Stability of Descriptive Models for the Term Structure of Interest Rates[1] [2]

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Abstract

This paper discusses the use of parametric models for the term structure of interest rates and their uses. The paper focuses on a potential problem which arises out of the use of certain models. In most cases the process of parameter estimation involves the minimization or maximization of a function (for example, least squares or maximum likelihood). In some cases this function can have a global minimum/maximum plus one or more local minima/maxima. As we progress through time this leads to a process under which parameter estimates and the fitted term structure can jump about in a way which is inconsistent with bond-price changes.

Here a number of models are identified as susceptible to this sort of problem. A new descriptive model (the restricted-exponential model) is proposed under which it is proved that the likelihood and Bayesian posterior functions have unique maxima: both in a zero-coupon bond market and in a low-coupon bond market. A counterexample shows that this result can break down for larger-coupon bond markets. An alternative Bayesian estimator in combination with the restricted-exponential model is shown to be free from the problem of catastrophic jumps in all coupon-bond markets.

Keywords: term-structure; multiple maxima; restricted-exponential model; likelihood function; posterior distribution; Bayesian estimator.

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1 Introduction

1.1 Descriptive models

A descriptive model takes a snapshot of the bond market as it is today. Generally, there is no reference to price data from other dates. The sole aim is to get a good description of today's prices: that is, of the rates of interest which are implicit in today's prices.

A descriptive model, on its own, gives us no indication of how the term structure might change in the future. We know that there is randomness in the future but this sort of model does not describe this feature. The description of the dynamics of the term structure falls into the domain of arbitrage-free equilibrium and evolutionary models (for example, see Baxter and Rennie, 1996, Rebonato, 1996, or Jarrow, 1996) or more general actuarial and econometric models which are not necessarily arbitrage free but which pay more attention to past history (for example, see Wilkie, 1995, or Mills, 1993).

Descriptive models have a number of uses:

- They give us a broad picture of market rates of interest which are implied by market prices (and, in particular, if there is only a coupon-bond market) (for example, see Nelson & Siegel, 1987, Svensson, 1994, and Dalquist & Svensson, 1996).
- They can be used to price forward bond contracts.
- They can assist in the analysis of monetary policy (Dalquist & Svensson, 1996).
- A forward-rate curve can be used as part of the input to a model based on the Heath, Jarrow & Morton (1992) framework or the more recent positive interest framework of Flesaker & Hughston (1996). Of course, here, it is also necessary to specify a volatility structure (for example, see Jarrow, 1996). Once this has been added to the descriptive model we have a full model which describes the dynamics of the term structure. The input forward-rate curve is often recalibrated each day.
- They can be used in the construction of yield indices (Feldman *et al.,* 1998).
- Finally descriptive models provide sufficient information for us to get a precise market value of a non-profit insurance portfolio or to price, for example, annuity contracts.

1.2 Parametric models

This paper will concentrate on the use of parametric models.

The alternative to such models is spline graduation (for example, see McCulloch, 1971, 1975, Mastronikola, 1991, Deacon & Derry, 1994, or Vasicek & Fong, 1982). Parametric curves aim to give a parsimonious description of the term structure (Svensson, 1994) providing a broad picture which shows the main features of the term structure. Spline graduations aim to give a detailed and highly-parametrized picture of the market: warts and all. It is questionable, however, if all this detail is accurate. For example, splines can overcompensate for small but genuine differences between observed prices and underlying theoretical prices (for example, due to liquidity problems, small variations in taxation or, simply, the bid/offer spread). Furthermore, the results of a spline graduation seem to be sensitive to the number and the location of the knots (Dalquist & Svensson, 1996, and, for example, see Deacon & Derry, 1994, Figures 3.2 to 3.5). In terms of confidence intervals for rates of interest this is likely to result in a relatively wide band at all maturities (that is, a relatively wide margin of error). Parametric models, on-the-other-hand, display a relatively wide margin of error only at the very short and the very long end of the maturity spectrum (Cairns, 1998). At all other maturities estimates of par yields, spot rates and forward rates are all relatively robust.

With parametric curves, the aim is to get a parsimonious description of the term structure: that is, we wish to use a model which captures as much of the detail of the market as possible. These two requirements clearly conflict: it is always possible to improve the fit of the model by increasing the number of parameters. For an additional parameter to be worthwhile the improvement in fit must exceed a specified amount (for example, according to the Schwarz-Bayes Criterion discussed in Wei, 1990, and Cairns, 1995, 1998). Such curves avoid the lumpiness of spline models. However, they can still be biased at certain maturities if there are small heterogeneities in the market not accounted for in the model and if bonds with certain characteristics are clustered. For example, in the UK there is a cluster of strippable gilts at the long end of the market.

1.3 Existing parametric models

1.3.1 Gross redemption yields

Dobbie & Wilkie (1978) proposed the model which is currently used in the construction of the UK yield indices published in the Financial Times. It is a model for gross redemption yields: that is,

$$y(t,t+s) = b_0 + b_1 e^{-c_1 s} + b_2 e^{-c_2 s}$$

is the gross-redemption-yield curve at time t for a coupon bond maturing at time $t+s$. This curve is fitted to low-, medium-, and high-coupon bands separately to take account of the old, UK coupon effect. Since 1996, however, income and capital gains on UK gilts have been taxed on the same basis making this yield curve approach obsolete and forward-rate curves more relevant.

1.3.2 Forward-rate curves

Nelson & Siegel (1987) proposed the following curve:

$$f(t,t+s) = b_0 + (b_{10} + b_{11}s)e^{-c_1 s}.$$

The curve is of the form of a constant *plus* a polynomial-times-exponential term. It allows for a single hump or dip in the curve.

Svensson (1994) generalised this by adding a further polynomial-times-exponential term:

$$f(t,t+s) = b_0 + (b_{10} + b_{11}s)e^{-c_1 s} + b_{21} s e^{-c_2 s}.$$

This curve can have up to two turning points.

Wiseman (1994) proposed another model of exponential type:

$$f(t,t+s) = b_0 + b_1 e^{-c_1 s} + \ldots + b_n e^{-c_n s}.$$

The order of the model n varies from one country to the next. The curve can have up to $n-1$ turning points.

Bjørk & Christensen (1997) generalised all of the previous forms of forward-rate curves by describing the *exponential-polynomial* class of curves:

$$f(t,t+s) = L_0(s) + \sum_{i=1}^{n} L_i(s) e^{-c_i s}$$

where each of $L_0(s), \ldots, L_n(s)$ is a polynomial in s:

$$L_i(s) = b_{i0} + b_{i1}s + \ldots + b_{ik_i} s^{k_i}.$$

In the *unrestricted* case all parameters (the b_{ij} and the c_i) are estimated. This is the case, for example, in Nelson & Siegel (1987), Svensson (1994) and Wiseman (1994).

1.4 Estimation

The parameters in a descriptive model are fitted by talking, first, a snapshot of the data. For example, this may give us a set of price data, P. For a given model, let ϕ be the set of parameters. For a given model and value for ϕ we have, for each i, a theoretical price $P_i(\phi)$ in addition to the observed price P_i. ϕ can be estimated by a number of means: for example, weighted least squares (Dobbie & Wilkie, 1978, Wiseman, 1994); maximum likelihood (Cairns, 1998); or Bayesian methods (Cairns, 1998). Least squares methods can be demonstrated to have a sound statistical basis (Cairns, 1998) but only the likelihood and Bayesian approaches can give a complete picture of the results. In particular, they give not only parameter estimates but also an indication of the level of parameter uncertainty and of the level of uncertainty in the estimates of various interest rates.

Let us take a specific example. Suppose that we are considering a zero-coupon bond market and that we wish to fit the Nelson & Siegel (1987) model using maximum likelihood.

For maximum likelihood we must specify a full statistical model. Here we assume for simplicity that the logarithm of the price of a zero-coupon bond has a Normal distribution with mean $\log \hat{P}_i$ and with a variance which depends upon the term to maturity:

$$\begin{aligned} \log P_i &\sim N\left(\log \hat{P}_i(\phi),\, \sigma^2(t_i)\right) \\ \text{where } t_i &= \text{term of stock } i. \end{aligned}$$

It is not necessary for us to specify the form of $\sigma^2(t)$ here. For possible definitions see Svensson (1994), Wiseman (1994) or Cairns (1998).

Now the form of the forward-rate curve in the Nelson & Siegel (1987) model means that $-\log \hat{P}_i(\phi)$ has the following simple form:

$$-\log \hat{P}_i(c_1, b) \quad = \quad b_0.t_i + b_{10}.\frac{(1-e^{-c_1 t_i})}{c_1} + b_{11}.\frac{1}{c_1^2}\left(1-(1+c_1 t_i)e^{-c_1 t_i}\right)$$

There are three components to this formula. Each component is of the form of a linear b-coefficient times a non-linear term involving t_i and c_1.

Because of this linearity in b it is easy to estimate $b = (b_0, b_{10}, b_{11})$ for this statistical model given a specific value of c_1.

We call the b's linear parameters and c_1 a non-linear parameter.

If we go back to the log-likelihood function this is a function of c_1 and b given P

$$l(c_1, b; P) = -\frac{1}{2}\sum_{i=1}^{n}\left\{\log[2\pi\sigma^2(t_i)] + \frac{(\log P_i - \log \hat{P}_i(c_1, b))^2}{\sigma^2(t_i)}\right\}$$

and we have to maximise this over c_1 and b. This function is quadratic in b so that, for a given value of c_1, the estimate $\hat{b}(c_1)$ is unique. For simplicity we write $\hat{l}(c_1)$ when the function has been maximised over b. $\hat{l}(c_1)$ is the profile log likelihood.

We then have to maximise this over c_1. This function has to be maximised numerically. Sometimes we have a problem with this because $\hat{l}$ occasionally can have more than one maximum: say one global maximum and another local maximum.

The same problem arises within a coupon-bond market. However, the so-called linear parameters, $\hat{b}(c_1)$, are no longer simple to estimate. Generally, $\hat{b}(c_1)$ is uniquely defined, while $\hat{l}(c_1)$ can still have more than one maximum.

In Figure 1 we give an example of this from the close of business on 31 May, 1995, in the UK coupon-bond market. The function $\hat{l}$ has a global maximum at about $c_1 = 0.6$ but it also has a local maximum at about $c_1 = 2.7$. The difference in likelihoods is not too large implying that the local maximum gives almost as good a fit as the global maximum.

1.5 Instability of parameter estimates

Let us think now about the consequences of having more than one maximum as we move through time.

Figure 2 gives a simple scenario of what might happen. As we go from one day to the next, the likelihood curve will gradually evolve. In particular, the two maxima will go up and down. From time to time one of them might disappear totally and from time to time the global maximum might jump from one location to the other as we see here. We start on day 1 with a unique maximum at A. Suppose also that the maximisation algorithm starts at yesterdays maximum and finds the local maximum. From days 2 to 5 there is a second

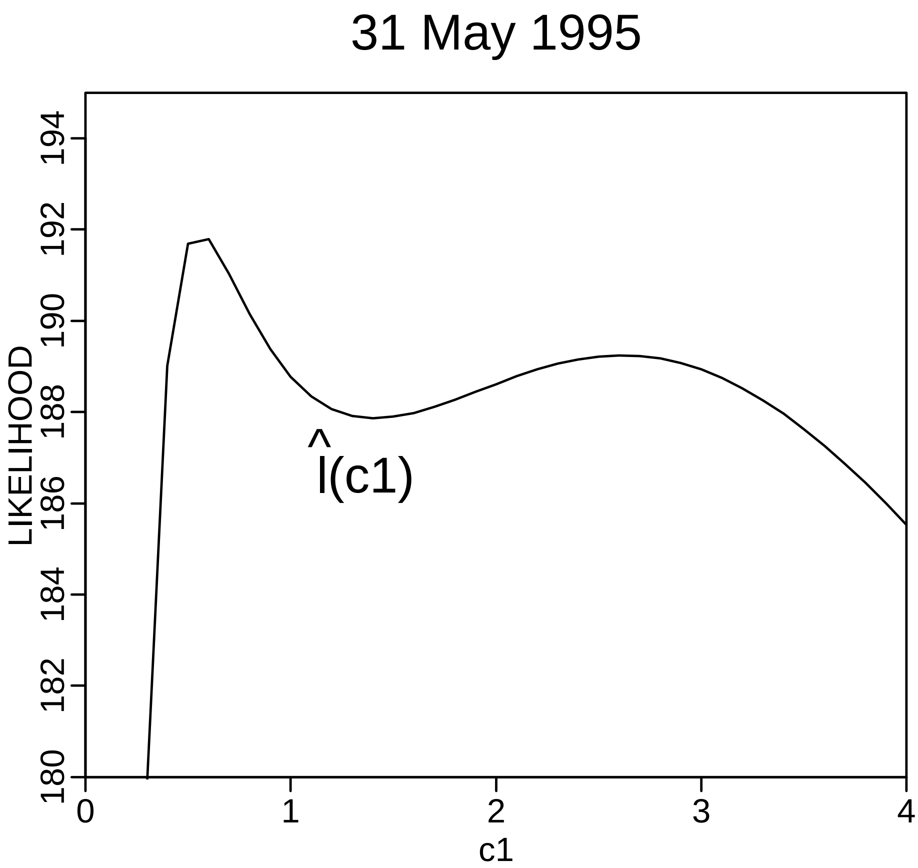

Figure 1: 31 May, 1995. UK coupon bond market. Profile log-likelihood function $\hat{l}(c_1)$ for the Nelson & Siegel model.

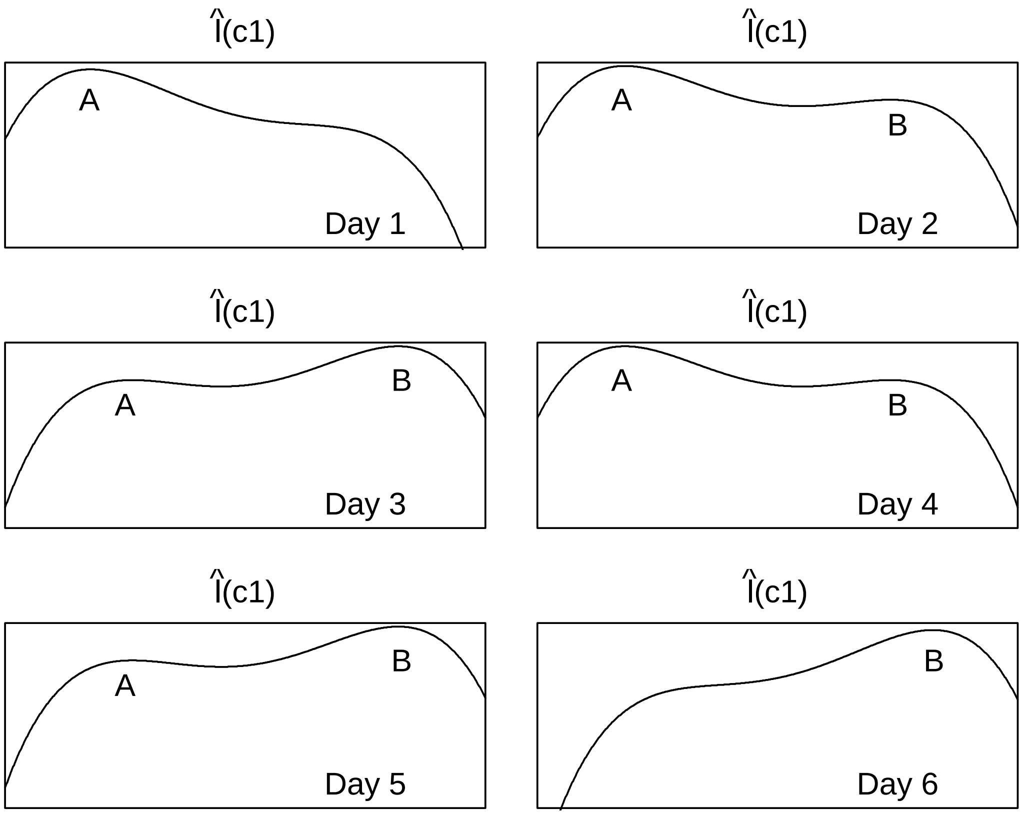

Figure 2: Possible development of the profile log-likelihood function through time.

maximum at B and indeed on days 3 and 5 this is the global maximum. The algorithm will continue until day 5, however, at maximum A. It is not until day 6 when the maximum at A disappears totally that the algorithm moves across to B. Other algorithms might jump more frequently, in particular if they are designed to find the global maximum.

What are the consequences of this problem?

We have identified that as we move from one day to the next the location of the maximum might jump. This is sometimes referred to as a *catastrophic* jump. When such a jump occurs, the size of the jump will typically be much larger than would be consistent with the corresponding changes in prices. For example, if prices follow a diffusion process then the parameter estimates should also follow a diffusion process and in particular should be a continuous process: this continuity will clearly be violated if there is a catastrophic jump.

If parameter estimates jump then a published yield index will also jump in an equally obvious way and the indices will start to lack credibility and fall into disuse. Equally if the curve is used as input to a Heath-Jarrow-Morton model with frequent recalibration it is essential that the recalibrated curves evolve in a way which is consistent with price changes. This is not the case if catastrophic jumps occur which will cause unexpected jumps, for example, in derivative prices.

The existence of more than one maximum can also lead to potential mispricing of such things as bonds, interest-rate derivatives or the pricing of annuities or other life insurance contracts.

All of the Dobbie & Wilkie (1978), Svensson (1994) and Wiseman (1994) models exhibit the same problem with multiple maxima (for example, see Cairns, 1998). In particular, this problem can arise in a zero-coupon bond market as well as in a coupon-bond market. In each case estimates for the linear, polynomial coefficients are unique and simple to derive while the multiple maxima show up in their profile log-likelihood functions. The problem arises on different dates, however, for different models and with varying degrees of magnitude.

In Figure 3 we return to the previous example. Here we have plotted forward-rate, spot-rate and par-yield curves for each of the two maxima in Figure 1. The largest differences occur between the two forward-rate curves. This is because these rates are the furthest from what we actually observe: which is coupon-bond prices. Par yields are closest to what we see on the market so the errors are smallest. The maximum difference here is about only 0.03% which does not sound very much. But if we take the issue of a long-dated stock with a duration, say, of 10 years, then this leads to an error of £3 million per £1 billion issued, which is not trivial. On other dates and for other models these errors can be bigger.

2 The restricted-exponential family

Given the shortcomings of the models described in the previous section, an alternative model is therefore appropriate. Here we describe the the restricted-exponential family. This is a simple family of forward-rate curves:

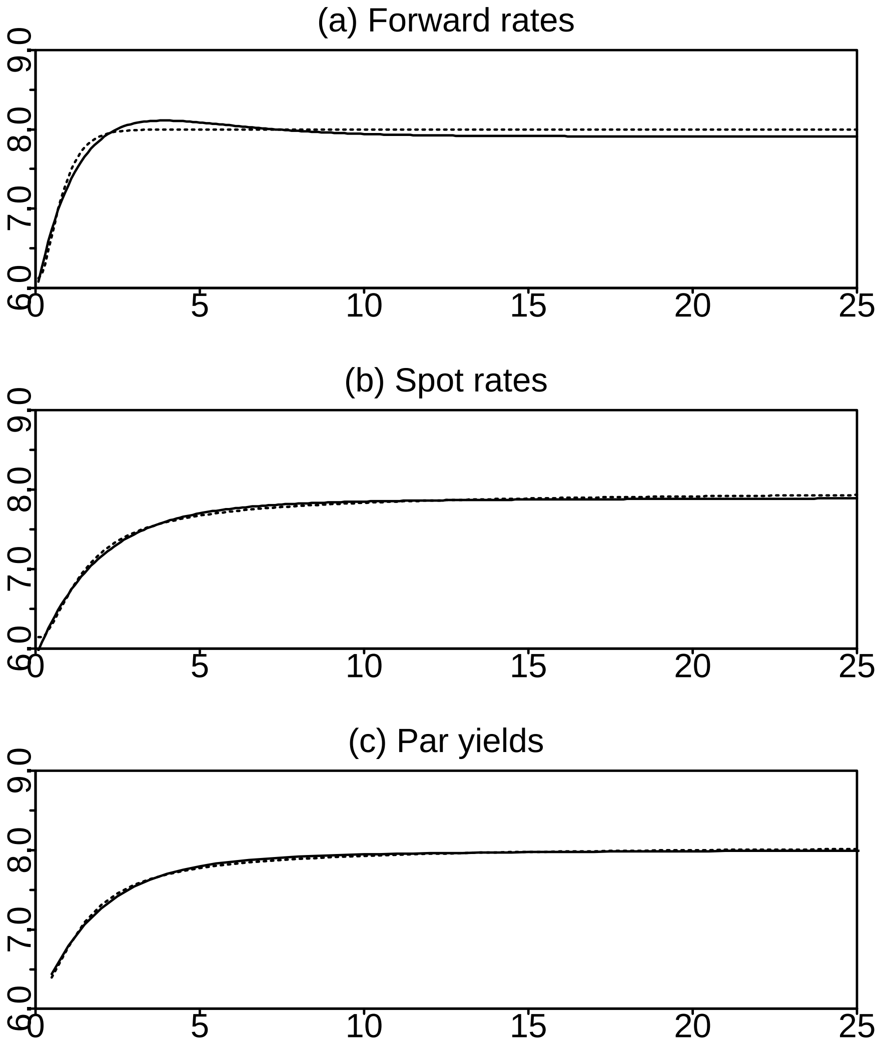

Figure 3: Forward rates, spot rates and par yields for the global maximum (solid line) and for the local maximum (dotted line).

$$\begin{aligned} f(t,t+s) &= b_0 + b_1 e^{-c_1 s} + \ldots + b_m e^{-c_m s} \\ \phi &= (b,c) \end{aligned}$$

The curve is a sum of a constant plus m constant-times-exponential terms and is superficially the same as the Wiseman (1994) model. The parameter set ϕ is divided up into subsets of linear terms $b = (b_0, \ldots, b_m)$ and non-linear terms $c = (c_1, \ldots, c_m)$. This suggests that there is likely to be the same problem as before. However, the approach taken here is a bit different.

In the previous models we estimated all of the parameters: linear and non-linear. In contrast, here we only estimate a subset of the parameter set. Thus, we fix the exponential parameters c at the outset and at no future point do we estimate their values. At any point in time we only estimate the linear parameters b.

2.1 Maximum-likelihood estimation

Proposition 2.1

Under the statistical model proposed in Section 1.4:

(a) In a zero-coupon bond market the resulting maximum-likelihood estimate $\hat{b}$ is unique.

(b) In a low-coupon bond market the log-likelihood function is concave within required region containing all possible maxima. Hence, the maximum-likelihood estimate $\hat{b}$ is unique.

Proof: See Appendix A. In particular, we define what we mean by a low-coupon bond market.

2.1.1 A counterexample for larger coupons

The likelihood and Bayesian posterior-density functions have been shown to have a unique maximum in a low-coupon bond market. Unfortunately, the result does not extend, at least theoretically, to markets with higher coupons as the following simple counterexample shows.

Suppose that we take a very simple case where $m = 1$: that is, $f(t,t+s) = b_0 + b_1 \exp(-c_1 s)$. Our market consists of two stocks:

Stock 1: annual coupon, rate g, term t_1 to maturity.

Stock 2: zero-coupon, term t_2 to maturity.

Stock 1 has an actual price of P_1 and a theoretical price given b_0, b_1 of $\hat{P}_1(b_0, b_1)$. Let L_1 be the set $\{(b_0, b_1) : \hat{P}_1(b_0, b_1) = P_1\}$. This is a downward sloping and *convex* curve (see Appendix A).

Stock 2 has an actual price of P_2 and a theoretical price of $\hat{P}_2(b_0, b_1)$. Let L_2 be the set $\{(b_0, b_1) : \hat{P}_2(b_0, b_1) = P_2\}$. L_2 is a straight line with a negative gradient.

Given the details of stock 1 it is possible to choose P_2 and t_2 such that L_1 and L_2 intersect in two points within the feasible region $\mathcal{B} = \{b : \; f(t,t+s) \geq 0 \text{ for all } s \geq 0\}$. (With $m = 2$, $\mathcal{B} = \{b : \; b_0 \geq 0, \; b_0 + b_1 \geq 0\}$.) Let the points of intersection be (b_{10}, b_{b11}) and (b_{20}, b_{b21}). At each point the theoretical prices equal the observed prices so clearly the likelihood function will be maximised at both points of intersection. These maxima will also, of course, be of the same height.

Numerical example:

Suppose that $c_1 = 0.2$. For stock 1 we have $P_1 = 1$, $t_1 = 20$ and $g = 0.08$, and for stock 2 we have $P_2 = 0.352478$, $t_2 = 13.3562$.
The there are two solutions in $b = (b_0, b_1)$: $b = (0.03, 0.137958)$ and $b = (0.11, -0.091620)$.

The experience of the UK gilts market suggests, however, that there is no problem with multiple maxima. The counterexample above just shows that we cannot rule out the possibility altogether. This is possibly because the problem diminishes as the number of stocks increases.

2.2 Bayesian estimation

2.2.1 Maximum-posterior-density estimation

Suppose instead we wish to use Bayesian methods with a prior distribution $g(b)$ for b and a 0-1 loss function. Then the log-posterior density function is $g(b|P) = g(b) + l(b;P) +$ constant, and the Bayesian estimator is the mode of the posterior distribution. (The 0-1 loss function thus gives an estimator which gives the best fit which is consistent with the prior distribution.)
There are two principal reasons for using Bayesian methods:

- We have introduced a constraint that the forward-rate curve should be positive at all maturities, on the basis that the risk-free rate will be non-negative at all times. If $f(t,t+s)$ is equal to zero for any value of s then this means that $r(t+s) = 0$ with probability 1. If we wish to exclude this possibility then we must require that $f(t,t+s)$ is strictly positive for all s. It is unreasonable to require that $f(t,t+s)$ has a minimum value higher than 0. However, if we use maximum likelihood and this finds that the maximum is at some b for which the forward-rate curve is negative at some maturities then the introduction of a constraint will mean that the forward-rate curve is still equal to 0 for some s. This problem can be avoided if we use Bayesian methods. In particular, if the prior density function, $g(b)$, tends to zero on the boundary of the feasible region (in which $f(t,t+s)$ remains positive) then the maximum of the posterior will give a strictly positive forward-rate curve at all maturities.

- Bayesian methods provide a coherent framework within which we can analyse parameter risk and construct confidence intervals for specified interest rates and so on.

Corollary 2.2
If the log-prior distribution function is concave then:

(a) In a zero-coupon bond market the resulting Bayesian estimate $\hat{b}$ is unique.

(b) In a low-coupon bond market the log-posterior density function is concave within required region containing all possible maxima. Hence, the Bayesian estimate $\hat{b}$ is unique.

2.2.2 Squared-error loss functions

Suppose instead that the loss function is of the form

$$L(b,\tilde{b}) = \int_0^\infty \nu(s)\{f(t,t+s;b) - f(t,t+s;\tilde{b})\}^2 ds,$$

where $\int_0^\infty \nu(s)ds < \infty$, $f(t,t+s;b) = b^T d'(s)$, and $d'(s)^T = (1,\ \exp(-c_1 s),\ \ldots,\ \exp(-c_m s)\ )$. The best estimator is $\hat{b}$ which maximizes the posterior expectation of the loss function. Thus:

$$E[L(b,\hat{b})|P] = \inf_{\tilde{b}} E[L(b,\tilde{b})|P].$$

Now

$$\begin{aligned}
e(\tilde{b}) = E[L(b,\tilde{b})|P] &= E\left[\int_0^\infty \nu(s)\left(b^T d'(s) - \tilde{b}^T d'(s)\right)^2 ds\ |P\right] \\
&= E\left[\int_0^\infty \nu(s)(b-\tilde{b})^T d'(s)d'(s)^T(b-\tilde{b})ds\ |P\right] \\
\Rightarrow \frac{\partial e}{\partial \tilde{b}}(\tilde{b}) &= E\left[2\int_0^\infty \nu(s)d'(s)d'(s)^T(b-\tilde{b})ds\ |P\right] \\
&= 2\int_0^\infty \nu(s)d'(s)d'(s)^T ds E[b-\tilde{b}\ |P].
\end{aligned}$$

Thus $e(\tilde{b})$ is minimized at $\hat{b} = E[b|P]$. This estimator is well defined relative to the problem of maximizing a function with more than one maximum. The estimator will also evolve without the risk of catastrophic jumps, since the form of the posterior distribution evolves in a way which is consistent with price changes. Interestingly, this estimate does not depend upon the form of $\nu(s)$.

If, on-the-other-hand, we are considering one of the models in the unrestricted exponential-polynomial class we have the same problems of non-linearity in the exponential parameters. This arises from the fact that the minimization problem here is essentially the same as the maximization problem described in Sections 1.4 and 1.5 for a zero-coupon bond market.

Clearly any loss function which is quadratic in b and $\tilde{b}$ will also have the same properties: for example, if we replace the forward-rate curve by the spot-rate curve.

3 Further remarks

3.1 Choice of m

To get a consistently good fit in the UK gilts market we require $m = 4$: that is, 4 exponential terms. Inevitably we require more terms than if we estimate both the linear and non-linear parameters. The new approach with 4 exponential terms is roughly equivalent to estimating b and c in a model with only 2 exponential terms (but in each case we are still only estimating 5 parameter values). However, the restricted exponential model has the advantage that it will fit much better on dates where more than one turning point in the forward-rate curve is apparent.

With $m = 4$ we can have a very rich or wide range of yield curves with up to 3 turning points.

3.2 Choice of c

The restricted-exponential fixes c at the outset. It is relevant, therefore, to ask the questions what is an appropriate choice for c and what quantities are sensitive to the choice of c?

One point that can be made at the outset is that it is reasonable to think of variation of the parameters c as variation within a continuous spectrum of models rather than parameter values. Normally we consider a collection of models to be a discrete and possibly finite collection. Here different values of c can be thought of as representing different models.

First, goodness of fit can be seen to be *not* sensitive to changes in c (Cairns, 1998). On the other hand, fitted very short and very long rates *are* sensitive. In between, fitted rates hardly vary if the value of c is changed. This, in fact, is an artefact of the data rather than the model since we are exptrapolating beyond the range of the data. The same would be true of other models.

Other quantities, for example, swaps or spreads might be sensitive to the choice of c but this is beyond the scope of this paper.

3.3 A more general family of curves

Bjørk and Christensen (1997) consider what families of curve are consistent with the evolution of certain models for the term structure. They describe a more general class of model: the restricted-exponential-polynomial family. Any curve in this family in which the polynomials are all of degree 0 is in the restricted-exponential family. A simple example is the Vasicek (1977) model. This is a one-factor model under which the forward-rate curve evolves within the family of curves $\{f(t,t+s) = b_0 + b_1 \exp(-c_1 t) + b_2 \exp(-c_2 t)\}_{b_0,b_1,b_2}$ where c_1 and c_2 are fixed and $c_2 = 2c_1$. There are of course further restrictions on the parameters b_0, b_1 and b_2 but the curve does evolve within this higher-dimensional family.

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Appendix A

A.1 Zero-coupon bond market

Define $Z(b;t) \equiv Z(b) \equiv Z(b;t) = \exp[-b^T d(t)]$ where $b^T = (b_0, \ldots, b_m)$, $d(t)^T = (d_0(t), \ldots, d_m(t))$, $d_0(t) = t$ and $d_k(t) = (1 - \exp(-c_k t))/c_k$ for $k = 1, \ldots, m$.

For a bond maturing at t_i write $d_i = d(t_i)$ and $d_{ik} = d_k(t_i)$.

Suppose that the observed prices are P_1, P_2, ..., P_N. The likelihood function is

$$\begin{aligned} l(b;P) &= -\frac{1}{2}\sum_{i=1}^{N}\left\{\log 2\pi\sigma^2(t_i) + (\log P_i - \log Z(b;t_i))^2 / \sigma^2(t_i)\right\} \\ &= -\frac{1}{2}\sum_{i=1}^{N}\lambda_i(\log P_i + d_i^T b)^2 + \text{constant} \\ \text{where } \lambda_i &= 1/\sigma^2(t_i). \end{aligned}$$

Maximising the likelihood is thus equivalent to minimising the function

$$\begin{aligned} g(P|b) &= \frac{1}{2}\sum_{i=1}^{N}\lambda_i(\log P_i + d_i^T b)^2 \\ \Rightarrow \frac{d^2 g}{db} &= \sum_i \lambda_i d_i d_i^T. \end{aligned}$$

The matrix of second derivatives is constant and positive definite.

Proposition 2.1(a)

Hence $g(P|b)$ is convex and has a unique minimum in b.

If we wish instead to use Bayesian methods it is necessary for us to specify also a prior density function. Suppose that the log-prior density function is denoted by $p(b)$. The log-posterior density function is then

$$p(b|P) = p(b) - g(P|b) + \text{constant}.$$

Corollary 2.2(a)

If $p(b)$ is convex then $-\partial^2 p(b)/\partial b^2$ is positive semi-definite. Thus the matrix of second derivatives of minus the log-posterior density function $-\partial^2 p(b|P)/\partial b^2$ is positive definite and there is a unique value of b for which $\partial p(b|P)/\partial b = 0$.

A.2 Coupon-bond market

Suppose that there are N bonds each with a nominal value of 1. Bond i has cashflows $c_{i1}, \ldots, c_{in_i}$ at times $0 < t_{i1} < \ldots < t_{in_i}$ respectively. Given b, the theoretical price of each bond is then:

$$
\begin{aligned}
\hat{P}_i(b) &= \sum_{j=1}^{n_i} c_{ij} Z(b;t_{ij}) \\
&= \sum_{j=1}^{n_i} c_{ij} \exp(-b^T d(t_{ij})) \\
\text{Write } d_{ij} &= d(t_{ij}) \\
d_{ijk} &= d_k(t_{ij}) \\
g(b) &= \frac{1}{2}\sum_{i=1}^{N} \lambda_i [\log P_i - \log \sum_{j=1}^{n_i} c_{ij} Z(b;t_{ij})]^2
\end{aligned}
$$

(Recall that the log-likelihood is $l(b;P) = -g(b) + \text{constant}$.)
Now

$$
\begin{aligned}
Z(b;t) &= \exp(-b^T d(t)) \\
\Rightarrow \frac{dZ(b;t)}{db} &= -Z(b;t)d(t) \\
\frac{d^2 Z(b;t)}{db^2} &= Z(b;t)d(t)d(t)^T. \\
\text{Thus } g'(b) &= -\sum_i \lambda_i [\log P_i - \log \sum_j c_{ij} Z(b;t_{ij})] \frac{\sum_j c_{ij} \frac{dZ(b;t_{ij})}{db}}{\sum_j c_{ij} Z(b;t_{ij})} \\
&= -\sum_i \lambda_i [\log P_i - \log \sum_j c_{ij} Z(b;t_{ij})] \sum_j f_{ij}(b) d_{ij} \\
&= -\sum_i \lambda_i [\log P_i - \log \sum_j c_{ij} Z(b;t_{ij})] \bar{d_i}(b) \\
\text{where } \bar{d_i}(b) &= \sum_j f_{ij}(b) d_{ij} \\
\text{and } f_{ij}(b) &= \frac{c_{ij} Z(b;t_{ij})}{\sum_j c_{ij} Z(b;t_{ij})} \\
g''(b) &= \sum_i \lambda_i \left(\frac{\sum_j c_{ij} \frac{dZ(b;t_{ij})}{db}}{\sum_j c_{ij} Z(b;t_{ij})} \right) \left(\frac{\sum_j c_{ij} \frac{dZ(b;t_{ij})}{db}}{\sum_j c_{ij} Z(b;t_{ij})} \right)^T - \sum_i \lambda_i [\log P_i - \log \sum_j c_{ij} Z(b;t_{ij})] V_i(b) \\
&= \sum_i \lambda_i \bar{b}_i(b) \bar{b}_i(b)^T - \sum_i \lambda_i [\log P_i - \log \sum_j c_{ij} Z(b;t_{ij})] V_i(b) \\
\text{where } V_i(b) &= \frac{d}{db} \left(\frac{\sum_j c_{ij} \frac{dZ(b;t_{ij})}{db}}{\sum_j c_{ij} Z(b;t_{ij})} \right) \\
&= \frac{(\sum_j c_{ij} Z(b;t_{ij}) d_{ij} d_{ij}^T)(\sum_j c_{ij} Z(b;t_{ij})) - (\sum_j c_{ij} Z(b;t_{ij}) d_{ij})(\sum_j c_{ij} Z(b;t_{ij}) d_{ij}^T)}{(\sum_j c_{ij} Z(b;t_{ij}))^2} \\
&= \sum_j f_{ij}(b) d_{ij} d_{ij}^T - (\sum_j f_{ij}(b) d_{ij})(\sum_j f_{ij}(b) d_{ij})^T
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_j f_{ij}(b)(d_{ij} - \bar{d_i}(b))(d_{ij} - \bar{d_i}(b))^T \\
&\geq 0
\end{aligned}
$$

Now write $X_i(b) = \log P_i - \log \sum_j c_{ij} Z(b;t_{ij}) = \log[P_i/\hat{P}_i(b)]$.

$$
\begin{aligned}
g'(b) &= \sum_i \lambda_i X_i(b) \bar{d_i}(b) \\
g''(b) &= \sum_i \lambda_i \left(\bar{d_i}(b) \bar{d_i}(b)^T - X_i(b) V_i(b) \right)
\end{aligned}
$$

Now in a zero-coupon bond market $V_i(b) \equiv 0$ and the $\bar{d_i}(b)$ are constant and do not depend on b. Thus $g''(b)$ is constant and positive definite confirming the simpler derivation above.

A.3 Assumptions

We make the following assumptions:

The range of acceptable values for b is denoted by $\mathcal{B}$. Each $b \in \mathcal{B}$ must satisfy the following criterion: for all $0 < t < s$, $1 > Z(b;t) > Z(b;s)$. This criterion is equivalent to the assumption that the forward-rate curve $f(t, t+s)$ is non-negative for all t and for all $s > 0$: that is,

$$
\begin{aligned}
\mathcal{B} &= \{ b : \; b^T d'(t) \geq 0 \text{ for all } t \} \\
\text{where } \; d'(t)^T &= (1, \; \exp(-c_1 t), \; \ldots, \; \exp(-c_m t) \;).
\end{aligned}
$$

Suppose that $u > 0$ and that $b \in \mathcal{B}$. For any $0 < t < s$ we note that $1 > Z(b;t) > Z(b;s)$ and therefore $0 < b^T d(t) < b^T d(s)$. Thus

$$
\begin{aligned}
0 < u.b^T d(t) &< u.b^T d(s) \\
\Rightarrow 1 > \exp(-ub^T d(t)) &> \exp(-ub^T d(s)) \\
\Rightarrow 1 > Z(bu;t) &> Z(bu;s)
\end{aligned}
$$

Thus $b \in \mathcal{B}$ if and only if $bu \in \mathcal{B}$ for all $u > 0$. That is, $\mathcal{B}$ is a cone.

Furthermore, suppose that b_A and b_B are in $\mathcal{B}$. Then for any λ such that $0 < \lambda < 1$, and for any $0 < t < s$:

$$
\begin{aligned}
0 < b_A^T d(t) &< b_A d(s) \\
\text{and } \; 0 < b_B^T d(t) &< b_B d(s) \\
\Rightarrow 0 < (1-\lambda) b_A^T d(t) + \lambda b_B^T d(t) &< (1-\lambda) b_A^T d(s) + \lambda b_B^T d(s) \\
\Rightarrow 0 < [(1-\lambda) b_A + \lambda b_B]^T d(t) &< [(1-\lambda) b_A + \lambda b_B]^T d(s) \\
\Rightarrow [(1-\lambda) b_A + \lambda b_B] &\in \mathcal{B}
\end{aligned}
$$

Thus $\mathcal{B}$ is a convex cone.

A.4 Locality of possible minima

We define the following sets:

$$\begin{aligned} B_0 &= \{b \in \mathcal{B}: \ P_i < \hat{P}_i(b) \text{ for all } i\} \\ B_1 &= \{b \in \mathcal{B}: \ P_i > \hat{P}_i(b) \text{ for all } i\} \\ B_2 &= \mathcal{B} \backslash B_1. \end{aligned}$$

Clearly for all $b \in B_0$, $\hat{P}_i(bs) > P_i$ for all $0 < s \leq 1$. That is, $b \in B_0$ implies that $bs \in B_0$ for all $0 < s \leq 1$.

Similarly, $b \in B_1$ implies that $bs \in B_1$ for all $1 \leq s < \infty$.

Lemma A.1

There does not exist $b \in B_0$ or $b \in B_1$ such that $g'(b) = 0$ is positive definite.

Proof Suppose that there exists such a $b \in B_1$.

Let $s_0 = \inf\{s : bs \in B_1\}$. Thus $\hat{P}_i(bs) < P_i$ for all $s > s_0$ and $\hat{P}_i(bs)$ is decreasing with s for $s > s_0$. Hence $g(bs)$ is an increasing function of s for $s > s_0$ which indicates that there cannot be a minimum at b.

Similarly there does not exist such a $b \in B_0$.

Lemma A.2

B_1 is convex.

Proof

Note that $\hat{P}_i(b)$ is convex in b for all i.

Suppose b_A and b_B are members of B_1.

Then $\hat{P}_i(b_A) < P_i$ and $\hat{P}_i(b_B) < P_i$.

For $0 < \lambda < 1$: since $\hat{P}_i(b)$ is convex,

$$\hat{P}_i((1-\lambda)b_A + \lambda b_B) < (1-\lambda)\hat{P}_i(b_A) + \lambda\hat{P}_i(b_B) < P_i$$

Thus $b_A, b_B \in B_1 \Rightarrow (1-\lambda)b_A + \lambda b_B \in B_1$.

Given $b_A, b_B \in B_1$ this is true for all i. Thus B_1 is convex.

Proposition 2.1(b)

For small coupon rates there is a unique maximum.

Proof

We proceed as follows:

(a) Establish a means of moving continuously and smoothly from zero-coupon bonds, via a new parameter, γ ($\gamma = 0$ giving a zero-coupon bond market and $\gamma = 1$ giving us the true coupon-bond market).

(b) Establish that $B_2 = \mathcal{B} \backslash B_1$ is finite, and let B_3 be some finite expansion of B_2 and which contains B_2 for all values of $\gamma : 0 \leq \gamma \leq 1$.

(c) Within B_3, the shape of the log-likelihood function $g(b)$ can only deform slightly as we increase γ from 0. In particular, for small γ there will still only be one maximum.

(a) We start with a coupon-bond market with N stocks. For stock i we have an observed price P_i. The n_i future cashflows under this stock are $c_{i1},\ldots,c_{in_i}$ at times $t_{i1},\ldots,t_{in_i}$. The final payment is made up of nominal capital of 1 and a final coupon payment of $c_{in_i}-1$.

Now specify an arbitrary forward-rate curve $\tilde{f}(t,t+s)$: for example, the curve fitted to yesterday's prices. This has a corresponding set of zero-coupon bond prices $\tilde{Z}(s)$ for maturity in s years.

The actual coupon bond prices are P_i which we can write as

$$P_i = \left(\sum_{j=1}^{n_i} c_{ij}\tilde{Z}(t_{ij})\right)e^{\varepsilon_i}$$

for some $\varepsilon_1,\ldots,\varepsilon_N$.

We define the γ-coupon bond market as follows. For each stock we multiply the original coupon payments for each stock by γ but retain the full redemption payment. Thus the cashflows for stock i are $c_{i1}(\gamma),\ldots,c_{in_i}(\gamma)$ where

$$c_{ij}(\gamma) = \begin{cases} \gamma c_{ij}, & 1 \le j \le n_i - 1 \\ 1+\gamma(c_{in_i}-1), & j = n_i \end{cases}$$

The price of stock i in the hypothetical γ-coupon bond market is defined as

$$Q_i(\gamma) = \left(\sum_{j=1}^{n_i} c_{ij}(\gamma)\tilde{Z}(t_{ij})\right)e^{\varepsilon_i}$$

Clearly $\gamma = 0$ represents a zero-coupon bond market while $\gamma = 1$ returns us to the original coupon-bond market.

Proof of (b):

Let $R(t,t+s;b) = b^T d(s)/s$ be the spot rate at time t for maturity at time $t+s$.

Let t_0 be the shortest dated time to a coupon or a redemption payment.

Let $r_m = \inf\{R(t,t+t_0;b):\ |b| = 1, b \in \mathcal{B}\}$ and let $B_m = \{b:\ R(t,t+t_0;b) = r_m\}$.

Let $E(\lambda) = \{b:\ b \in \mathcal{B}, |b| = \lambda\}$, $\tilde{E}(\lambda) = \{b:\ b \in \mathcal{B}, |b| \le \lambda\}$.

$E(1)$ is a closed set so that r_m is attainable: that is, B_m is non-empty.

$r_m > 0$. Otherwise there exists b such that $R(t,t+t_0;b) = 0$. This implies that $f(t,t+s;b) = 0$ for $0 < s < t_0$. This can only be true if $b \equiv 0$.

Let $\tilde{\lambda}(\gamma) = \sup_i \frac{1}{r_m t_0} \log \frac{F_i(\gamma)}{Q_i(\gamma)}$ where $Q_i(\gamma)$ is the price of stock i in the γ-coupon bond market and $F_i(\gamma) = \sum_{i=1}^{n_i} c_{ij}(\gamma)$ is the total amount of the future payments under that stock.

Consider the γ-coupon bond market. For all $b \in \mathcal{B}$, $|b| \ge \tilde{\lambda}$, for all i, the theoretical price of stock i is:

$$
\begin{aligned}
\hat{Q}_i(\gamma)(b) &= \sum_{j=1}^{n_i} c_{ij}(\gamma)\exp(-R(t,t+t_{ij};b)t_{ij}) \\
&= \sum_{j=1}^{n_i} c_{ij}(\gamma)\exp(-R(t,t+t_{ij};b/|b|)t_{ij}|b|) \\
&\leq \sum_{j=1}^{n_i} c_{ij}(\gamma)\exp(-r_m t_0|b|) \\
&\leq F_i(\gamma)\exp(-r_m t_0\tilde{\lambda}) \\
&\leq Q_i(\gamma)
\end{aligned}
$$

Thus $B_2(\gamma) = \mathcal{B}\backslash B_1(\gamma) \subset \mathcal{B}\cap\tilde{E}\left(\tilde{\lambda}(\gamma)\right)$.

Choose some λ' such that $\sup_{0\leq\gamma\leq 1}\tilde{\lambda}(\gamma) < \lambda < \infty$.
Let $B_3 = \mathcal{B}\bigcap\tilde{E}(\lambda')$.

Write $g(b,\gamma)$ for the log-likelihood function for the γ-coupon bond market.
Clearly $g(b,\gamma)$ is infinitely differentiable.
For any γ $(0\leq\gamma\leq 1)$, for any i and for any $b\in\mathcal{B}\setminus B_3$, $\hat{Q}_i(\gamma)(b) < Q_i(\gamma)$.
Thus, by Lemma A.1, there can be no $b\in\mathcal{B}$ outside $\tilde{E}(\lambda')$ such that $\partial g/\partial b(b,\gamma) = 0$ at b.
Hence no local maxima can come sliding in from infinity as soon as $\gamma > 0$.
Let $\hat{b}\in B_3$ be the unique value of b such that

$$
\frac{\partial g}{\partial b}(\hat{b},0) = 0.
$$

Note that $-\partial^2 g/\partial b^2(b,0)$ is constant and strictly positive definite. (This leads to the unique maximum $\hat{b}$ mentioned above for the zero-coupon bond market.)
Hence there exists $\gamma_1 > 0$ such that $-\partial^2 g/\partial b^2(b,\gamma)$ is strictly positive definite for all $0<\gamma<\gamma_1$ and for all $b\in B_3$ (since g is C^∞ and B_3 is finite).
Thus for each γ $(0<\gamma<\gamma_1)$ there is a unique $\hat{b}(\gamma)\in B_3$ (and hence $\mathcal{B}$) which maximises $g(b,\gamma)$.

Corollary 2.2(b)

Furthermore, if a prior distribution for b is such that -1 times the log-prior density function is positive semi-definite then -1 times the log-posterior density function is also positive definite within B_3.