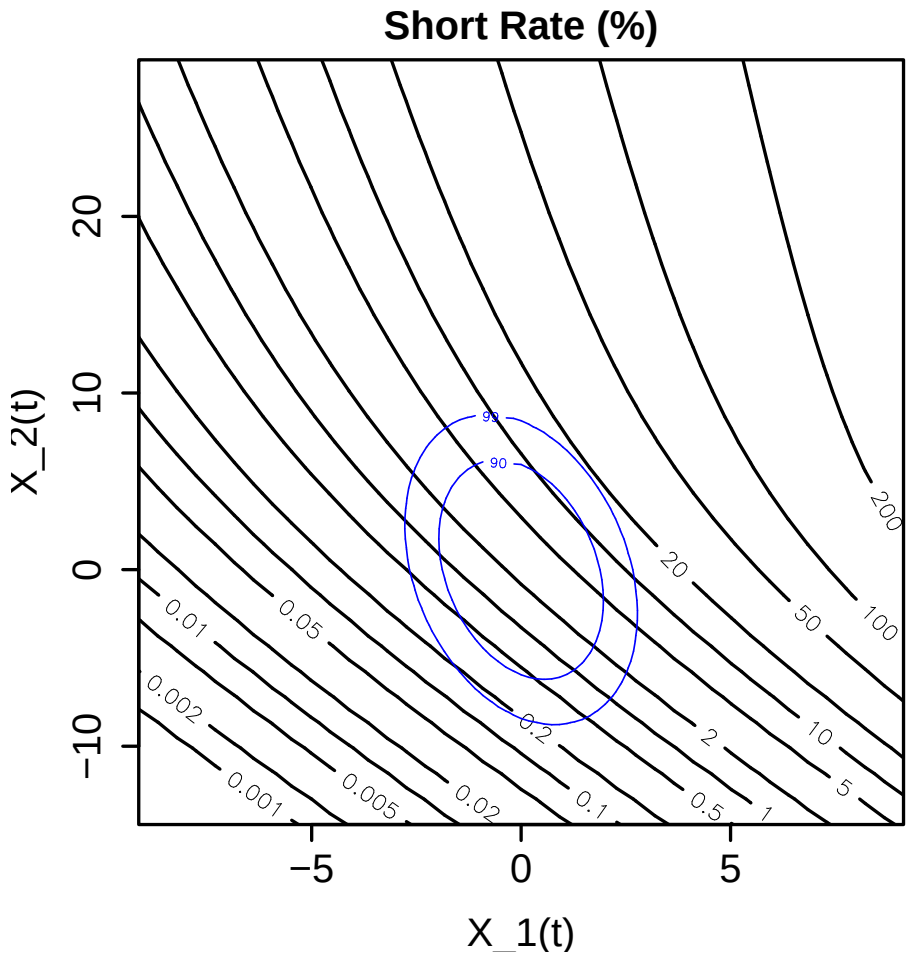
- Modelling assumption

  Real-world measure:

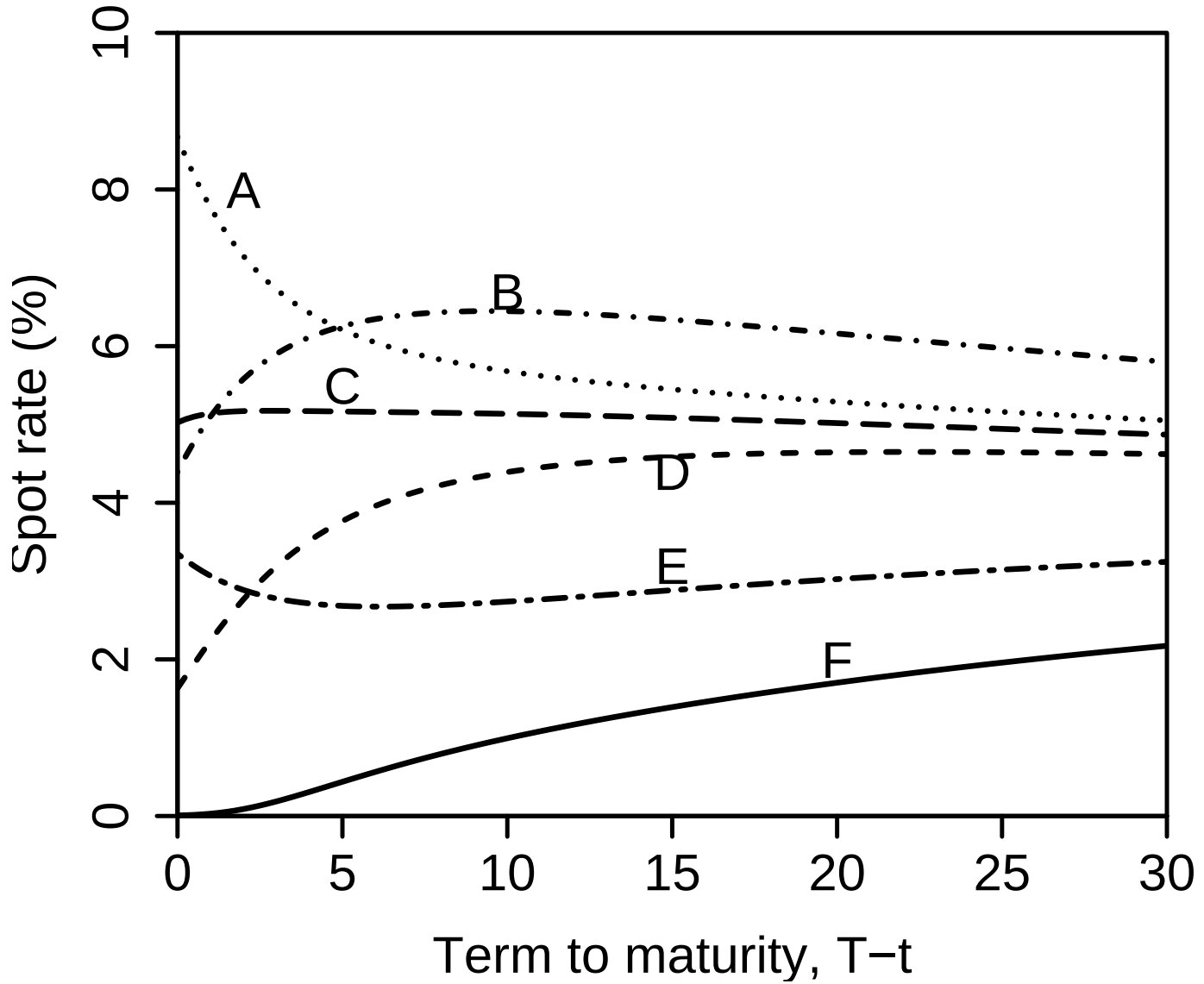
  $P \Rightarrow X_1(t),\ X_2(t)$ mean reverting to $\mu_1,\ \mu_2$.


X_2(t)
X_2(t)
0
5
15
A
B
0
100
200
300
400
Year, t
Risk−free Rate, r(t)
Interest rate (%)
0
5
15
A
B
0
100
200
300
400
Year, t
Long−bond yield
Interest rate (%)
0
5
15
A
B
0
100
200
300
400
Year, t


Long−dated Swap Rate (%)
X_2(t)
X_1(t)
20
10
5
2
100
50
99
90
−10
0
10
20
−5
0
5


Short Rate (%)
X_2(t)
X_1(t)
200
100
50
20
10
5
2
1
0.5
0.2
0.1
0.05
0.02
0.01
0.005
0.002
0.001
99
90
−10
0
10
20
−5
0
5

Two-factor model: Possible spot-rate curves



Mixed volatility state processes ($\Rightarrow$ stochastic volatility)

Example: $n = 3$

$$dX_1(t) = \alpha_1(\mu_1 - X_1(t))dt + \sigma_1\sqrt{X_1(t)}\sum_{j=1}^{3} c_{1j}d\hat{W}_j(t)$$

$$dX_2(t) = -\alpha_2 X_2(t)dt + \sqrt{X_1(t)}\sum_{j=1}^{3} c_{2j}d\hat{W}_j(t)$$

$$dX_3(t) = -\alpha_3 X_3(t)dt + \sqrt{X_1(t)}\sum_{j=1}^{3} c_{3j}d\hat{W}_j(t)$$

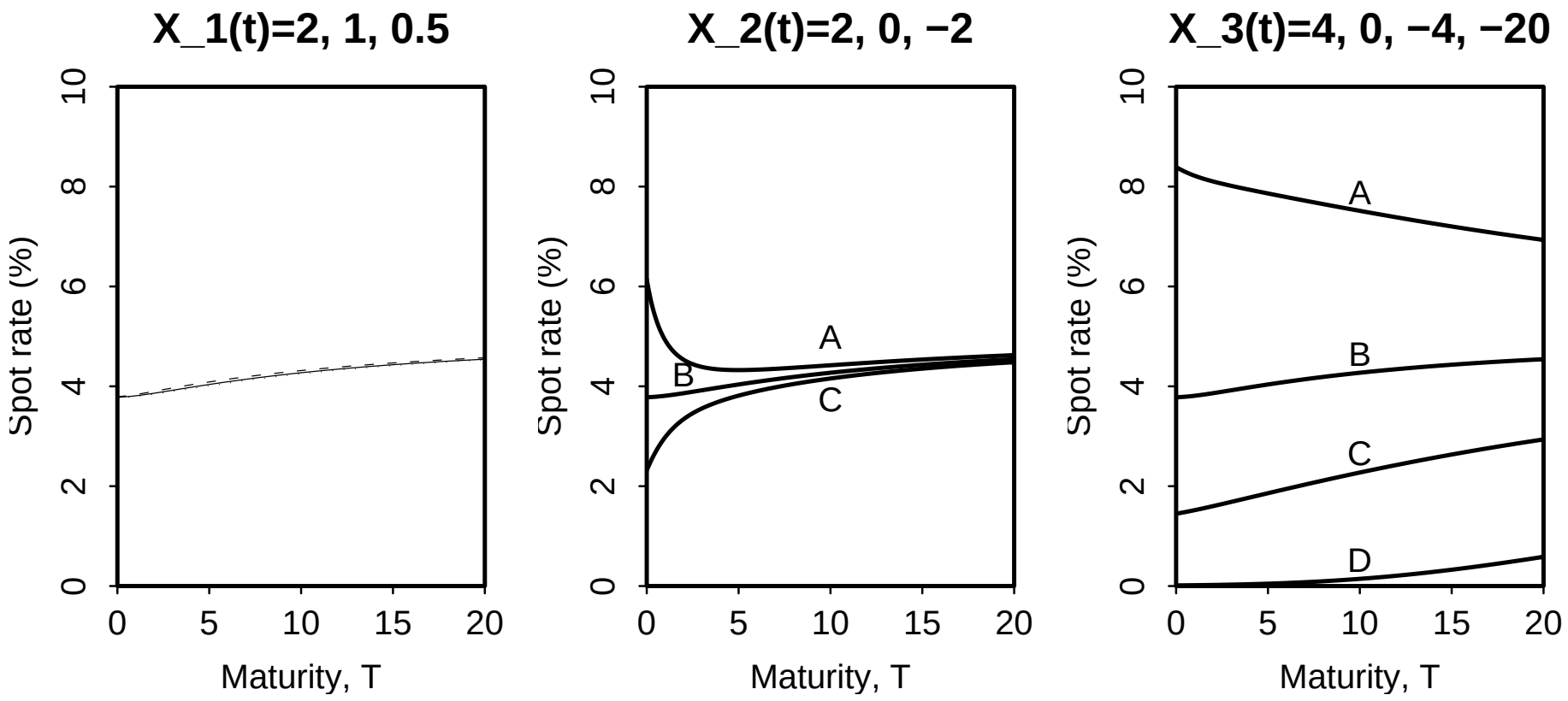
$\hat{W}_1(t)$, $\hat{W}_2(t)$ and $\hat{W}_3(t)$ independent Brownian motions

Numerical examples:

- $\beta = 0.05$ (long-term spot-rate)
- $\mu_1 = 1,\ \alpha_1 = 2,\ \sigma_1 = 0.3$ (square-root process)
- $\alpha_2 = 1.5,\ \ \alpha_3 = 0.075$
- $B_1(0) = 0.01,\ \ B_2(0) = 0.25,\ \ B_3(0) = 0.35$
- $\rho_{12} = \rho_{13} = 0,\quad \rho_{23} = -0.8$

Spot-rate curves: sensitivity to changes in $X(t)$:

Central case: $X(t) = (1, 0, 0)'$



Volatility term structure

$f(t,T) =$ instantaneous forward-rate curve

$$df(t,T) = f(t,T)\sqrt{X_1(t)}\sum_{j=1}^{3} s_{fj}(t,T,X(t))d\hat{W}_j(t) + O(dt)$$

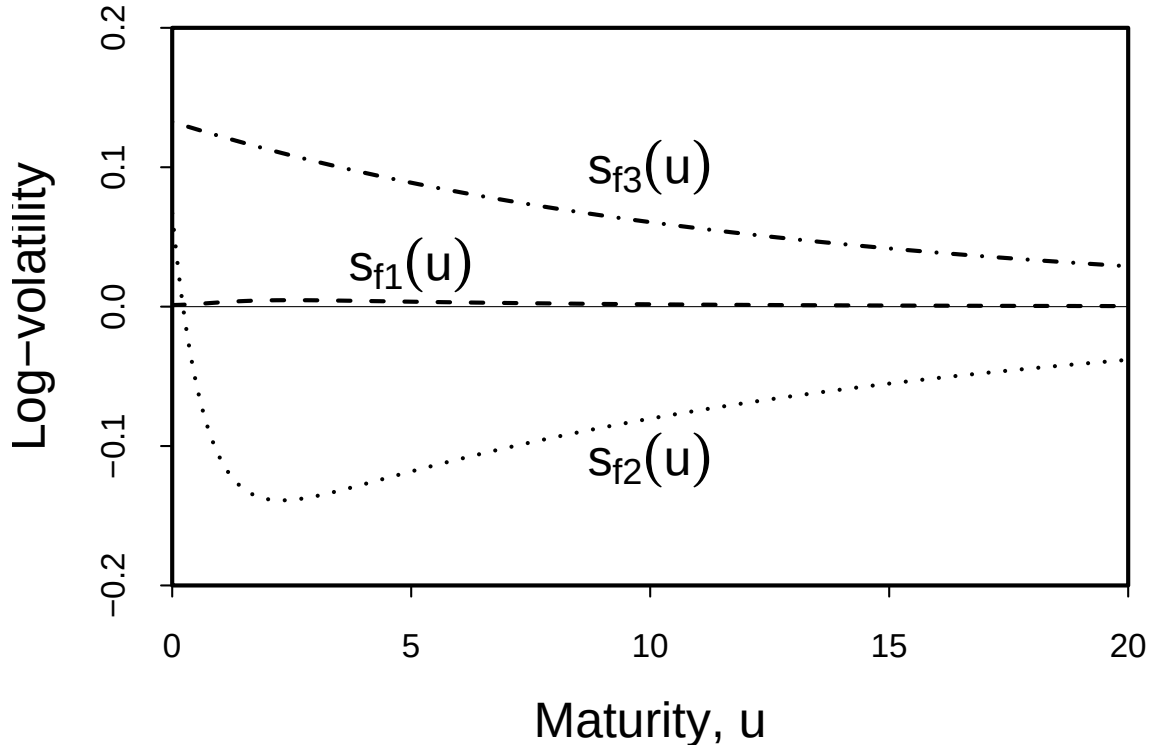
Independent log-volatility functions:

$$s_{fj}(t,T,X(t)) = s_{Mj}(T-t) - \frac{D_j(t,T,X(t))}{D(t,T,X(t))}$$

$s_{Mj}(T-t)$ is deterministic and the dominant term

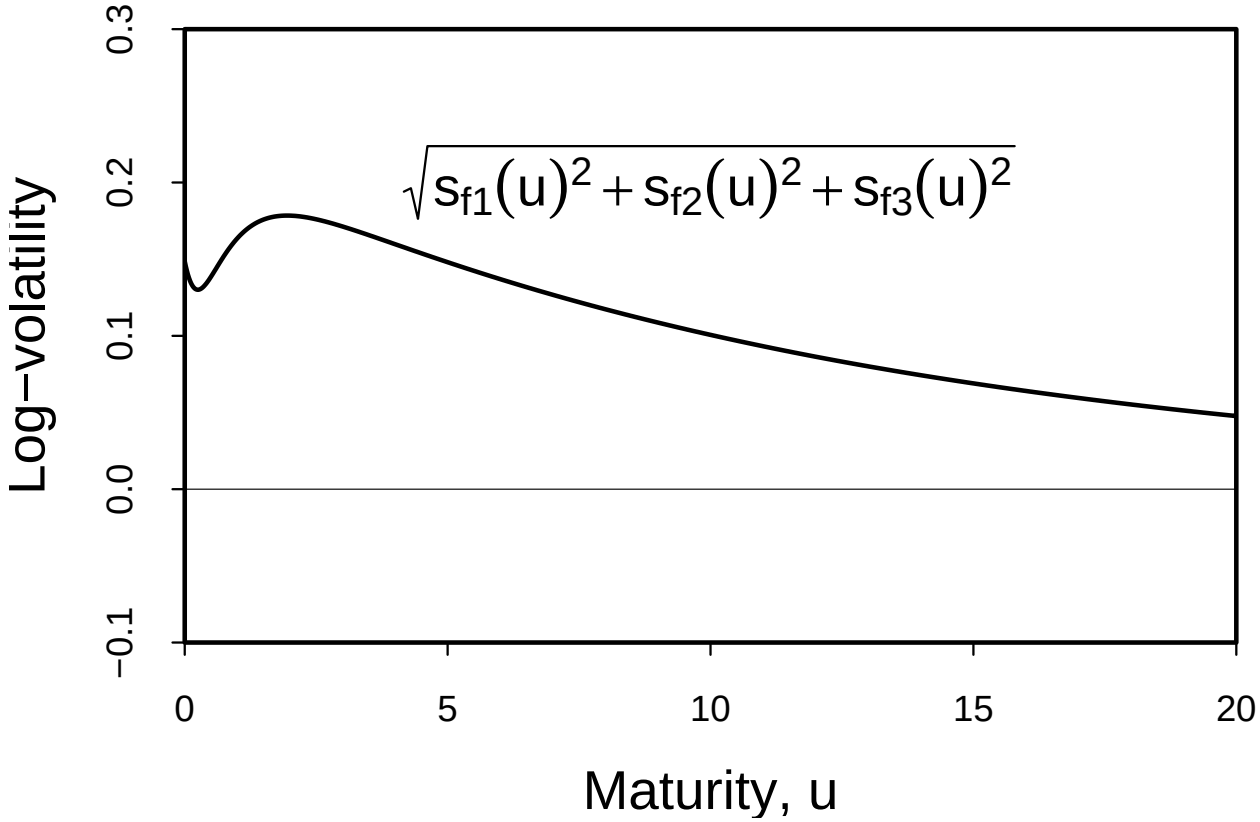
$D_j(t,T,x) = \partial D(t,T,x)/\partial x_j$

Independent log-volatility functions



- volatility in $X_1(t)$ not important in short term
- magnitude of $X_1(t)$ important (volatility of $X_2(t)$, $X_3(t)$)
- different parameters produce different results

Total log-volatility function



- general shape similar to market data
- different parameters produce different shapes

Example: volatility smiles

$n = 2$ factors

$$dX_1(t) = \alpha_1(\mu_1 - X_1(t))dt + \sigma_1\sqrt{X_1(t)}\sum_{j=1}^{2} c_{1j}d\hat{W}_j(t)$$

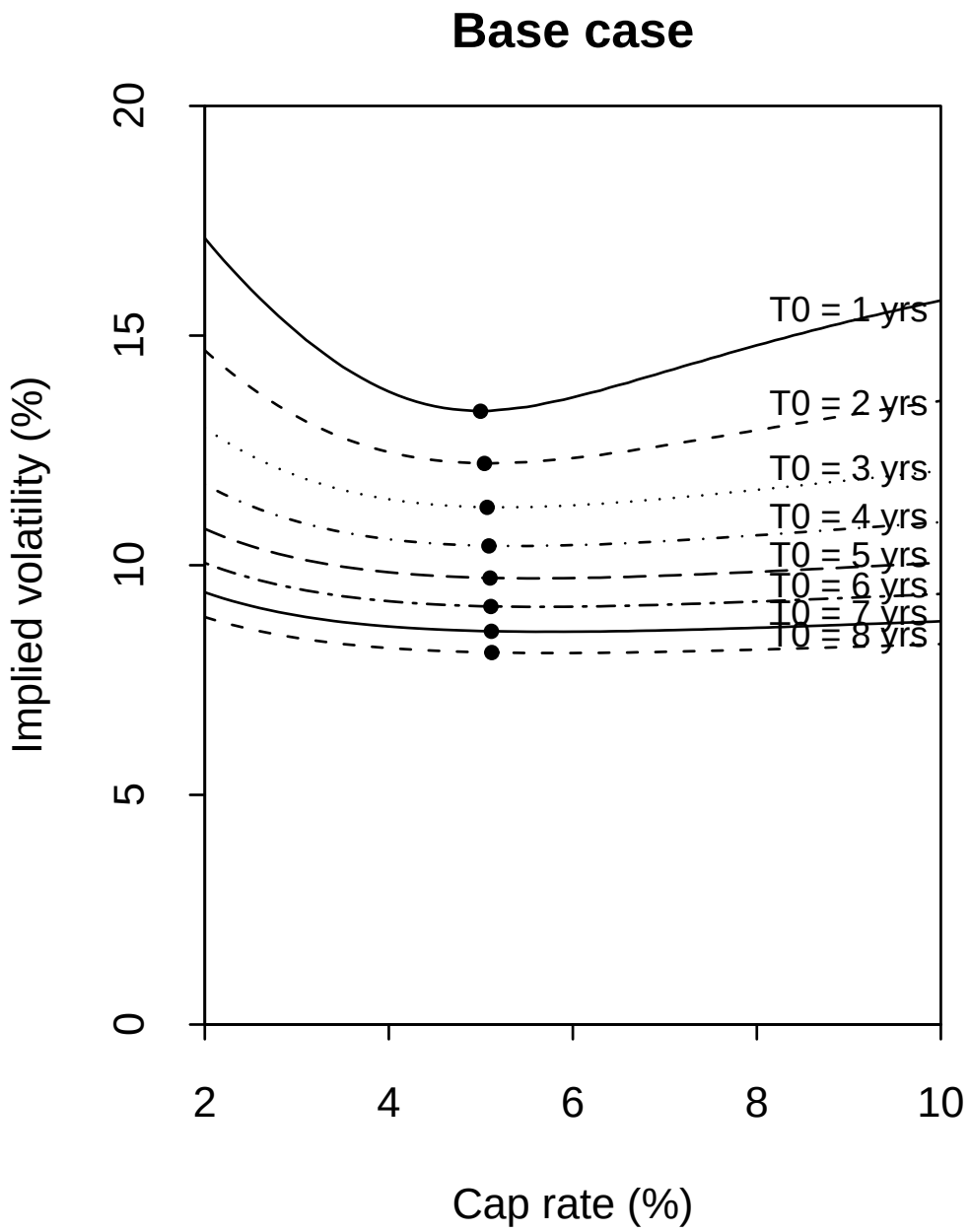
$$dX_2(t) = -\alpha_2 X_2(t)dt + \sqrt{X_1(t)}\sum_{j=1}^{2} c_{2j}d\hat{W}_j(t)$$

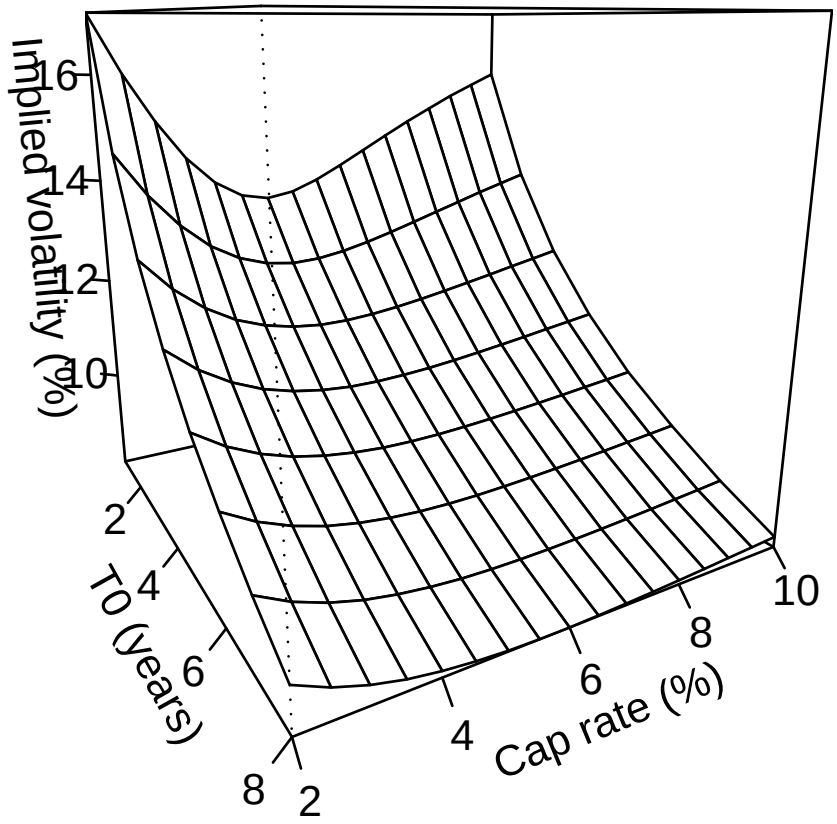
$\hat{W}_1(t)$, and $\hat{W}_2(t)$ independent Brownian motions

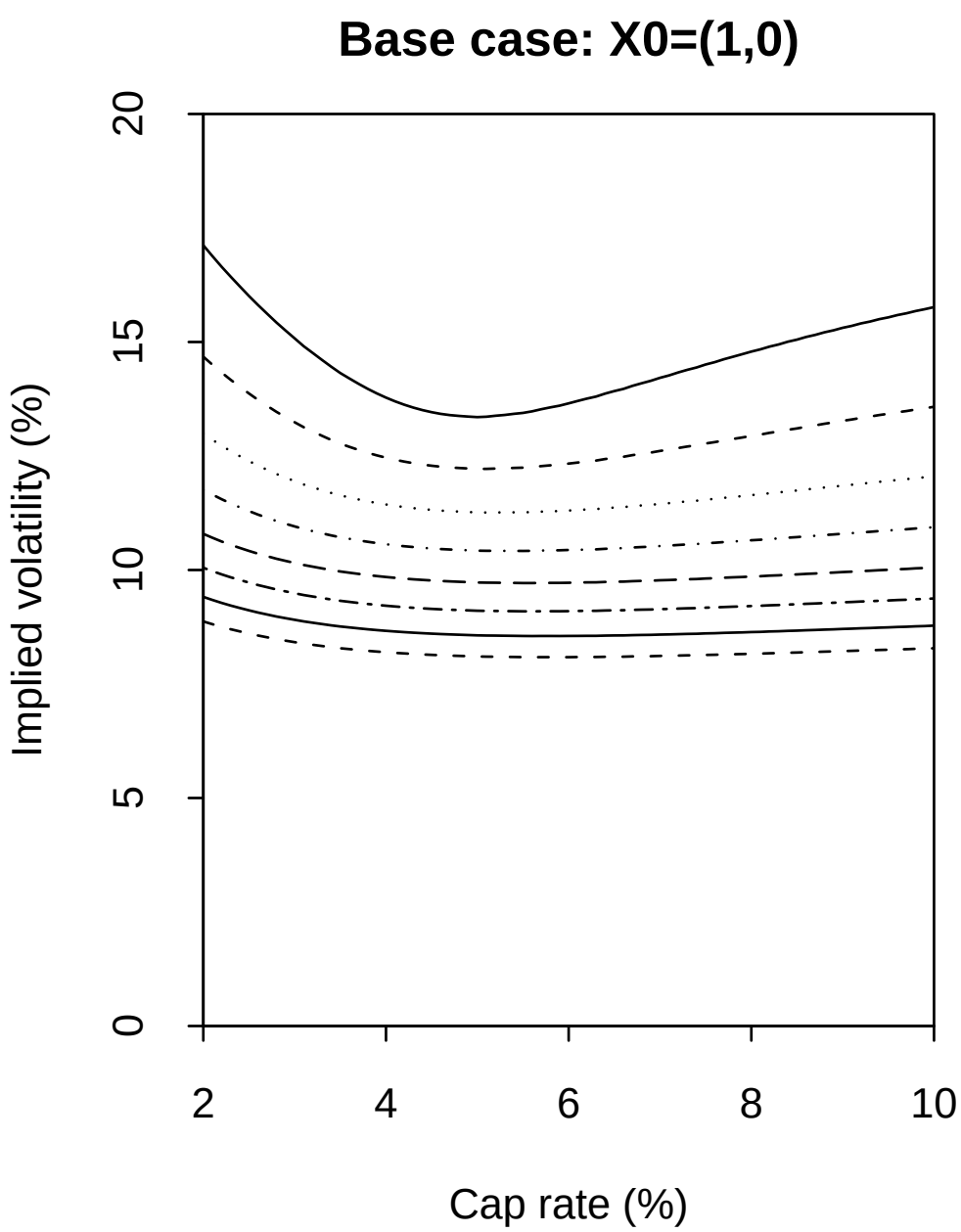
Example: Caplet implied volatilities

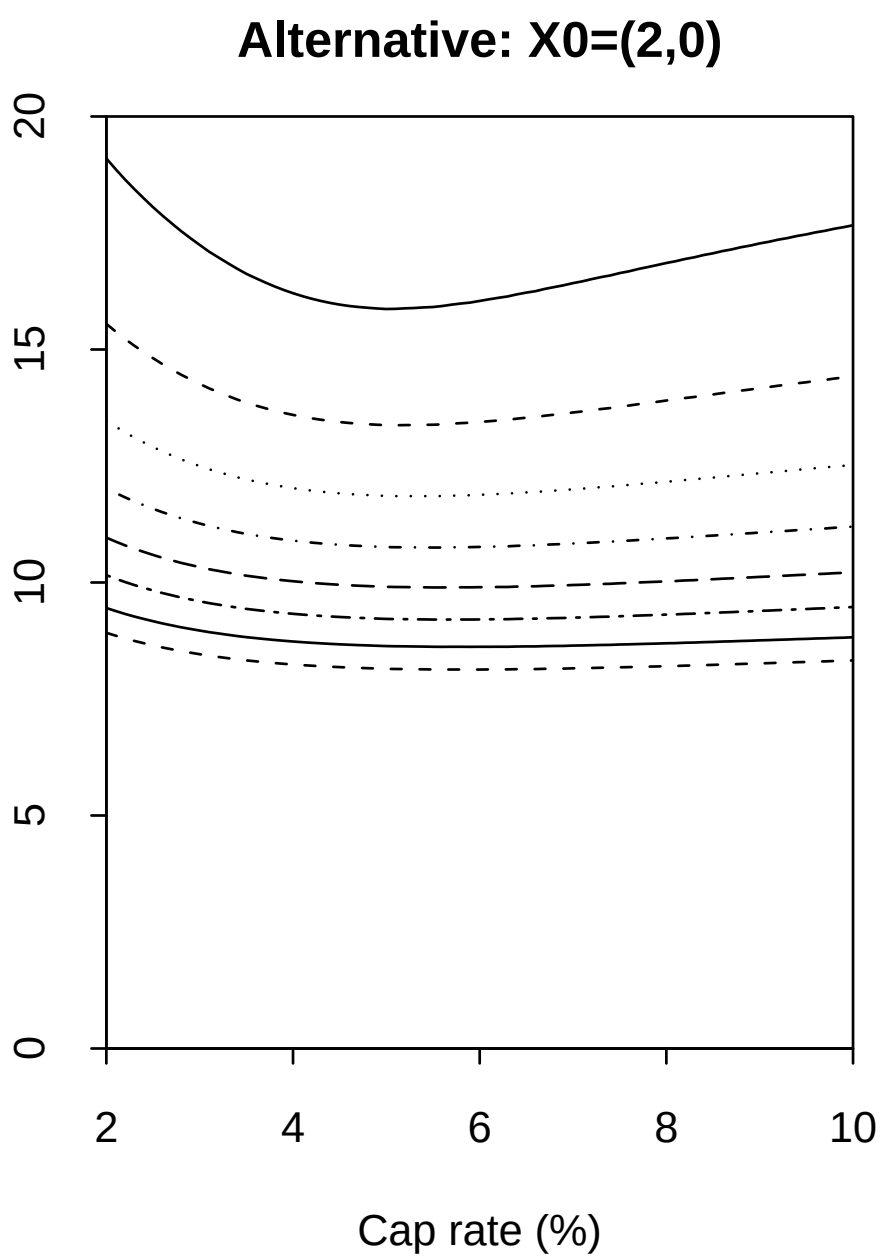
- 12-month LIBOR
- Exercise dates: $T = 1, 2, \ldots, 8$
- Payoff $V(T, X(T))$ at T
- Value at t is

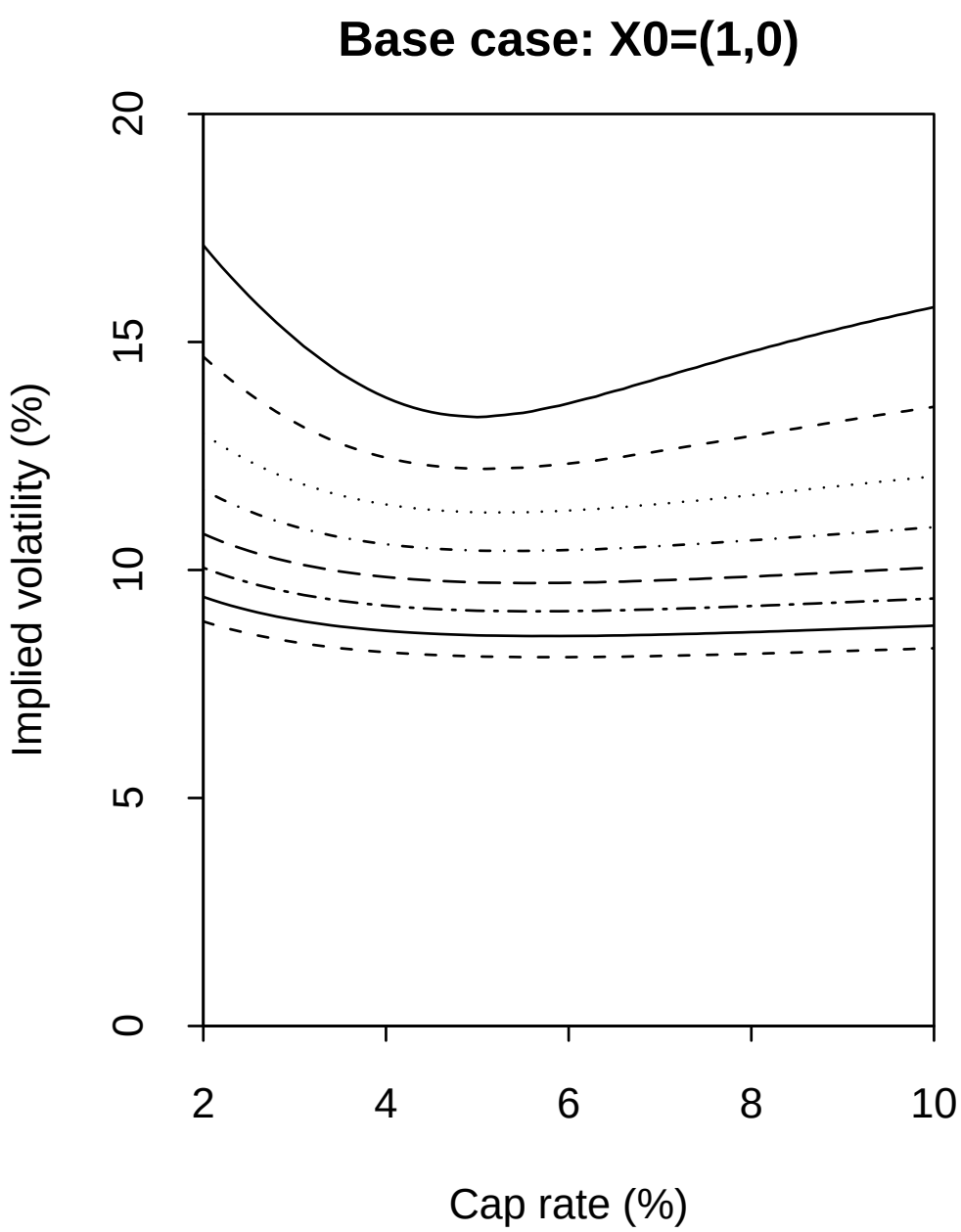
$$V(t, X(t)) = \frac{E_{\hat{P}}[\, V(T, X(T))\; D(T, T, X(T)) \mid \mathcal{F}_t]}{D(t, t, X(t))}$$

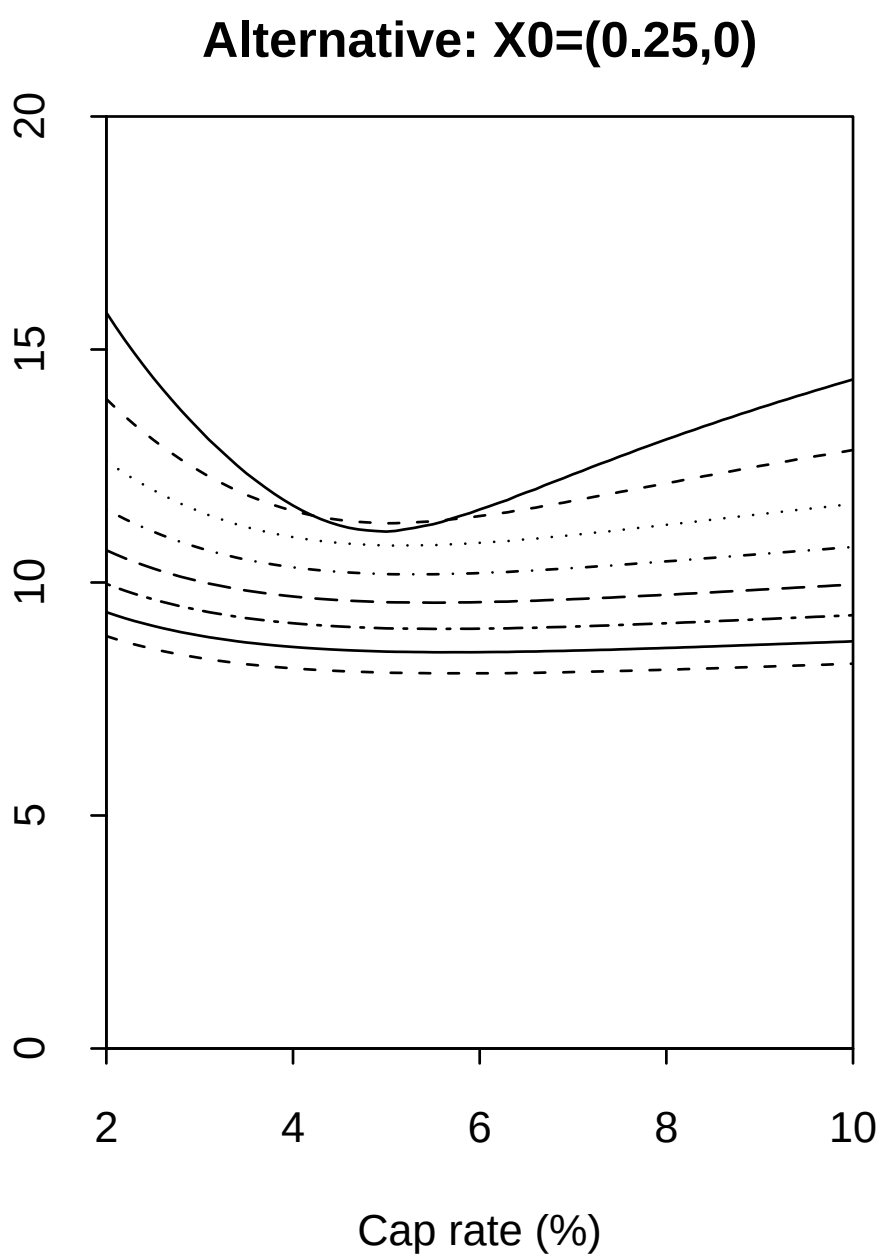

Base case
Implied volatility (%)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)
T0 = 1 yrs
T0 = 2 yrs
T0 = 3 yrs
T0 = 4 yrs
T0 = 5 yrs
T0 = 6 yrs
T0 = 7 yrs
T0 = 8 yrs

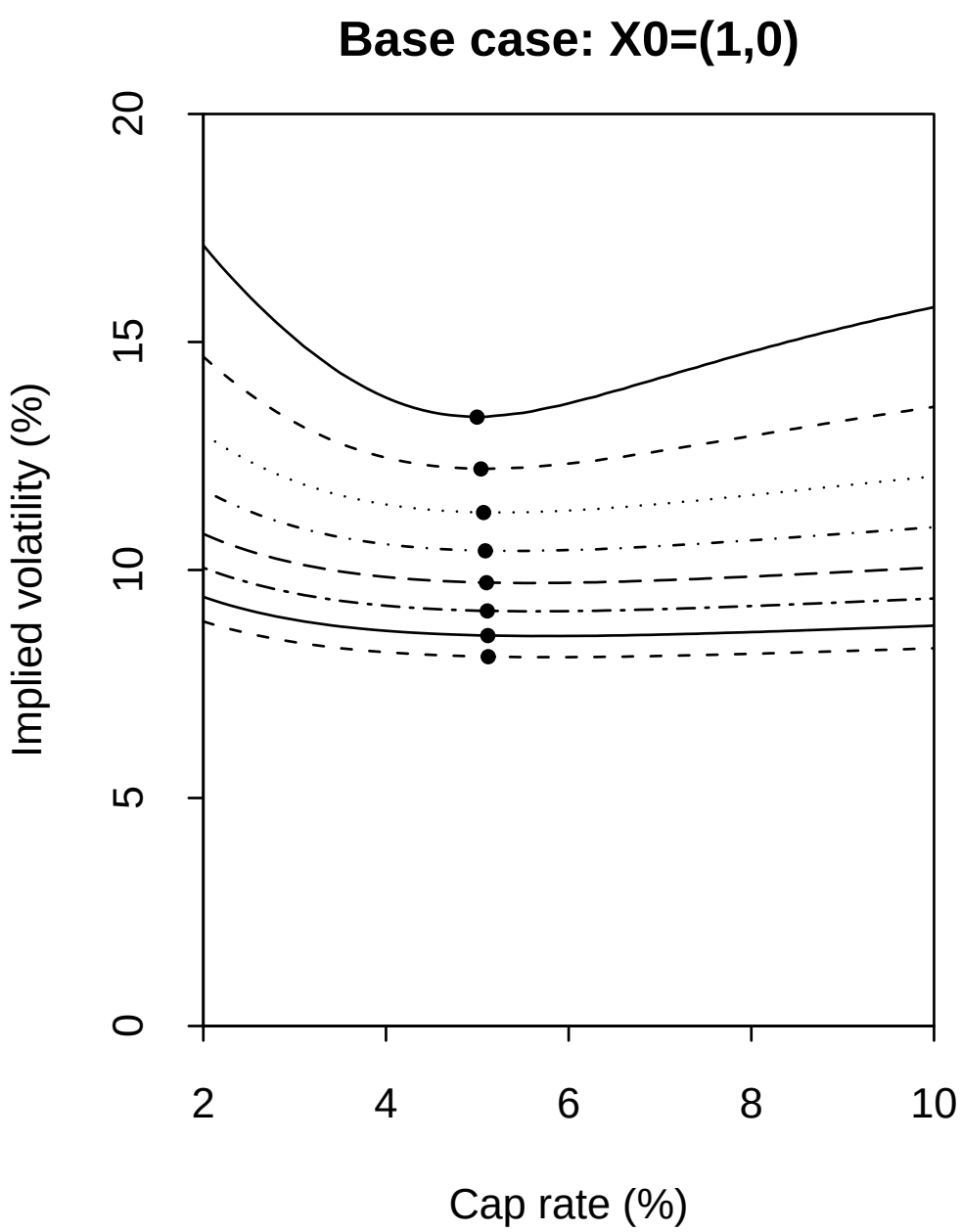

Implied volatility (%)
10
12
14
16
T0 (years)
2
4
6
8
Cap rate (%)
2
4
6
8
10

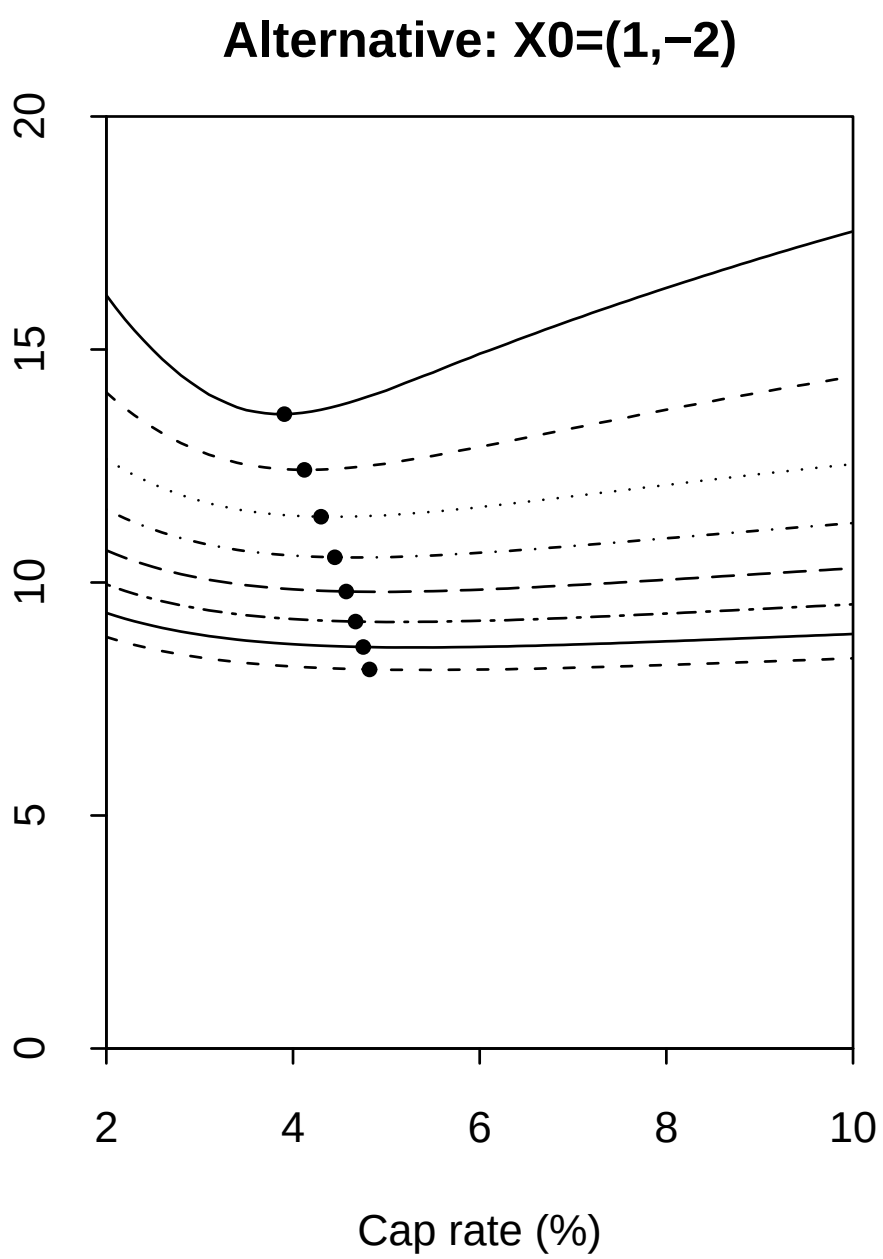

Base case: X0=(1,0)
Implied volatility (%)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)

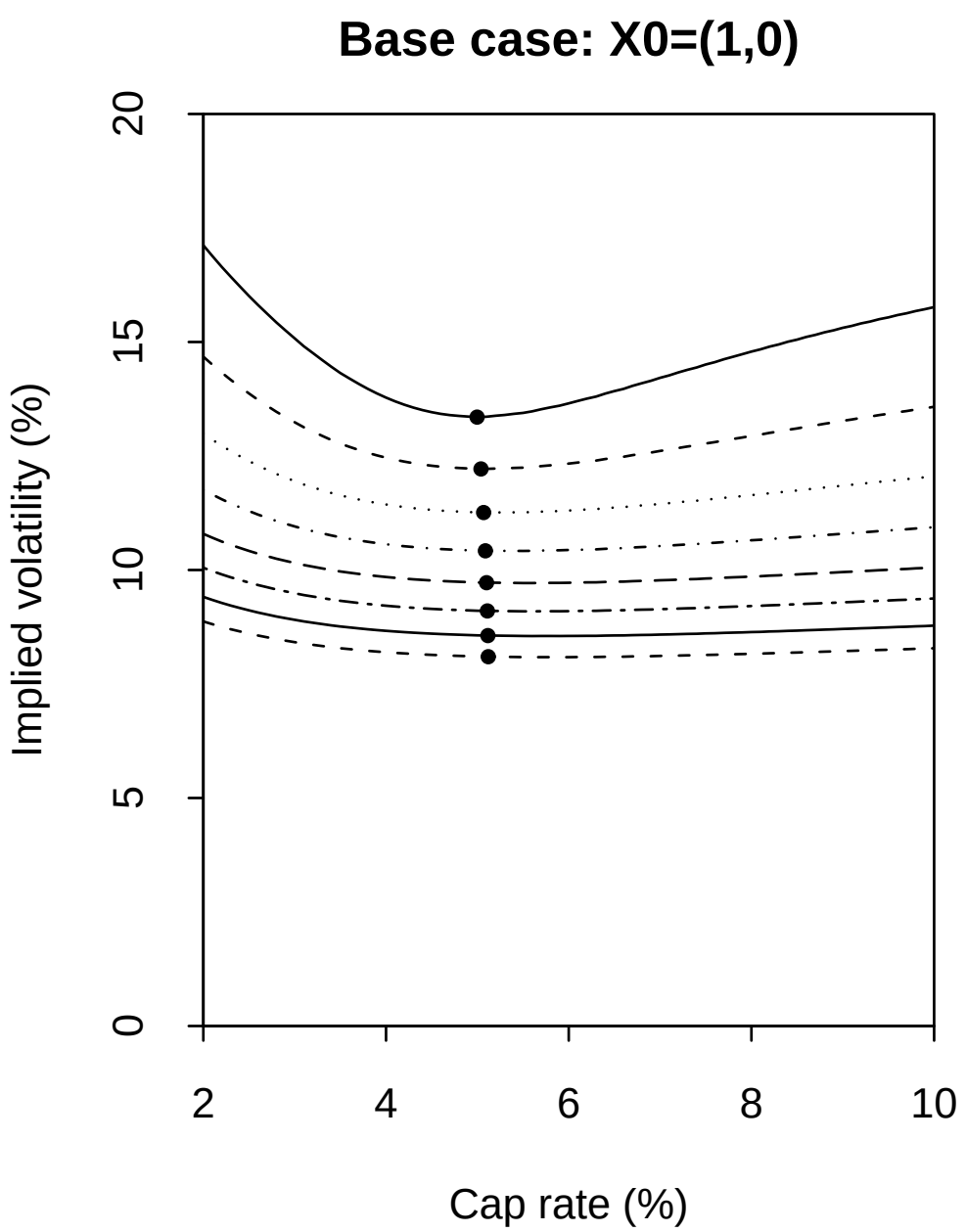

Alternative: X0=(2,0)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)

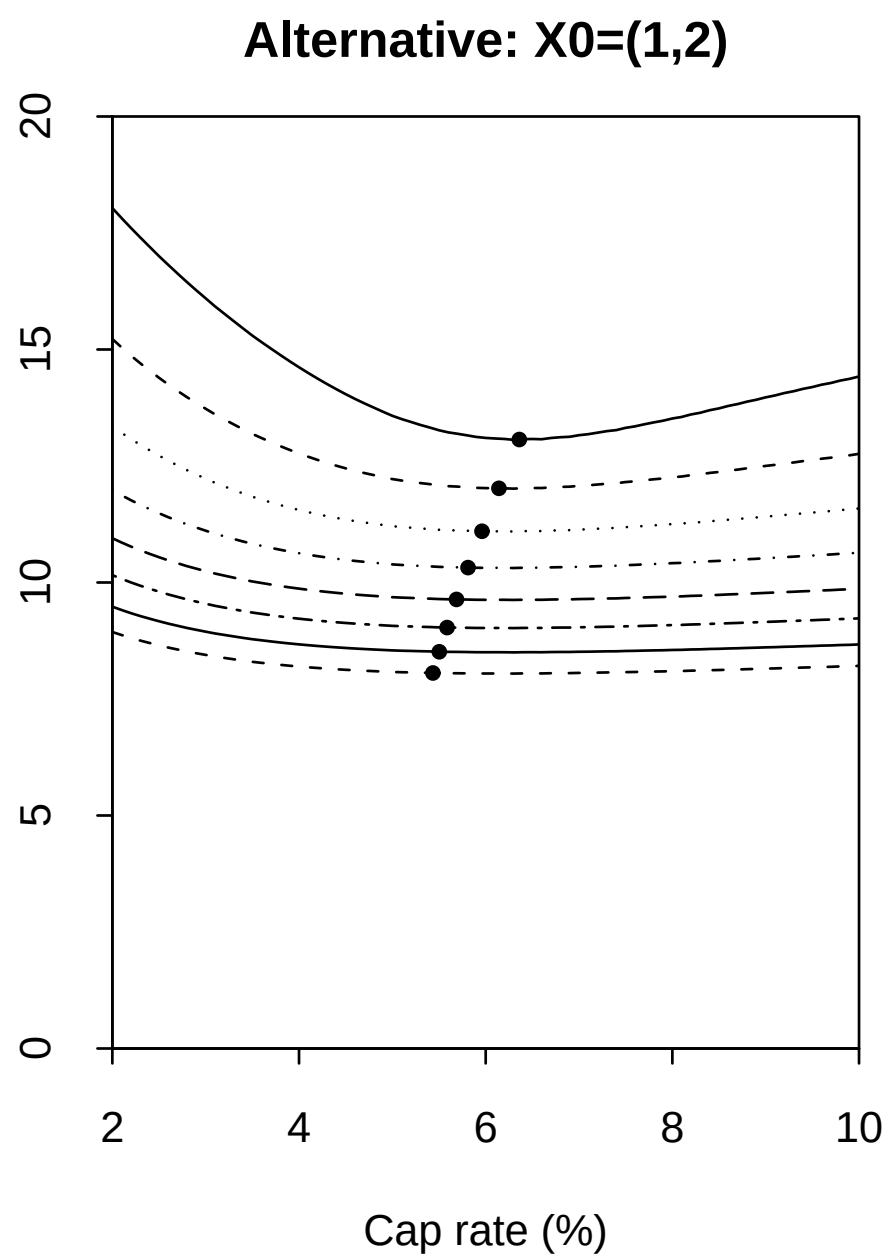

Base case: X0=(1,0)
Implied volatility (%)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)

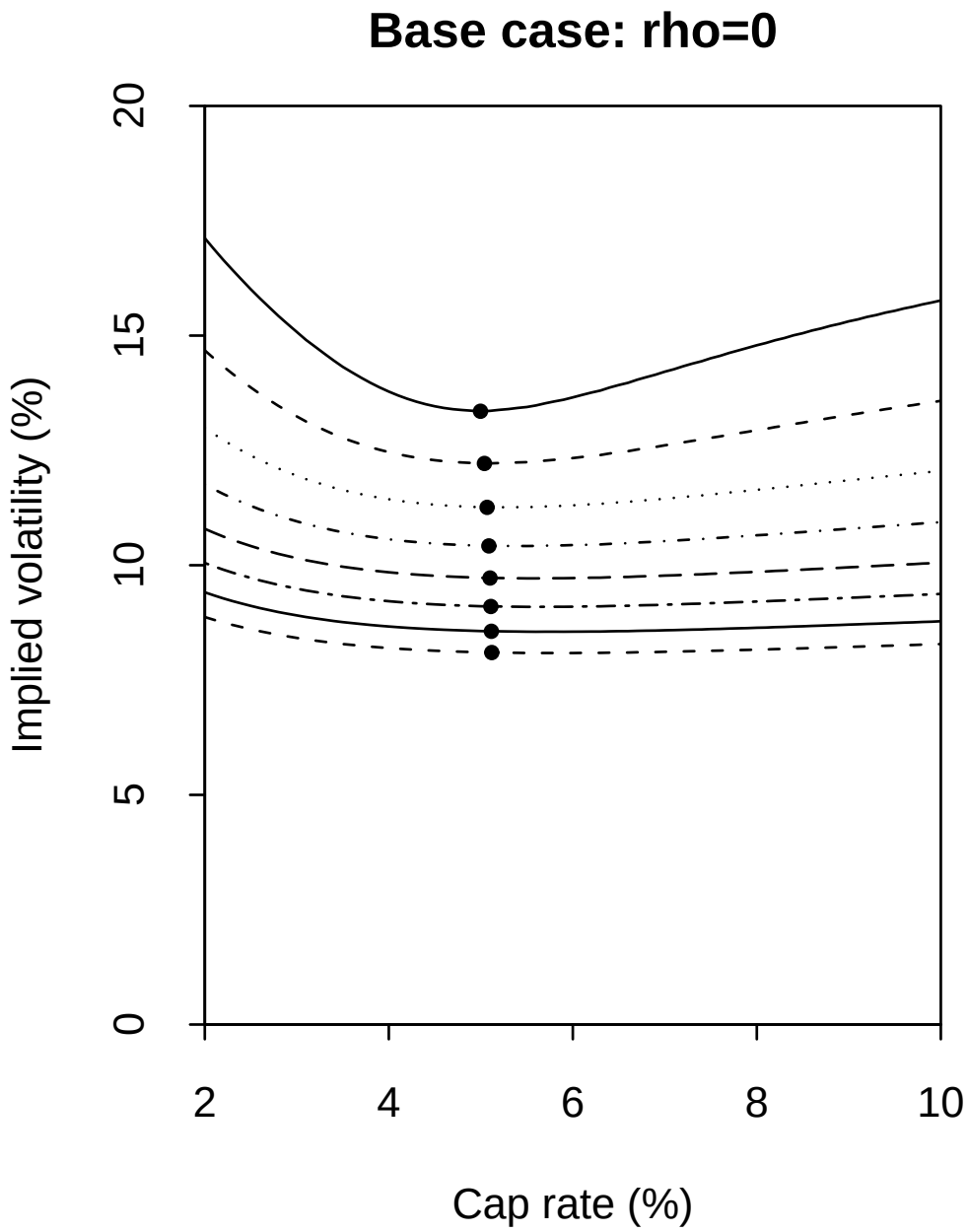

Alternative: X0=(0.25,0)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)

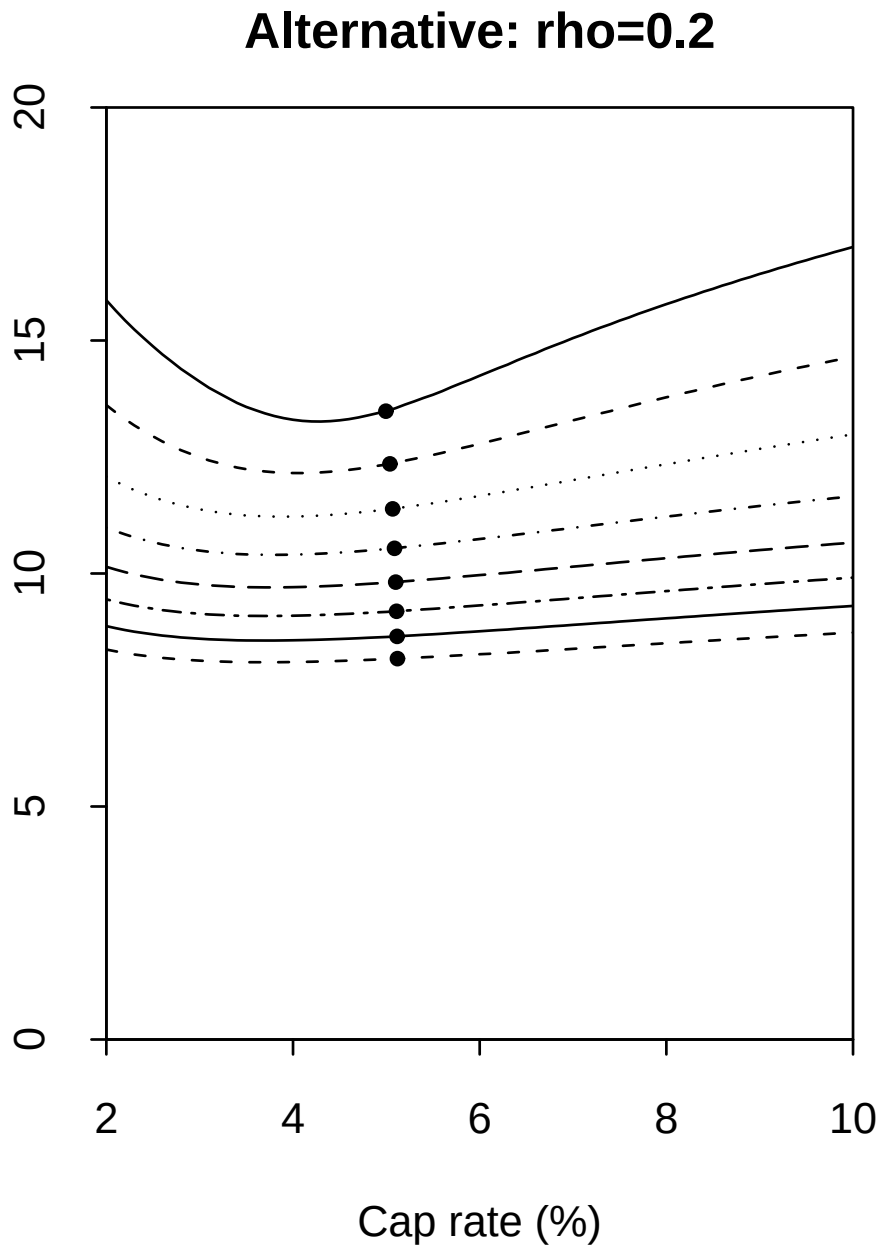

Base case: X0=(1,0)
Implied volatility (%)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)

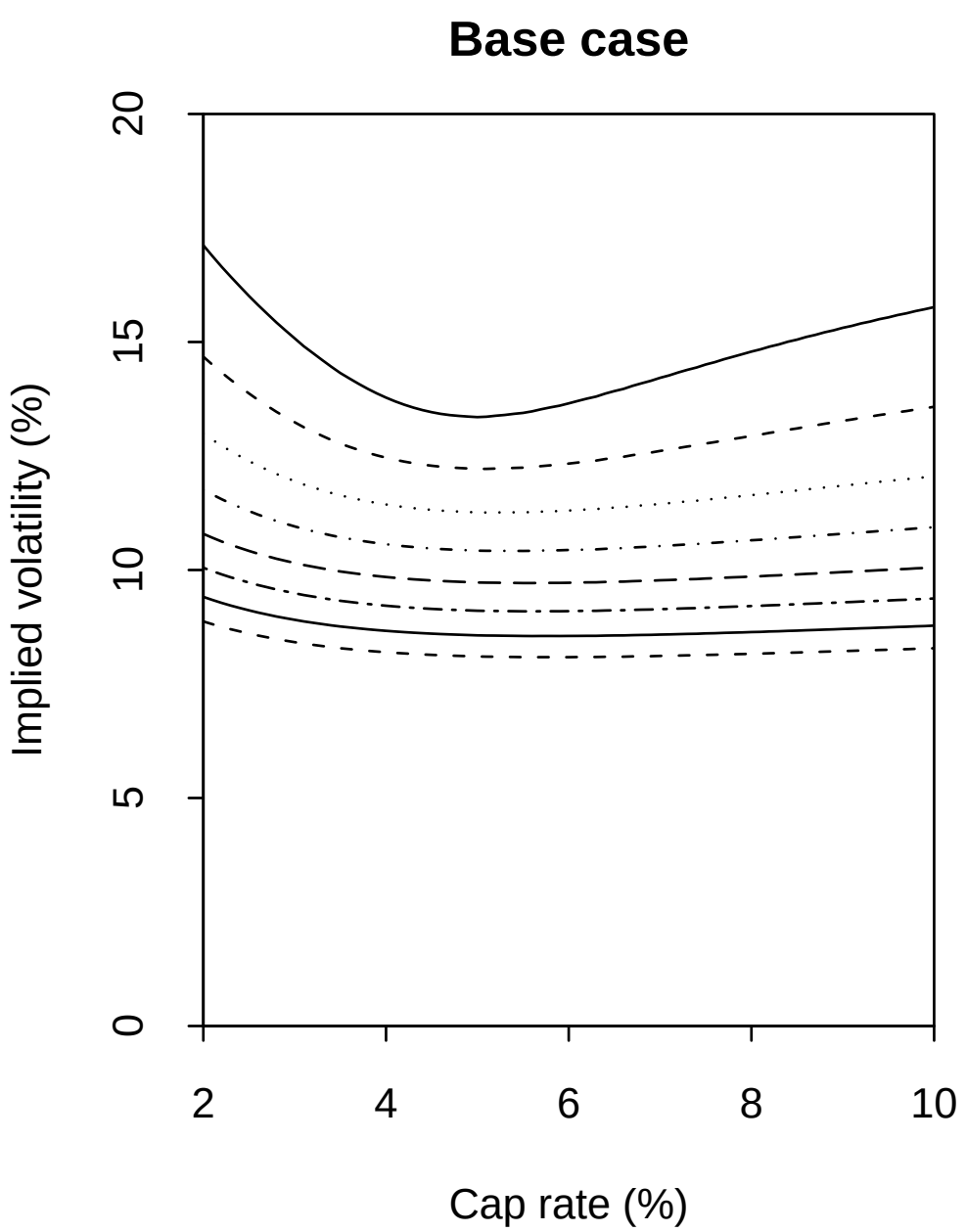

Alternative: X0=(1,−2)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)

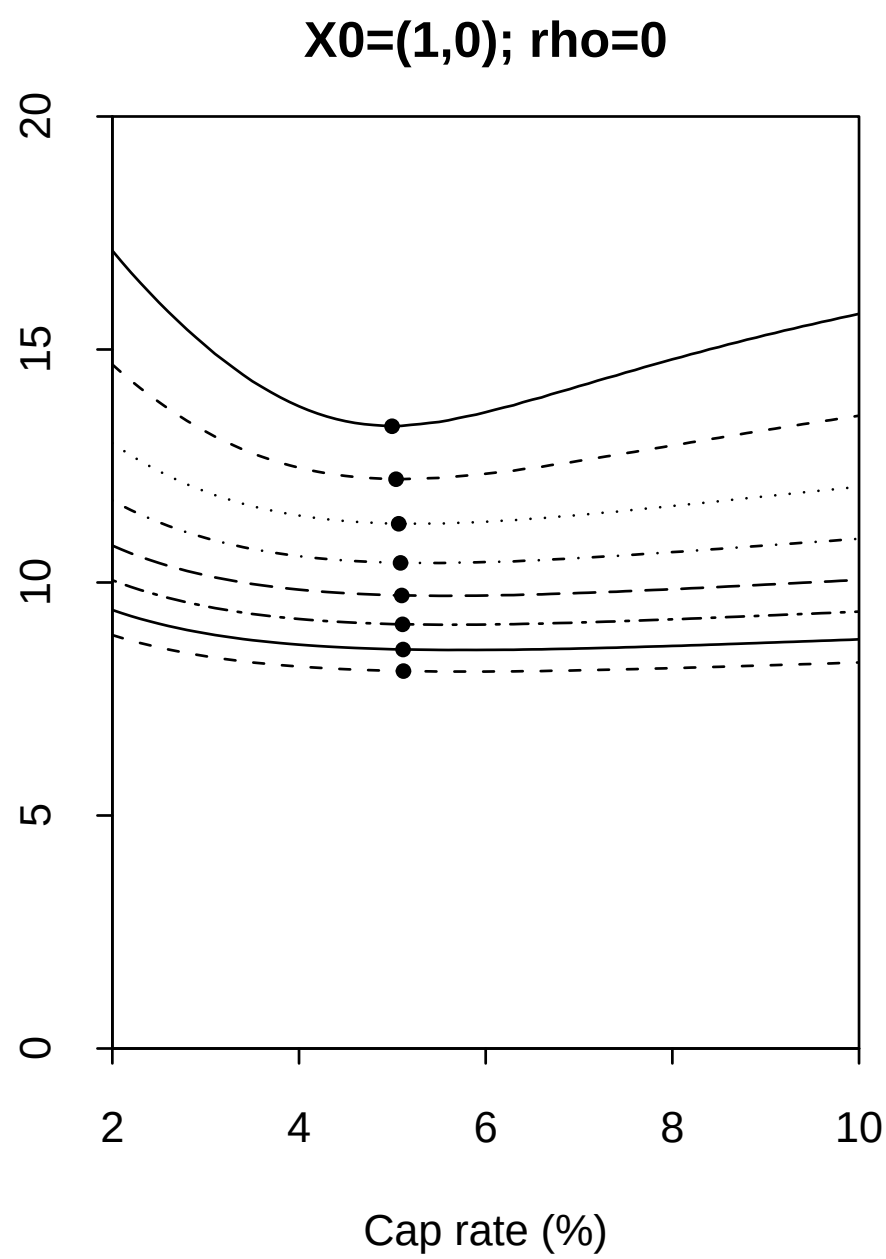

Base case: X0=(1,0)
Implied volatility (%)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)


Alternative: X0=(1,2)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)


Base case: rho=0
Implied volatility (%)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)


Alternative: rho=0.2
0
5
10
15
20
2
4
6
8
10
Cap rate (%)


Base case
Implied volatility (%)
0
5
10
15
20
2
4
6
8
10
Cap rate (%)


X0=(1,0); rho=0
0
5
10
15
20
2
4
6
8
10
Cap rate (%)

3-factor model again

$$dX_1(t) = \alpha_1(\mu_1 - X_1(t))dt + \sigma_1\sqrt{X_1(t)}\sum_{j=1}^{3} d\hat{W}_1(t)$$

$$dX_2(t) = -\alpha_2 X_2(t)dt + \sqrt{X_1(t)}d\hat{W}_2(t)$$

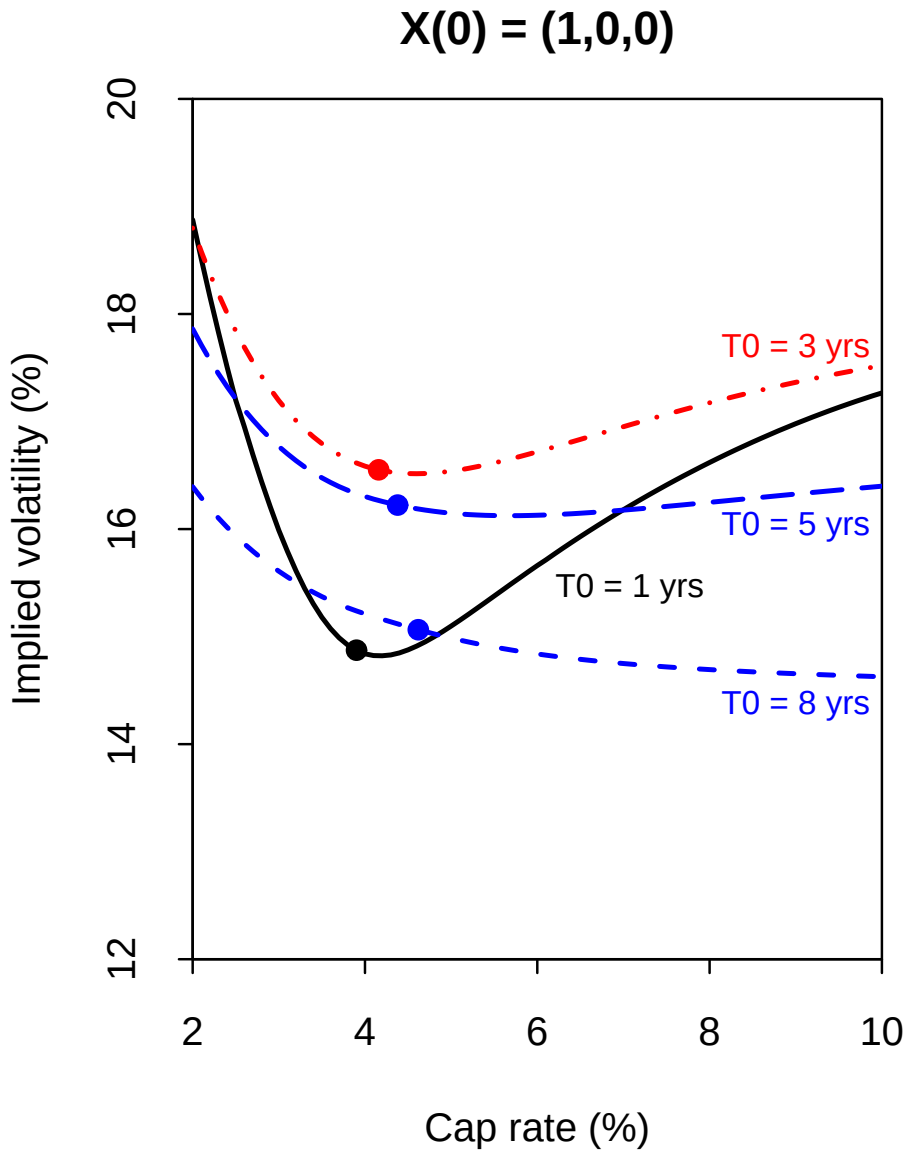
$$dX_3(t) = -\alpha_3 X_3(t)dt + \sqrt{X_1(t)}\left(\rho d\hat{W}_2(t) + \bar{\rho} d\hat{W}_3(t)\right)$$

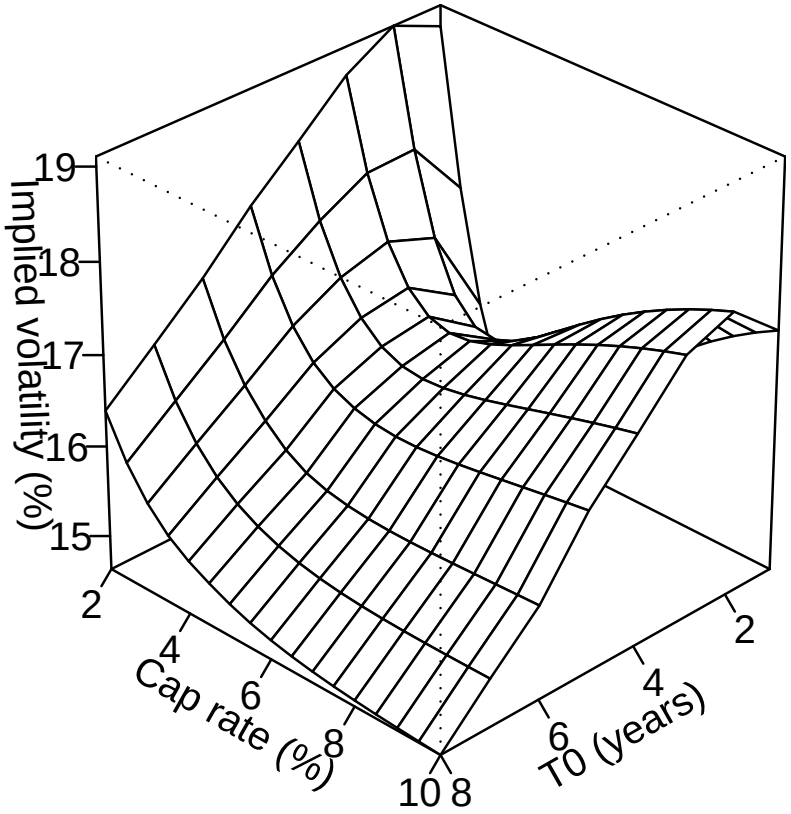
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Numerical examples:

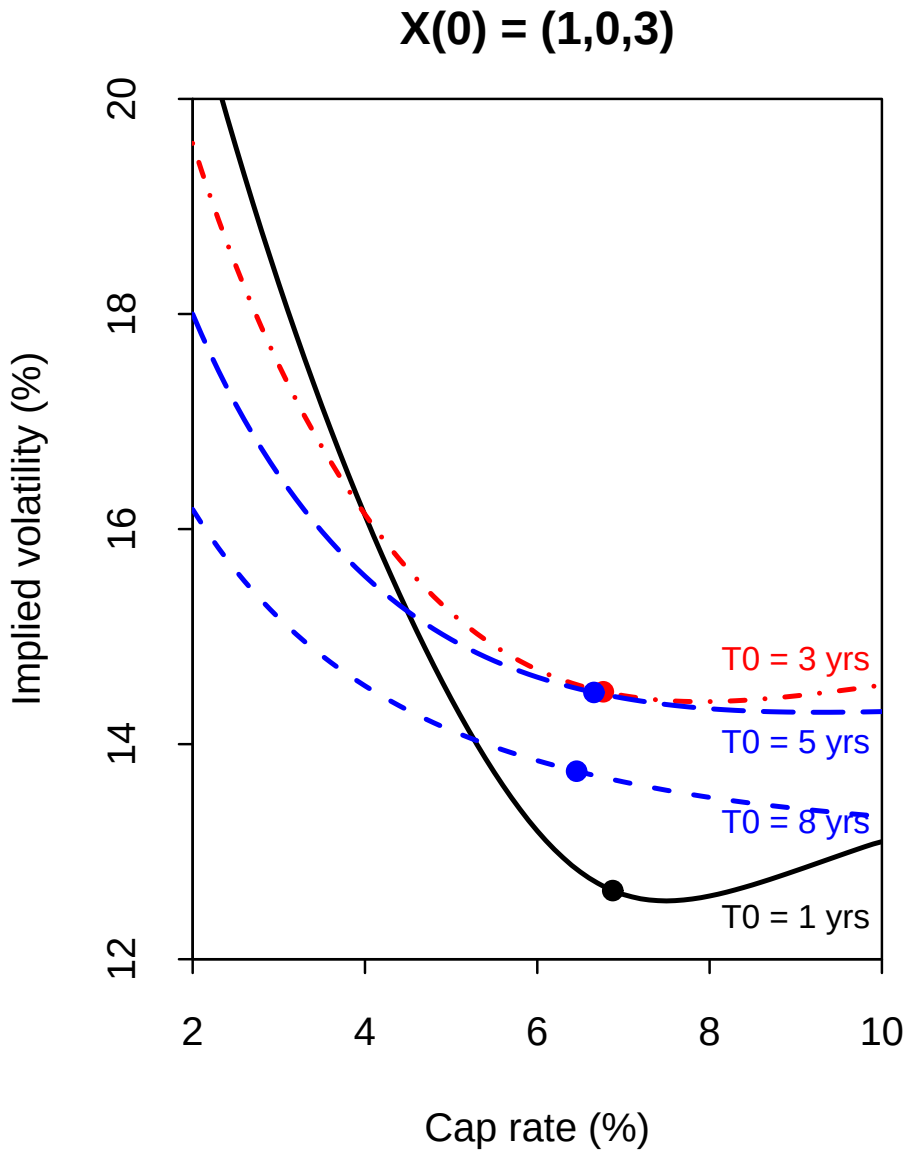
- $\beta = 0.05$ (long-term spot-rate)
- $\mu_1 = 1,\ \alpha_1 = 1,\ \sigma_1 = 1$ (square-root process)
- $\alpha_2 = 1.5,\ \ \alpha_3 = 0.075$
- $B_1(0) = 0.00,\ \ B_2(0) = 0.25,\ \ B_3(0) = 0.35$
- $\rho = -0.8$

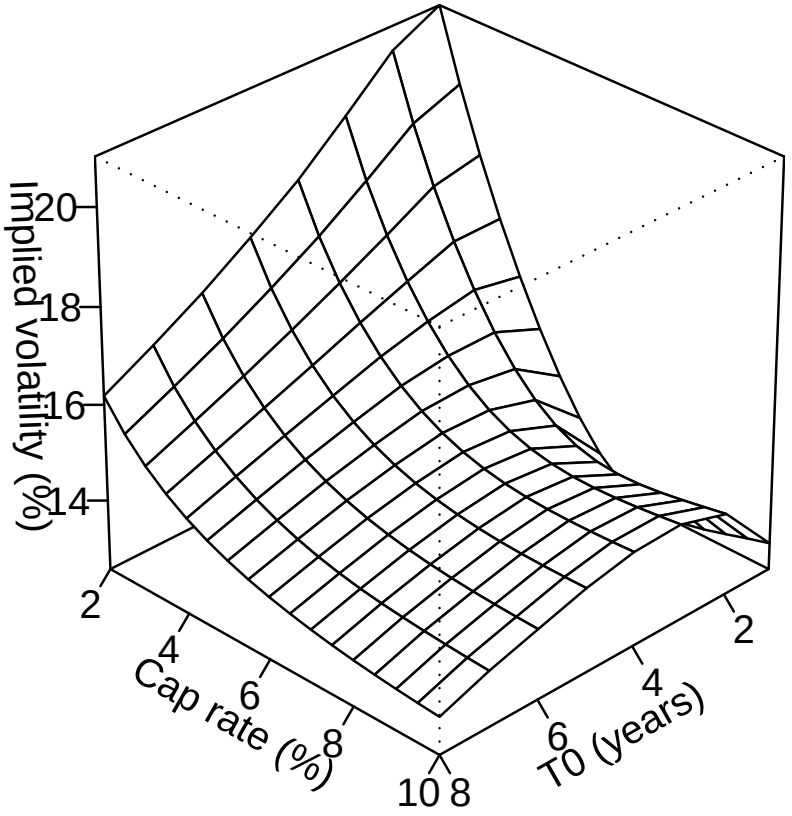
Caplet implied volatilities, 6-month LIBOR $X(0) = (1, 0, 0)$





Caplet implied volatilities, 6-month LIBOR $X(0) = (1, 0, 3)$

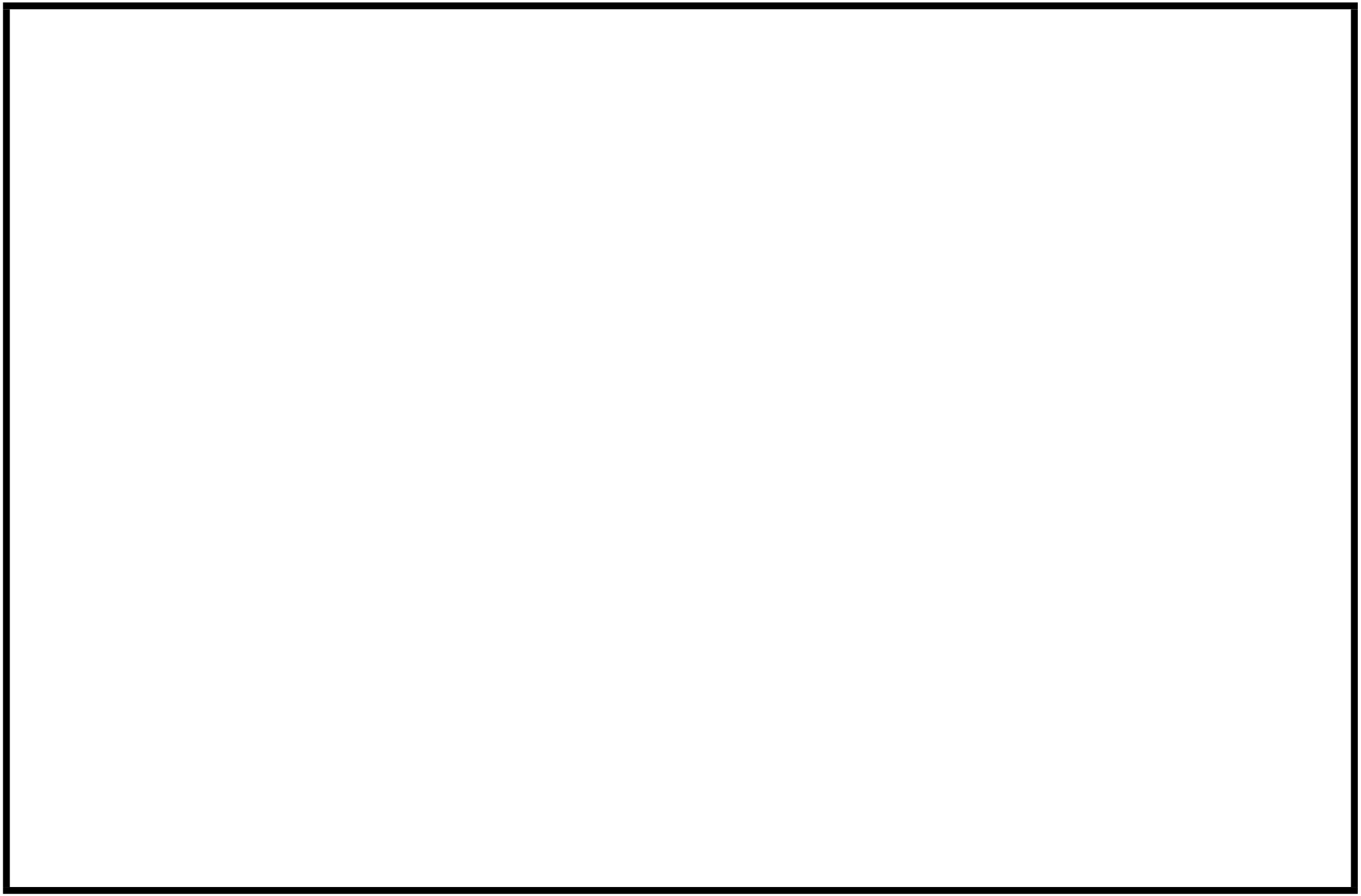




Concluding remarks

- Model can produce a wide range of yield curves including Japan, without the need to recalibrate
- Ongoing work includes parameter estimation
- Calibration:
  - Frequent recalibration of constant model parameters not required
  - Caplets and swaptions: no data to test!!!!

Paper online at: http://www.ma.hw.ac.uk/∼andrewc/papers/



Numerical calculations

Recall

$$V(t, X(t)) = \frac{E_{\hat{P}}[\, V(T, X(T))\; D(T, T, X(T)) \mid \mathcal{F}_t]}{D(t, t, X(t))}$$

- For this parametrisation, and for caplets $V(T, X(T))$ and $D(T, T, X(T))$ relatively insensitive to $X_1(t)$
- Notation: $\mathcal{F}_t^{(1)} = \sigma\{\hat{W}_1(u) : 0 \leq u \leq t\}$

- Given $\mathcal{F}_T^{(1)}$, $(X_2(T), X_3(T))'$ is bivariate normal with known mean ($\rho_{12} = \rho_{13} = 0$) and (co)variances dependent on $\mathcal{F}_T^{(1)}$.
- Calculate $E_{\hat{P}}[V(T, X(T))D(T, T, X(T))|\mathcal{F}_T^{(1)}]$ over lattice for $(X_2(T), X_3(T))$.
- Average over simulated $\mathcal{F}_T^{(1)}$.
- As checks on programming:
  - $V(T, X(T)) = 1$:
    $P(0, T, X(0)) = E_{\hat{P}}[D(T, T, X(T))]/D(0, 0, X(0))$
  - $V(T, X(T)) = P(T, S, X(T))$: $P(0, S, X(0)) =$
    $E_{\hat{P}}[P(T, S, X(T))D(T, T, X(T))]/D(0, 0, X(0))$

Concluding remarks

- Model can produce a wide range of yield curves including Japan, without the need to recalibrate
- Ongoing work includes parameter estimation
- Calibration:
  - Frequent recalibration of constant model parameters not required
  - Caplets and swaptions: no data to test!!!!

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