PENSIONMETRICS 2: STOCHASTIC PENSION PLAN DESIGN DURING THE DISTRIBUTION PHASE[1]

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Abstract

We consider the choices available to a defined contribution (DC) pension plan member at the time of retirement for conversion of his pension fund into a stream of income in retirement. In particular, we compare the purchase at retirement age from a life office of a conventional life annuity (that is, a bond-based investment) with distribution programmes that involve differing exposures to equities during retirement. The residual fund at the time of the plan member's death can either be bequested to his estate or, in exchange for the payment of survival credits while alive, reverts to the life office.

We find that the best programme depends (unsurprisingly) on the plan member's attitude to risk, with more risk-averse individuals preferring a greater investment in bonds or, in the limit, a conventional life annuity. For plan members with different degrees of risk aversion, we quantify the cost of holding a sub-optimal portfolio.

More surprisingly, we find:

- For the central values chosen for the bequest utility function, the plan member will favour an equity-linked annuity over an income drawdown policy.

- For a given level of relative risk aversion, the optimal choice of programme is robust with respect to:

  - the weight attached by the plan member to the provision of a bequest;
  - differences between life-office mortality experience and the plan member's subjective assessment of his own mortality prospects.

Finally when we allow for a one-off switch from an initial, equity-linked programme into a fixed, level annuity, we find that the optimal switching age is critically dependent on both the plan member's level of risk aversion and on recent equity performance.

Keywords: Stochastic pension plan design; defined contribution; discounted utility; life annuity; income drawdown; asset-allocation; optimal annuitisation age.

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1 Introduction

In many countries, such as the UK, the principal retirement income vehicle in defined contribution (DC) pension plans is the life annuity (which is basically a bond-based investment with longevity insurance). This is the *only* financial instrument that protects the annuitant from outliving his resources: no other distribution programme will *guarantee* to make payments at a specified rate for however long an individual lives. Indeed Yaari (1965) shows that in a world with a single safe asset and no bequest motive, it is optimal for an individual to use *all* his wealth to purchase *actuarial notes* (a generalised form of annuity).

In the UK, the accumulated pension fund must be used to buy a life annuity from a life office when the plan member reaches a certain age (namely, 75). The amount of the annuity will depend on the size of the fund, the long-term bond yield on the purchase date, the type of annuity (that is, whether the payments are fixed or variable[5]), the age, sex and (occasionally) state of health of the annuitant, and a margin to cover the life office's profit and costs of marketing, administration, and investment management.

Recently, however, annuities have come in for considerable criticism. They involve a range of risks for both the buyer and the seller. The plan member bears the risk of retiring when interest rates are low, so that the retirement annuity is permanently low. After he retires, he bears inflation risk if he purchases a level annuity: the risk of losses in the real value of his pension due to subsequent, unanticipated inflation. For their part, the life offices selling annuities face reinvestment risk (the risk of failing to match asset cash flows with expected liability outgo) and mortality risk (the risk that their pool of annuitants has lighter mortality than allowed for, e.g., because of an underestimate of mortality improvements).[6] But the strongest criticisms in recent years have centred on their perceived poor value and on their suitability as investment vehicles in the post-retirement period.

A number of papers have addressed the question of why (particularly in the USA) so few individuals voluntarily choose to annuitise their DC fund at the time of retirement.[7] Friedman & Warshawsky (1990) suggest that one reason for this is a high loading factor in quoted annuity rates compared with the actuarially fair rate for each individual. This loading factor arises from life-office expenses and profit, and also from the effect of self-selection. However, Mitchell *et al.* (1999), using US data, and Finkelstein & Poterba (2002), using UK data, show that the money's worth of annuities is much higher than is commonly supposed. Bernheim (1991) shows that many people have a strong bequest motive and this reduces their

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[5]It has become possible for life offices to sell index-linked annuities (which link payments to the variability in the retail price index) as a result of the introduction of long-dated index-linked government bonds that provide the essential matching assets for life offices. For more on index-linked bonds, see, for example, Anderson *et al.* (1996) and Cairns (1997).

[6]For a more detailed analysis of the problems facing annuity markets and some potential solutions to these problems, see Blake (1999).

[7]In the USA there is no mandatory requirement to purchase an annuity by any age. Brown & Warshawsky (2001) predict that the switch in employer sponsorship in the USA from defined benefit (DB) plans to DC plans will lead to even lower annuity purchases in the future.

desire to annuitise their wealth if they can avoid it. On the other hand, Brown (2001) finds that the bequest motive is not a significant factor in the annuitisation decision. Brown (2001) and Finkelstein & Poterba (2002) show that people in poor health try to avoid buying annuities if they can; these findings confirm the predictions from Brugiavini's (1993) theoretical model in which the health status of the individual follows a stochastic model the parameters of which are known only to the individual.

Many DC plans switch quite suddenly from an accumulation programme that is typically very heavily weighted in equities to a distribution programme that is almost entirely invested in bonds (see, for example, Blake, Cairns & Dowd, 2001, and Cairns, Blake & Dowd, 2000). Recently commentators have begun to question whether it is sensible to have such a substantial bond-based investment over such a long period. After all, the substantial improvements in longevity over the last century mean that retirees can typically expect to live for 15 years or more, and there are likely to be further improvements in the future. The issue of whether to rely so heavily on bond investments has also become more pressing in many countries following the substantial falls in bond yields over the last decade.[8]

The perceived poor value of traditional annuities has motivated a search for new investment-linked retirement-income programmes that involve the provision of retirement income from a fund with a substantial equity component. The attraction of such vehicles is obvious: there are very few historical periods where equities do not outperform bonds over long horizons.[9] Nevertheless, equity prices tend to be much more volatile than bond prices, so the higher expected returns from equities is gained at the cost of greater risk. Further, some of these alternatives do not by themselves hedge mortality risk and so do not satisfy the basic requirement of a pension plan to provide an income in retirement for however long the plan member lives. For example, any income drawdown programme which draws an annual fixed income from a fund heavily invested in equities has a strictly positive probability of ruin: that is, the fund runs down to zero before the plan member dies. This particular combination of longevity and equity risk can be mitigated in two ways. First, we can require that the amount drawn from the fund be linked to the size of the fund at each point in time. Second, we can impose the requirement that the plan member annuitises by a certain age, as happens, for example, in the UK. One of the aims of this study is to assess the impact of such a regulation.

In this paper we compare in terms of discounted utilities three different distribution programmes[10] for a male DC plan member retiring at age 65:

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[8]For example, in the UK long-bond yields reached a forty-year low in 1999 pushing up annuity prices to corresponding highs.

[9]Siegel (1997) shows that US equities generated higher average returns than US Treasury bonds and bills in 97% of all 30-year investment horizons since 1802. CSFB (2000) shows that similar results hold for the UK.

[10]In an earlier version of this paper (Blake, Cairns and Dowd, 2000), we analysed a larger range of distribution programmes. These included: (a) A programme in which income is fixed. The result is stability of income coupled with the risk of ruin before death. With most forms of plan member utility function analysed the possibility of ruin results in very low discounted expected utility (in some cases minus infinity) making such programmes extremely unattractive. (b) Variants on the equity-linked annuity and income-drawdown programmes involving the use of derivatives to limit

- *Purchased life annuity* (PLA) – The plan member transfers his retirement fund immediately on retirement at age 65 to a life office in return for a level pension. No bequest is payable at the time of death of the plan member. This is the benchmark programme against which the programmes below are compared.

- *Equity-linked annuity* (ELA)[11] with a level, life annuity purchased at age 75 – The assets are held in a managed fund containing both equities and bonds and the plan member is protected from running out of money before age 75, since the annuity income will fall in line with any falls in the value of the fund. We consider five different levels of equity exposure in the managed fund: 0%, 25%, 50%, 75% and 100%. At the start of each year, the life office pays an actuarially fair *survival credit* to the plan member if he is still alive. The survival credit accounts for anticipated mortality over the coming year and acts to offset *mortality drag*, the extra return built into an annuity in comparison with other investments that arises from the mortality risk-sharing implicit in an annuity: those who die early on create a profit for the life office which is shared amongst those annuitants who live longer than average. The mortality drag in any year is equal to the expected proportion of the surviving group of annuitants who die in that year, and is therefore increasing in age[12]. In return for these payments the residual fund *reverts* to the life office at the time of the death of the plan member.

- *Equity-linked income-drawdown* (ELID) with a level, life annuity purchased at age 75 – Such programmes, in principle, permit the plan member to draw any level of income at any point in time resulting in the possibility that the fund will become exhaused before he dies. In some countries, where pensions savings have benefited from tax breaks during the accumulation phase, the level of income drawn down is constrained by law and this prevents the fund from being completely used up. The programme proposed here satisfies this regulatory requirement. The residual fund is paid as a *bequest* to the plan member's estate if he dies before age 75.

We do not consider explicitly in this paper income deriving from other sources such as personal savings. In some countries, pension income accounts for the bulk of the total resources available to most people after retirement: for example, state and private pensions in the UK constituted 86% of the total income of a typical retired person in 1997-98.[13] In other countries, such as the USA, personal savings are quite significant for large segments of the population. Many people also have other free

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downside risk in the fund. Such programmes were found to give similar results to, but slightly worse than, funds excluding derivative investments with a similar annual standard deviation in returns. (c) A programme which purchased at 65 a deferred annuity from age 75 and consumed the remaining fund entirely between ages 65 and 75. None of these alternatives proved to be as effective as those discussed in detail in this paper. In some cases the proposed programme was significantly worse.

[11]The ELA is known in the USA as a variable annuity.

[12]For more details, see Blake (1999).

[13]Department of Social Security (2000, Table 1).

assets in the form of a residential home. For such people the issues described above are reversed: should they retain their free assets or should they use some or all of them to purchase an annuity? A number of authors have addressed this question.

Milevsky & Robinson (2000) and Albrecht & Maurer (2001) consider the case where the free assets are used to follow a risky investment strategy with no annuitisation and where a level pension equal to the equivalent annuity is drawn. They both find that the probability of ruin can be significant. Such analyses are important for educating pensioners about the risks associated with certain strategies.

Merton (1983) looks at optimal consumption and investment with a power utility function when there is a single safe asset, a single risky asset and uncertain future lifetimes. Under certain circumstances (no bequest motive and actuarially fair survival credits), optimal consumption and investment profiles are identical to the case in which there is no mortality risk (Merton, 1971). Merton finds that the optimal strategy maintains fixed proportions in the two assets through continuous rebalancing, while the income stream is equal to a predetermined rate (dependent on future mortality expectations) multiplied by the present fund size. Consequently the optimal level of income fluctuates with the fortunes of the underlying investments. Charaput & Milevsky (2002) derive a similar result although they do not acknowledge their result as being a special case of Merton's (1983) analysis with restrictions on both the asset strategy and the form of the mortality function.

In the present paper, we treat any non-pensions-related personal savings straightforwardly by adding them to the pension fund at retirement.[14] In terms of the residential home, we treat this as a fixed asset which becomes a bequest on death. This bequest is the same under all of the programmes described below and so has no differential effect. The value function defined in Section 4 is assumed, implicitly, to have incorporated this fixed bequest. Finally we assume for simplicity that the pensioner consumes all his pension income each year.[15]

The layout of this paper is as follows. Section 2 explains the stochastic framework underlying our analysis, Section 3 analyses the three key pension distribution programmes available to a DC plan member, Section 4 presents and discusses numerical results, and Section 5 concludes.

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[14]Any differential tax treatment of pension and non-pension savings inevitably complicates matters. However, we ignore this here as it would divert attention from the qualitative observations made below.

[15]A simple alternative that would not change the results would be to assume that any unconsumed pension is reinvested in the same assets as the pension fund.

2 Stochastic assumptions

2.1 Asset returns

We assume that there are two assets available for investment: risk-free bonds and equity. The bond fund, $M(t)$, grows at the continuously compounded constant risk-free rate of r per annum, so that at time t, $M(t) = M(0)\exp(rt)$, where $M(0)$ is the opening balance. Equity units, $S(t)$, constitute a risky investment which follows the log-normal or geometric Brownian motion model, so that $S(t) = S(0)\exp(\mu t + \sigma Z(t))$, where $S(0)$ is the opening balance and $Z(t)$ is standard Brownian motion. We work on an annual basis throughout. It follows that annual returns on equities are independent and identically distributed log-normal random variables with mean $\exp(\mu + \frac{1}{2}\sigma^2)$ and variance $\exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]$.

For simplicity, we assume that the pension is drawn at the start of each year and that pension plan assets are rebalanced annually to maintain predetermined proportions in each asset. We assume that the annual life office charge on equity investment for all distribution programmes is constant at 1% of fund value, implying a reduction in yield of 1%.[16]

In our simulations, we used the following parameters (net of expenses):

Parameter	Symbol	Nominal value (proportion)
Risk-free return	r	0.0296
Expected log return on equities	μ	0.0746
Standard deviation of the log return on equities	σ	0.244

This implies that the expected gross return on equities is 1.11 and the standard deviation of the total return is 0.275. These parameter values are consistent with historical returns on UK Treasury bills and equities over the last half century and include the 1% reduction in yield on equity returns.

2.2 Mortality and financial functions

We also make use of the following mortality and financial functions:

- q_x and p_x are the year-on-year mortality and survival probabilities respectively.[17] The values of these probabilities are the same as those used in the most appropriate UK mortality table for compulsory-purchase, male annuitants (that is, PMA92Base: see McCutcheon *et al.* (1998, 1999)).

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[16]Some of the distribution programmes that we consider, such as income drawdown, can be very expensive with charges considerably in excess of 1% p.a. Nevertheless, to preserve comparability, we assume a 1% charge for income drawdown as well. See Blake (1999) or Appendix A of Blake, Cairns & Dowd (2000)

[17]Thus, p_x is the probability that an individual aged x survives for one year, and $q_x = 1 - p_x$ is the corresponding probability of death.

- $_t p_x = p_x \times \ldots \times p_{x+t-1}$ is the probability that the pensioner survives to age $x + t$, given that he is alive at age x.

- $_t|q_x = {}_t p_x - {}_{t+1} p_x$ is the probability that the pensioner will die between ages $x + t$ and $x + t + 1$, given that he is alive at age x.

- $\ddot{a}_y = \sum_{t=0}^{\infty} {}_t p_y e^{-rt}$ is the fair price at age y of a level single-life life annuity of £1 per annum, payable annually in advance.

We assume in this study that mortality rates will not improve over time.[18]

3 Three distribution programmes

Our analysis is based on a typical 65-year old male who is assumed to have accumulated a personal pension fund on his retirement date (denoted $t = 0$ below) of $F(0) = £100,000$ and is about to retire. Our plan member has to choose between the following three distribution programmes.

3.1 Programme specifications

3.1.1 Programme 1: Purchased life annuity (PLA)

In Programme 1, $F(0)$ is used immediately to purchase a level life annuity at a price of $\ddot{a}_x$ per £1 of pension (where the rate of interest applied to $\ddot{a}_x$ is the risk-free rate, r). The pension is therefore $P(t) = P_B = F(0)/\ddot{a}_x$ for $t = 0, 1, 2, \ldots$ and is payable until death. No bequest is payable, but the plan member receives implicit survival credits instead.

3.1.2 Programme 2: Equity-linked annuity (ELA)

Programme 2 is designed to benefit from equity investment but also adjusts the pension paid to remove the possibility of running out of funds before age 75. Under this programme, the pension, $P(t)$, is adjusted each year to reflect both the fund size available at the beginning of the year and changing mortality rates as the plan member ages. The procedure for calculating each year's pension payment ensures that both $P(t)$ and $F(t)$ are always positive. The programme also allows for different degrees of equity weighting (ω).

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[18]For recent work on stochastic mortality improvements, see Milevsky & Promislow (2001), Yang (2001) and Wilkie *et al.* (2003).

Hence, for $t = 0, 1, \ldots, 9$ and $\delta = 0$,

$$P(t) = \frac{F(t)}{\ddot{a}_{x+t}} \tag{3.1.1}$$

$$B(t) = (1 - \delta)\frac{q_{x+t}}{p_{x+t}}(F(t) - P(t)) \tag{3.1.2}$$

$$F(t+1) = \left(\omega\frac{S(t+1)}{S(t)} + (1 - \omega)e^r\right)(F(t) - P(t) + B(t)) \tag{3.1.3}$$

$$D(t+1) = \delta F(t+1) \tag{3.1.4}$$

$$\text{where} \quad \delta = \begin{cases} 0 & \text{if survival credits are payable (ELA)} \\ 1 & \text{if bequests are payable (ELID).} \end{cases} \tag{3.1.5}$$

$P(t)$ is the pension payable at time t. $B(t)$ is the actuarially fair *survival credit* paid into the plan member's fund at the start of every year if the plan member is still alive (that is, the fund is increased from $F(t)$ to $F(t) + B(t)$ at time t if the plan member is still alive at that time). $F(t + 1)$ is the residual fund which reverts to the life office if the plan member dies during the year. $\delta = 0$ indicates that survival credits are payable, and $\delta = 1$ indicates that a bequest is payable.

At time $t = 10$ (age 75), if the plan member is still alive, the residual fund, $F(10)$, is used to purchase a level, single-life annuity. Thus, for $t = 10, 11, \ldots$, $P(t) = F(10)/\ddot{a}_{x+10}(10)$. No bequests are payable after time 10, but the plan member continues to receive the survival credits implicit in the annuity.

Note that when $\omega = 0$ we get $P(t) = P_B$ for all t, that is, the PLA and ELA programmes are identical in this case.

3.1.3   Programme 3: Equity-linked income-drawdown (ELID)

The above set of equations can also be used to describe Programme 3 if we set $\delta = 1$. $D(t + 1)$ represents the bequest payable to his estate at $t + 1$ if the plan member dies between times t and $t + 1$.

This programme will provide a level pension if the return on the assets is equal to the risk-free rate plus the mortality drag: that is, if:

$$\omega\frac{S(t+1)}{S(t)} + (1 - \omega)e^r = \frac{e^r}{p_{x+t}}. \tag{3.1.6}$$

Milevsky (1998) considered a similar framework and, by taking the expectation of the random component $S(t + 1)/S(t)$, derived a decision rule for switching from the ELID programme to the PLA. This rule indicates that the ELID programme continues only so long as the expected excess return on equities exceeds the mortality drag. This requirement on returns in equation (3.1.6) to achieve a level pension lies in contrast with Programme 2. In Programme 2 risky assets need only generate a return equal to the risk-free rate (i.e., $S(t+1)/S(t) = e^r$) to provide a level pension. This is because the survival credits already embodied in the ELA have the effect of enhancing the return on the fund for each year that the plan member survives. If the actual return on the fund exceeds this threshold, the pension will grow over time. However, this lower return threshold comes at the expense of no bequests.

4 Numerical results

4.1 The value function

To compare these programmes, we have used the log-normal distribution of $S(t + 1)/S(t)$ and the translated log-normal distribution of $(1 - \omega)e^r + \omega S(t + 1)/S(t)$ to calculate the plan member's discounted lifetime utility. This notion of utility captures the plan member's welfare throughout retirement and is similar to that employed by Merton (1990) and others.

We measure value relative to the standard life annuity which pays a fixed amount $P_B = F(0)/\ddot{a}_{65}$ per annum for life with no bequest (i.e., Programme 1: PLA). We refer to this annuity as the *benchmark* pension. For the given parameters and mortality rates and for an initial fund of £100,000, $P_B = £7551.53$ p.a. For Programmes 2 and 3 we consider fixed 0%, 25%, 50%, 75% and 100% equity investment. We recognise that strategies based on dynamic optimisation might give superior results, but they might also be difficult to implement in practice. Instead we choose to restrict ourselves to a set of programmes which are very straightforward to implement and which can be understood easily by plan members.

Now let K be the *curtate future lifetime*[19] of the plan member from age 65. The plan member's value function or expected discounted utility takes the form:

$$V(s, f) = E\left[\sum_{t=s}^{K} e^{-\beta t} J_1(P(t)) + k_2 e^{-\beta(K+1)} J_2(D(K+1)) \,\middle|\, F(s) = f, \text{ alive at } s\right]$$
(4.1.1)

$$\text{where} \quad J_1(P(t)) = h_1(\gamma_1)\left(\frac{P(t)}{P_B}\right)^{\gamma_1}$$
(4.1.2)

$$J_2(D(t)) = h_2(\gamma_2)\left[\left(\frac{D(t) + d_2}{d_2}\right)^{\gamma_2} - 1\right]$$
(4.1.3)

$$\text{with} \quad d_2 > 0.$$

$J_1(P(t))$ comes from the constant relative risk aversion (CRRA) class of utility functions (that is, power and log utility functions). The plan member's relative risk aversion (RRA) parameter is $1 - \gamma_1$. $J_2(D(t))$ comes from the hyperbolic absolute risk aversion (HARA) class (which includes the CRRA class as a special case). The parameter β measures the plan member's subjective rate of time preference and k_2 is used to specify the appropriate balance between the desire for income and the desire to make a bequest.

The functions $h_1(\gamma_1)$ and $h_2(\gamma_2)$ have a considerable impact on the analysis, unless $k_2 = 0$, in which case the magnitudes and the forms of $h_1(\gamma_1)$ and $h_2(\gamma_2)$ are irrelevant (since they have no impact on the outcome of the exercise). An essential requirement for each function is that it takes the same sign as its argument. A conventional parameterisation of J_1 and J_2 would employ $h_1(\gamma_1) = 1/\gamma_1$ and

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[19]That is, the random future lifetime of the plan member rounded down to the previous integer.

$h_2(\gamma_2) = 1/\gamma_2$. We experimented with these forms in preliminary work, but found that they resulted in strongly dichotomous preferences. Plan members with a low RRA had a very strong preference for the ELA programme (offering no bequests), while plan members with a high RRA had a very strong preference for the ELID programme (offering bequests). At each extreme, any alternative programme, apart from the one chosen, proved very costly to the plan member in terms of lower expected utility. However, we do not believe that real world behaviour is so extreme: we would expect to see some individuals with low RRA still wishing to make a bequest and *vice versa*.

We believe it is important to choose forms for $h_1(\gamma_1)$ and $h_2(\gamma_2)$ which avoid such extreme swings in preference over the range of RRA parameter values. We have used one form which achieves this, namely:

$$h_1(\gamma_1) = \frac{1}{1 - d_1^{\gamma_1}} \qquad (4.1.4)$$

$$\text{and} \quad h_2(\gamma_2) = \frac{1}{\left(\frac{F(0) + d_2}{d_2}\right)^{\gamma_2} - 1}. \qquad (4.1.5)$$

The parameter, d_1, can be freely determined within the range $0 < d_1 < 1$ for $h_1(\gamma_1)$ to take the correct sign. The bequest utility, $J_2(D(t))$ tends to minus infinity as $D(t)$ tends to $-d_2$. The parameter d_2 can be interpreted as the value of the plan-member's non-pension assets such as his house. The plan member could in theory borrow against the value of these other assets with the debt on the pension fund being repaid from these assets at the time of death, although in practice we do not permit this.

It follows that we can make the following remarks about the values of $J_1(\cdot)$ and $J_2(\cdot)$:

- $J_1(d_1 P_B) = d_1^{\gamma_1}/(1 - d_1^{\gamma_1}) = J_1(P_B) - 1$. Thus, $J_1(P)$ increases in absolute terms by 1 when P increases from $d_1 P_B$ to P_B.

- $J_2(0) = 0$. This is a consequence of our requirement that $d_2 > 0$.[20] Imposing $J_2(0) = 0$ means that when we value the benchmark PLA programme, the $J_2(\cdot)$ component of the discounted utility is unaffected by the timing of death. Additionally, any strictly positive bequest has a strictly positive impact on the discounted utility function relative to the benchmark.

- $J_2(F(0)) = 1$. It follows that $J_2(\cdot)$ increases by 1 as the size of the bequest changes from 0 to $F(0)$ (i.e., if the member died on the same day that the programme started).

The value of k_2 (and, to a lesser extent, γ_2) will reflect the family characteristics of the plan member: for example, a married man with young children is likely to

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[20]Suppose, in contrast, we used $d_2 = 0$ in combination with $\gamma_2 < 0$. This implies that $J_2(0) = -\infty$. However, we know that many plan members do choose to annuitise their entire liquid assets thereby leaving a bequest of zero. This is inconsistent with the assumption that $J_2(0)$ might be $-\infty$.

have a greater bequest motive and hence a higher value of k_2 than a single man with no children.

So far as the authors are aware, no standard definition exists for the relative risk aversion parameter attached to the value of a series of cashflows rather than to a single cashflow. We therefore propose the following definition. Consider the PLA programme where $P(t) = P_B$ is constant and no bequest is payable (that is, $P_B = F(0)/\ddot{a}_{65}$). Then the value function is a function of P_B only[21]:

$$\tilde{V}(P_B) = \tilde{V}(F(0)/\ddot{a}_{65}) \equiv V(0, F(0)) = E\left[\sum_{t=0}^{K} e^{-\beta t} J_1(P_B) \,\middle|\, F(0),\ \text{alive at time } 0\right].$$

We define the relative risk aversion parameter to be $-P_B.\tilde{V}''(P_B)/\tilde{V}'(P_B)$, where primes indicate derivatives. For the value function in Equation (4.1.1) the relative risk aversion parameter is therefore $1 - \gamma_1$ for all P_B. Where a bequest is payable we still take $1 - \gamma_1$ as the RRA parameter while acknowledging that this reflects only the pre-death risks.

The optimal programme maximises the value function $V(0, F(0))$.

4.2 General comments on the results

In the following experiments we have assumed that $\gamma_1 = \gamma_2 = \gamma$, and, as central values, that $k_2 = 5$, $d_1 = 0.75$ and $d_2 = 10000$.[22] Mortality is assumed to be independent of the investment scenario. For the moment we also assume that the true mortality model for (and known to) the plan member is the same as that used by the life office in calculating annuity rates. We refer to this as *standard mortality*. We evaluated the value function $V(0, F(0))$ across a range of values for the RRA parameter, $1 - \gamma$, varying from 0.25 to 25. This range embraces both very risk-averse and very risk-tolerant behaviour by plan members and is consistent with values found in other studies by Blake (1996) and Brown (2001). In addition, we fixed the value of β at $\log 1.05$.[23]

Numerical results are presented in Table 4.2.1 for the discounted utilities of the different programmes when the RRA coefficient is 3.96.[24] This value falls in the middle of the likely range according to some research (for example, Brown (2001) covering US pension plan members), although other research (for example, Blake (1996), covering UK investors) suggests that a value of 3.96 is relatively low.

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[21]The bequest utility function is not included here since the PLA programme means that $D(k + 1) = 0$ for all k_2, and $J_2(0) = 0$.

[22]The sensitivity of our results to these particular parameter values is assessed below.

[23]Blake (2000) estimates the marginal rate of time preference of a typical UK household to be about 3%. This is consistent with the use here of $\beta = \log 1.05$ in combination with an assumed rate of inflation of 2% (although an inflation assumption is not required for the present analysis).

[24]Technically, the PLA and ELA 0% programmes differ in terms of how they are administered. However, from the plan member's point of view they deliver identical pensions.

Programme	Equity %	$V(0, F(0))$
PLA	0	-8.82
ELA	0	-8.82
ELA	**25**	**-6.96**
ELA	50	-7.35
ELA	75	-9.99
ELA	100	-19.99
ELID	0	-11.98
ELID	25	-9.42
ELID	50	-10.10
ELID	75	-14.79
ELID	100	-33.11

Table 4.2.1: Expected discounted utilities for different distribution programmes when the plan member's relative risk aversion is 3.96. Parameter values: $k_2 = 5$, $d_1 = 0.75$, $d_2 = 10000$, $\beta = 0.0488$, standard mortality, annuitisation at age 75.

The values in Table 4.2.1 were calculated using a backwards recursion:

$$
\begin{aligned}
V(s, F(s)) \;=\; & e^{-\beta s} J_1(P(s)) + p_{65+s} E[V(s+1, F(s+1))|F(s), \text{ alive at } x+s+1] \\
& + q_{65+s} e^{-\beta(s+1)} E[\delta F(s+1)|F(s), \text{ alive at } x+s].
\end{aligned}
\tag{4.2.1}
$$

The optimal strategy for the plan member in this case is to select the ELA programme with 25% in equities.

Figure 4.2.1 shows the plot of the differences between the value functions for each equity-linked programme and the benchmark PLA programme: $V_P(0, F(0)) - V_B(0, F(0))$ over a range of values for the RRA parameter $1 - \gamma$. Panel (a) shows the ELA programme, and panel (b) the ELID programme. The top bold line indicates the best programme for a given degree of relative risk aversion.

Figure 4.2.1 reveals the following:

- The best programme is an equity-linked annuity with 100% equities if RRA is less than about 1.25. The ELA programme remains optimal for RRA parameters up to about 10, provided the equity weighting is gradually lowered from 100% to 0% as RRA increases. For more risk averse plan members, the PLA is optimal.

- For $k_2 = 5$, all the optimal policies use survival credits rather than bequests. We investigate below how robust this result is to changes in the value of k_2.

The optimal equity proportions for different RRAs are shown in Figure 4.2.2. The solid line indicates the best portfolio mix in the unrestricted range 0% to 100% as a function of RRA. With $k_2 = 5$, the optimal distribution programme for all
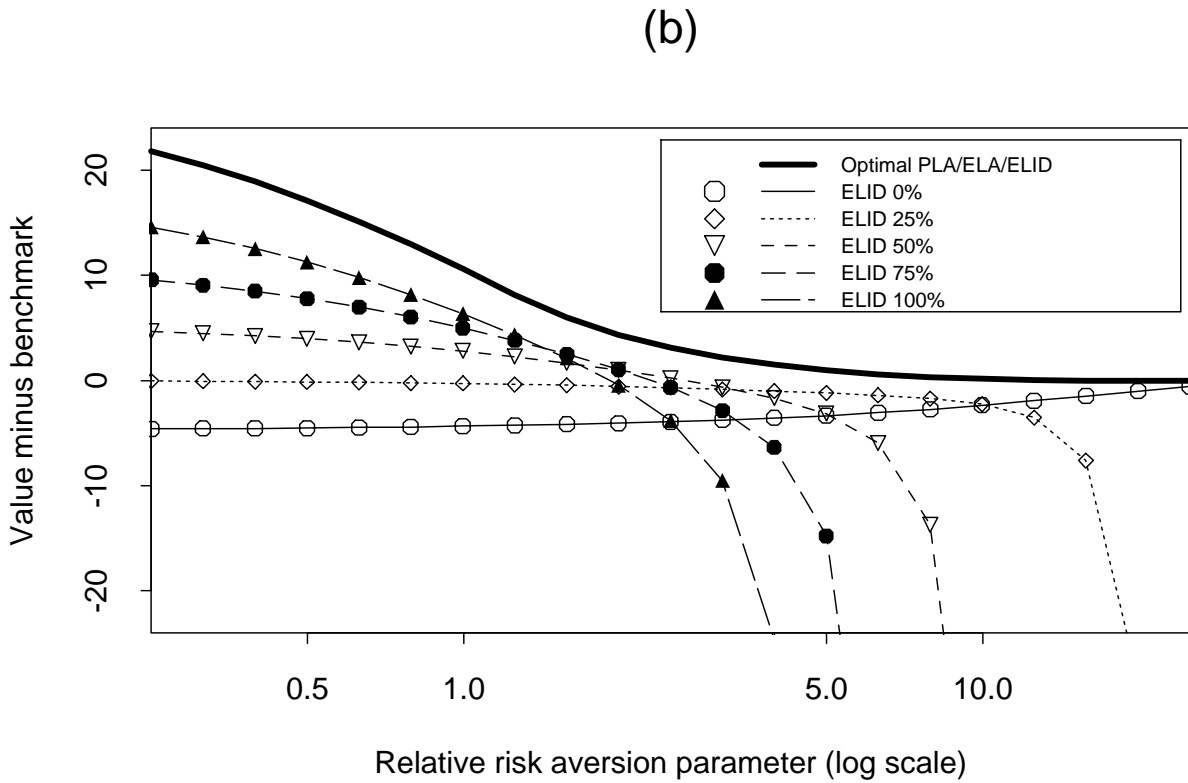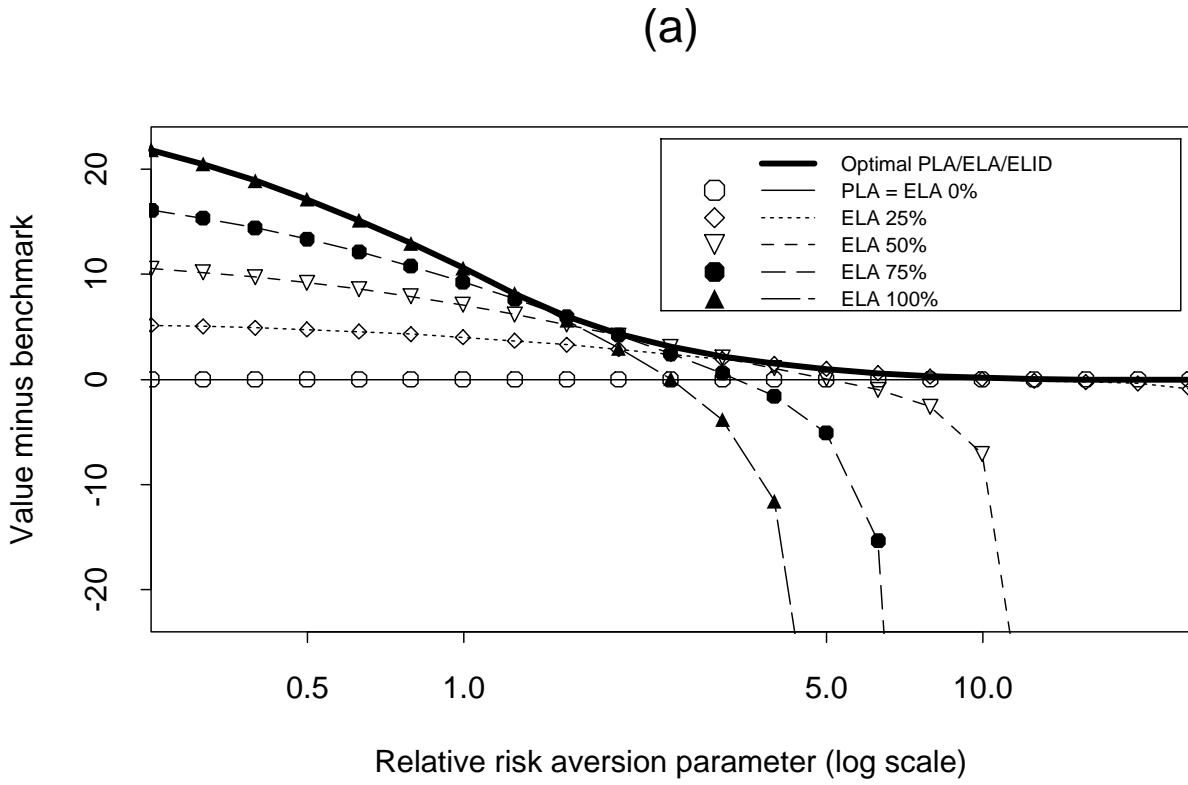
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Figure 4.2.1: Expected discounted utilities for different equity-linked distribution programmes (relative to the purchased life annuity) as a function of the relative risk aversion parameter: (a) annuity programmes paying survival credits, (b) income drawdown programmes paying bequests. Standard mortality, $k_2 = 5$, $d_2 = 10000$, $\beta = 0.0488$.
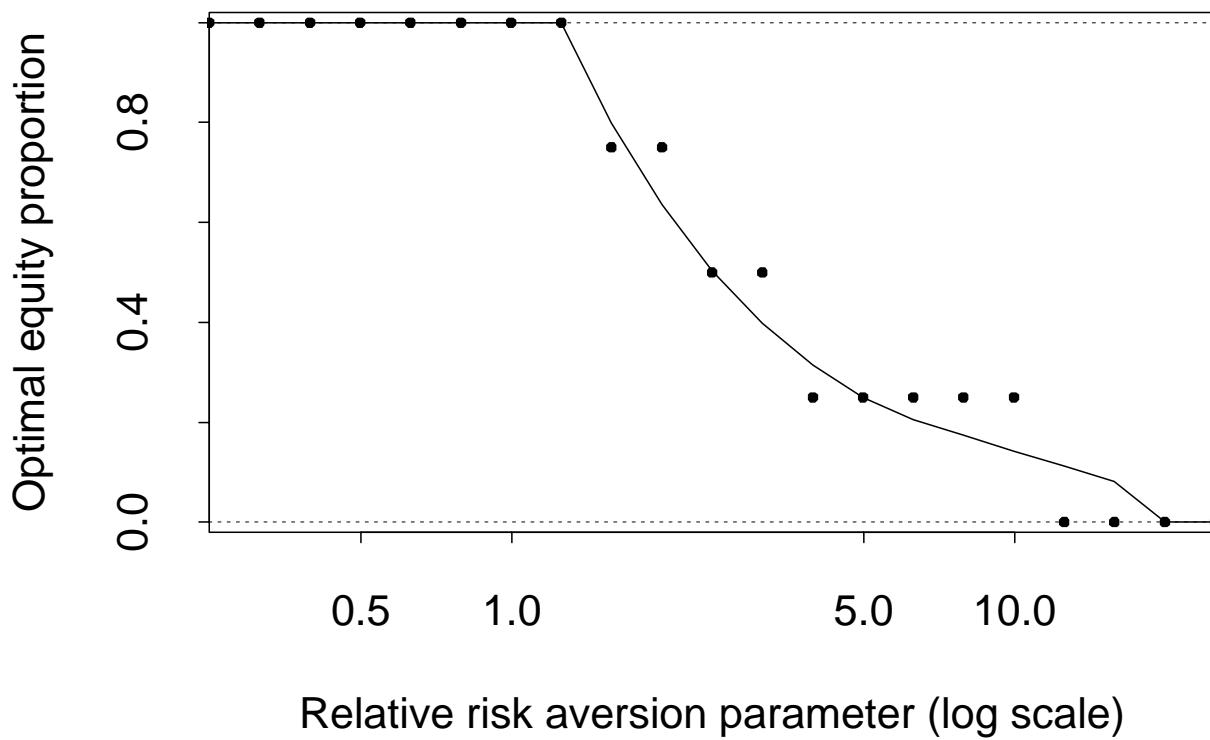
Figure 4.2.2: Optimal equity proportions in equity-linked distribution programmes as a function of the relative risk aversion parameter. Solid line: optimal relationship when the equity proportion can take any value between 0% and 100%. Dots: relationship when the equity proportions are constrained to 0%, 25%, 50%, 75% or 100%. Standard mortality, $k_2 = 5$, $d_2 = 10000$, $\beta = 0.0488$.

levels of RRA is the ELA rather than the ELID programme. The dots indicate the best portfolio mix when the plan member is restricted to choosing from one of the five ELA programmes with 0%, 25%, 50%, 75% and 100% equities. For example, the best available programme for a plan member with a RRA of 10, who would otherwise choose an equity proportion of 15% if free to do so, is the ELA with an equity proportion of 25%. The best available programme for a plan member with a slightly higher RRA is the one with a 0% equity proportion.

These results are, of course, not surprising. What is novel in this study are the findings that some individuals definitely prefer 100% equities and others prefer the purchased life annuity (PLA), and also how rapidly (in terms of increasing RRA) plan members switch between these choices.

Table 4.2.2 shows the values of RRA parameters for UK investors estimated by Blake (1996) in different financial wealth ranges using data on financial asset portfolios collected in the UK Financial Resources Survey 1991-92. Each RRA is a point estimate derived from grouped data. Within each wealth group, therefore, there will be a spread of values for the RRA. Nevertheless, these UK estimates for the RRA are generally higher than those found by Brown (2001) for the USA.

	Financial wealth range (£, 1991-92 values)					
	50-454	455-3,499	3,500-7,904	7,905-14,999	15,000-36,799	36,800+
Percentage of total population in range	25	25	15	5	4	1
Relative risk aversion parameter	47.60	34.33	28.82	17.90	12.44	7.88

Table 4.2.2: Average RRA parameters in different wealth ranges (Blake, 1996).

We can compare Blake's range of RRA parameters with the optimal choices illustrated in Figures 4.2.1 and 4.2.2. This suggests that the number of plan members choosing an equity-linked annuity could be quite small, only about 5% of the total.[25]

For any specific RRA, Figure 4.2.1 can be used to rank the programmes in order of preference, and Table 4.2.1 does this for a relative risk aversion parameter of $1 - \gamma = 3.96$. However, the differences in expected discounted utilities give us no feel for how much worse, say, the ELID with 75% equities programme is compared with the optimal ELA with 25% equities programme. It would therefore be useful to know the answer to the following question: how much extra cash would a plan member need at time 0 in order for the ELID programme with 75% equities to have the same expected discounted utility as the optimal ELA programme with $F(0) = £100,000$ and 25% equities? This question is answered in Figure 4.2.3: we would require an extra 25% approximately (Figure 4.2.3(b), ELID programme with 75% equities) or £25,000 to match the optimal ELA programme for a plan member with a RRA parameter of 3.96.

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[25]Of course, the percentage of total pension wealth held in equities during the decumulation
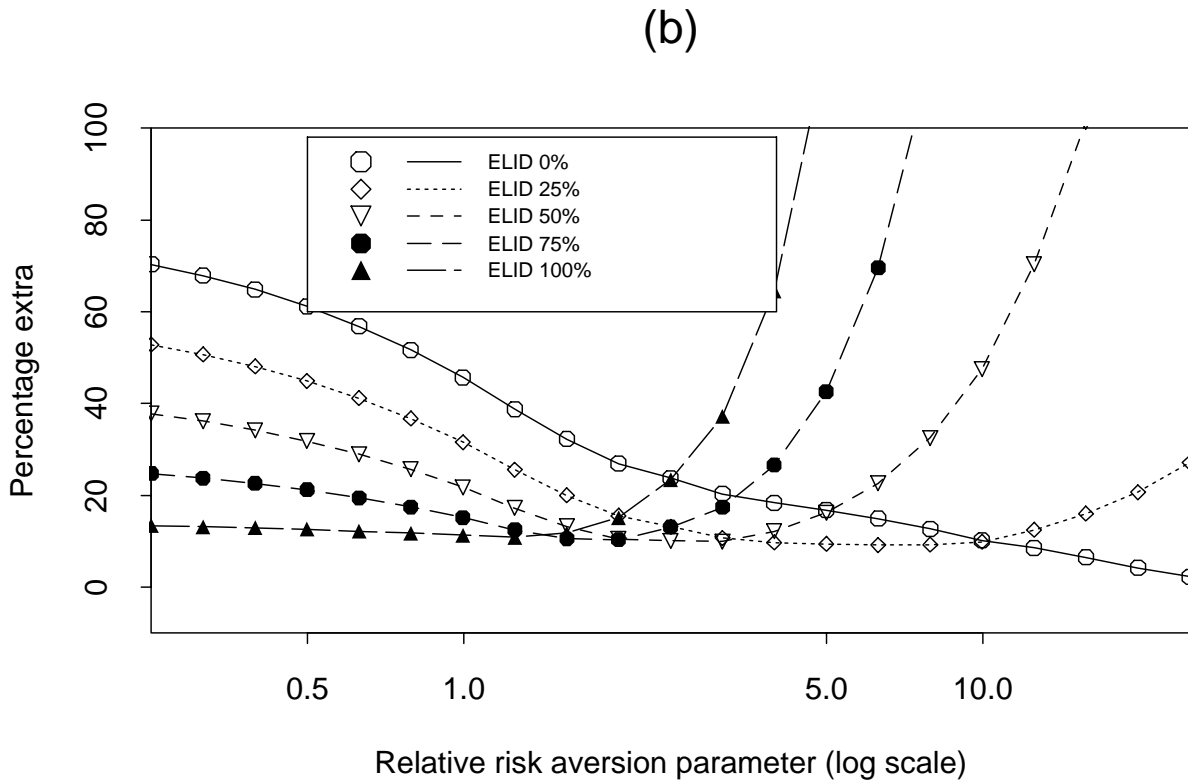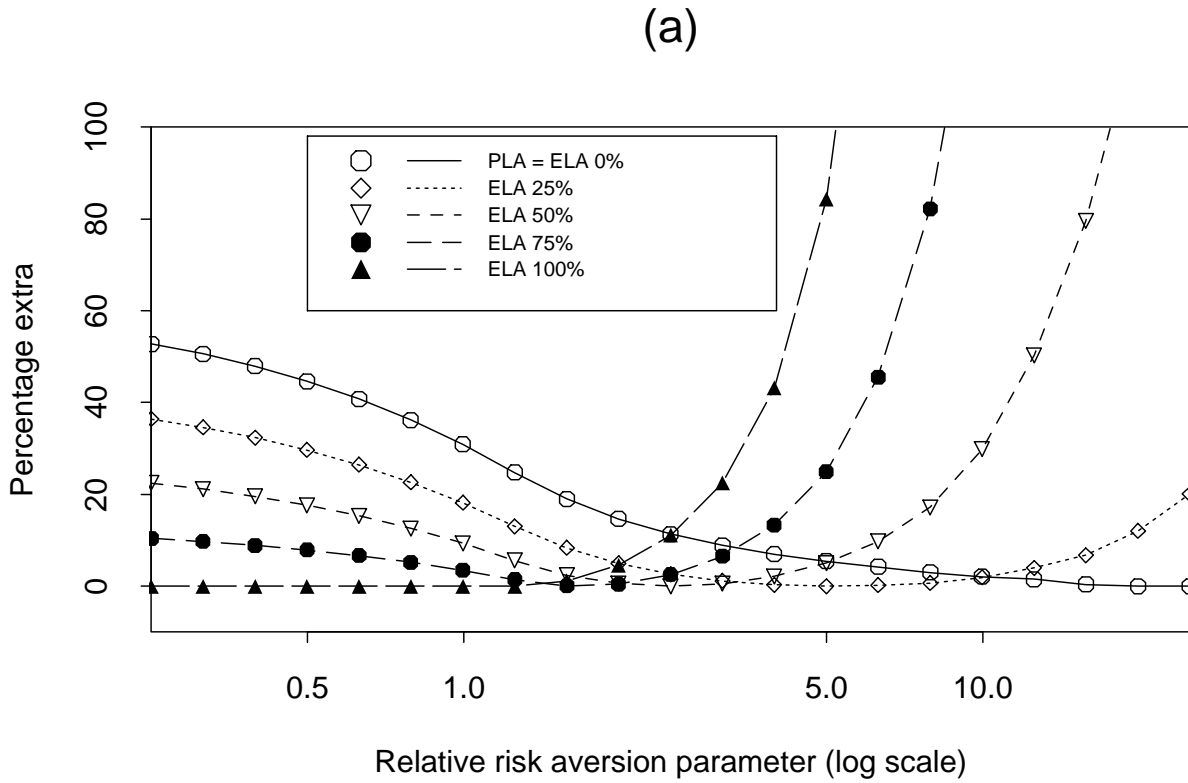
Figure 4.2.3: Extra cash required at time 0 for different distribution programmes to match the expected discounted utility of the optimal ELA programme as a function of the relative risk aversion parameter: (a) annuity (PLA and ELA) programmes paying survival credits versus the optimal ELA programme, (b) income drawdown programmes paying bequests versus the optimal ELA programme. Standard mortality, $k_2 = 5$, $d_2 = 10000$, $\beta = 0.0488$.

Figure 4.2.3 is much more informative than Figure 4.2.1 because it gives a much better impression of the relative quality of each programme. For example, we can see that for plan members with a relatively strong appetite for risk (low values of RRA of around 0.25), some programmes would require as much as 50% to 70% extra cash (Programmes 1 (PLA), 2 (ELA with 0% equities) and 3 (ELID with 0% equities)) to match their preferred Programme 2 (ELA with 100% equities). This finding should not be surprising: programmes with lower proportions of equities are evaluated unfavourably by risk-loving plan members because they have lower expected returns (important when RRA is low) as well as lower risk (not important to a plan member with a low RRA).

At the other end of the spectrum, Figure 4.2.3 shows that a very risk-averse plan member must opt for Programme 1 (PLA) (or equivalently the ELA programme with 0% equities), as all other programmes require significant additional amounts of cash at time 0 to give them the same expected discounted utility: indeed programmes with more than 25% in equities go off the scale due to the plan member's extreme aversion to risk.

Figure 4.2.2 illustrated the difference between the optimal programme with an unrestricted optimal equity proportion and the optimal programme when the equity proportion is restricted to a multiple of 25%, as a function of RRA. Figure 4.2.3(a) plots results for the PLA and restricted ELA programmes relative to the optimal unrestricted programme. It can be seen that the restriction costs very little in terms of extra cash (about 2%) required at time 0. This indicates that it is not unreasonable for a pension plan provider to simplify choices for plan members by also offering a restricted range of programmes. Figure 4.2.3(b) shows that, with $k_2 = 5$, the extra cash needed to compensate a plan member for taking a sub-optimal ELID programme paying bequests is around 10% across all levels of RRA.[26]

In summary, we note that the plan member has two decisions to take: the programme type (PLA, ELA or ELID); and the equity proportion. Overall we observe from our analysis that of the two decisions the choice of equity proportion is by far and away the most important. A poor choice can be very costly in terms of lower expected discounted utility.

4.3 Importance of bequests

We investigated the relative importance of the bequest by varying the parameter k_2 in Equation (4.1.3). Comparing Figure 4.2.3(b) ($k_2 = 5$) with Figure 4.3.1 ($k_2 = 10$, income-drawdown programmes with bequests only), we see that the change in k_2 has only a small effect on the relative costs. For the ELID programme to be preferred to the ELA programme, we require k_2, the weight attached to the bequest utility, to be at least 20: that is, when the RRA=3.96 and k_2 exceeds 20, the optimal ELID

stage will be much higher than 5%.

[26] These findings are a consequence of the functional form chosen for $h_1(\gamma_1)$ and $h_2(\gamma_2)$ (see Section 4.1) which was designed to provide consistency over the range of RRA parameters.
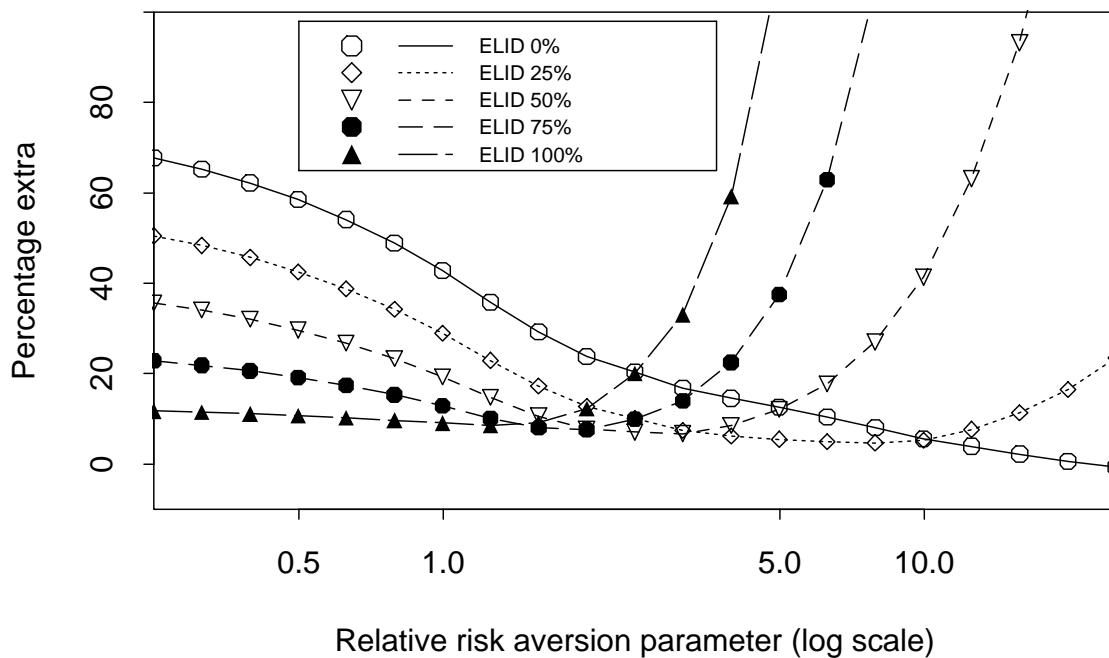
Figure 4.3.1: Extra cash required at time 0 for different income drawdown programmes to match the expected discounted utility of the optimal ELA programme as a function of the relative risk aversion parameter. Standard mortality, $k_2 = 10$, $d_2 = 10000$, $\beta = 0.0488$.

value exceeds the optimal ELA value.[27]

Empirical studies (see Brown (2001) and the references cited therein) are inconclusive on the importance attached by retirees to bequests, so it is difficult to judge with the current state of knowledge what typical values for k_2 should be. However, it is useful to know that a misspecification of k_2 is unlikely to have substantial cost implications for a plan member relative to the potential cost of choosing the wrong equity mix.

4.4 Adverse mortality selection – impaired lives

The calculations in the preceeding sections assumed that the plan member's mortality probabilities equal those used by the life office to calculate annuity prices. However, a typical group of plan members will include some in good health and others in poor health. For the latter group, the purchase of a life annuity at retirement at standard rates represents poor value relative to other plan members in better health. It is often suggested, therefore, that those in poor health should choose to defer annuitisation for as long as possible. We investigate the desirability of this here.

We consider an individual for whom mortality rates are approximately four times those assumed by the life office.[28] This degree of impairment is consistent with, for example, an individual who has just been diagnosed as suffering from Alzheimer's Disease (Macdonald & Pritchard (2000)).[29] Results for such an individual are presented in Figure 4.4.1 which should be compared with Figure 4.2.3 which was constructed using the same set of parameters ($k_2 = 5$, $d_2 = 10000$, $\beta = 0.0488$). We make the following observations.

First, if the choice of programme (ELA or ELID) is given, the optimal equity proportion (0%, 25%, 50%, 75% or 100%) for plan members with different levels of risk aversion are unaffected by differences between impaired and standard mortality. This is a direct consequence of power utility which generates constant portfolio proportions.

Second, compare Figures 4.4.1(a) and 4.2.3(a). We see that if we are restricted to the use of an ELA programme then the optimal equity mix is, unsusprisingly, largely unaltered. However, we can also observe that the cost of choosing a sub-optimal ELA programme with impaired mortality is about one-third lower than with standard mortality.

Third, Figure 4.4.1(b) compares each of the ELID programmes with the best ELA programme for each level of risk aversion. The cost of choosing a sub-optimal ELID programme with impaired mortality is about 50% or more lower than with standard

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[27]The equivalent thresholds for other levels of risk aversion are $k_2 = 50$ when the RRA=0.25 and $k_2 = 10$ when the RRA=25. This range of thresholds can be reduced by increasing d_1.

[28]Strictly we assume that, $p_x^{\text{Impaired}} = (p_x^{\text{PLA}})^4$, that is, the force of mortality is 4 times the standard force used by the life office in its annuity pricing. For small q_x^{PLA} this implies that $q_x^{\text{Impaired}} \approx 4q_x^{\text{PLA}}$.

[29]In Macdonald & Pritchard (2000) the ratio $q_x^{\text{Actual}}/q_x^{\text{PLA}}$ varies between 1 and 3 and is typically around 2.5 for the critical early years after diagnosis.
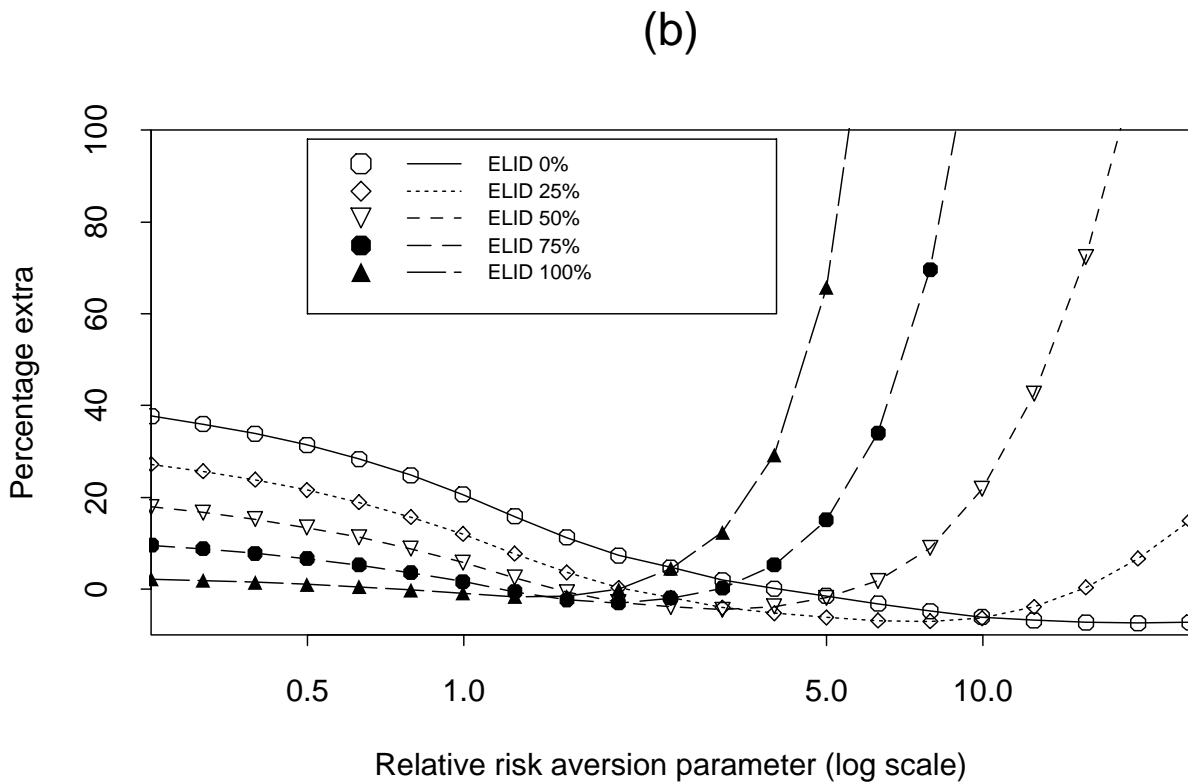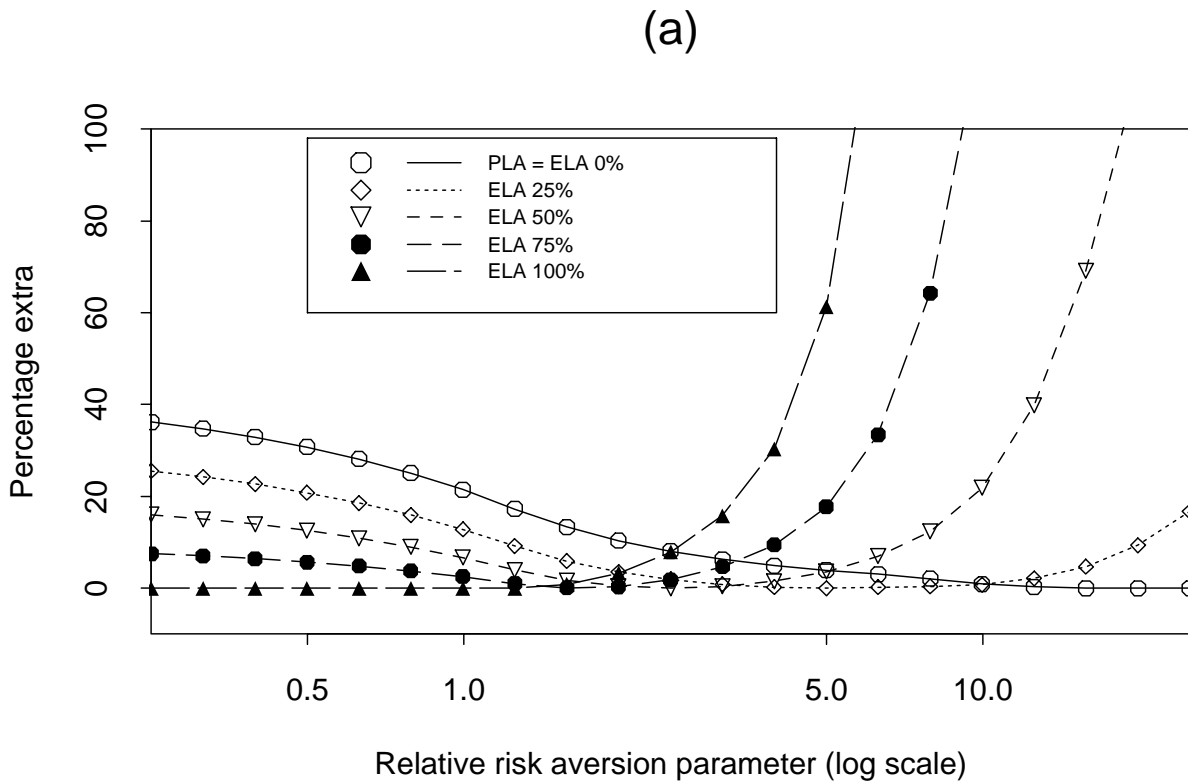
Figure 4.4.1: Extra cash required at time 0 for different impaired-life distribution programmes to match the expected discounted utility of the optimal ELA programme as a function of the relative risk aversion parameter: (a) annuity (PLA and ELA) programmes paying survival credits versus the optimal ELA programme, (b) income drawdown programmes paying bequests versus the optimal ELA programme. Impaired mortality, $k_2 = 5$, $d_2 = 10000$, $\beta = 0.0488$.

mortality (cf, Figure 4.2.3(b)). A negative value means that the ELID programme is preferred to the ELA programme. As the degree of impairment increases, plan members are more likely to prefer the income drawdown programme. However, the degree of impairment needs to be quite strong before the plan member switches from the ELA to the ELID programme. But for plan members with a stronger bequest motive (higher k_2), the degree of impairment does not need to be as strong before switching from ELA to ELID.

Fourth, it is possible that such an impaired life would benefit more from programmes which allowed for the accelerated payment of pension. The most beneficial improvement from the point of view of the plan member would be the payment of higher (that is, fairer) survival credits from the life office to reflect the higher mortality rates (as is the case with impaired life annuities) rather than from a programme with bequests. At the same time, we also observe that individuals with a lower degree of impairment are still likely to prefer the ELA programme, that is, the survival credits can still be worth having even though they are actuarially unfair.

4.5   The timing of annuitisation – the cost of regulation

We will now consider the impact and the cost in utility terms of the regulation which enforces annuitisation at age 75. To illustrate, we will allow the requirement to purchase an annuity to be delayed until age 85. We will look at the ELA and ELID programmes in turn and then look at the effect of the size of the fund. For both programmes, the optimal equity mix is unaltered by the change[30].

Figure 4.5.1 shows the impact of this change in the regulations on plan members who have opted for an ELA programme.[31] The raising of the compulsory annuitisation age increases expected discounted utility for plan members with low risk aversion. For example, a risk-neutral plan member (RRA = 0) will require 15% less cash at age 65 in order to achieve the same expected discounted utility as before, while a plan member with RRA of 0.5 will require 11% less cash. For plan members with higher RRAs, the cost is relatively small and, in the limit, zero for those who opt for 0% equity investment.

These observations are consistent with the continuous-time analysis of Merton (1990, Chapter 18). Merton considers only the ELA option with actuarially fair survival credit payments and power utility. He finds that it is optimal to maintain a fixed proportion in equities and not to switch gradually over time into bonds: that is, it is optimal to delay compulsory annuitisation for as long as possible.[32]

In Figure 4.5.2 we examine plan members who have opted for the ELID programme. In this case compulsory annuitisation at age 75 is best for almost all plan members. Indeed even delaying annuitisation to age 75 is very costly for very risk-averse plan members. This result follows because mortality drag increases sig-

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[30]Again this is a direct consequence of using power utility.

[31]Note that the ELA 0% programme with annuitisation at either of ages 75 or 85 delivers an identical income stream to the PLA.

[32]The payment in the Merton framework of an actuarially fair survival credit precisely cancels the effect of the mortality drag at all ages.
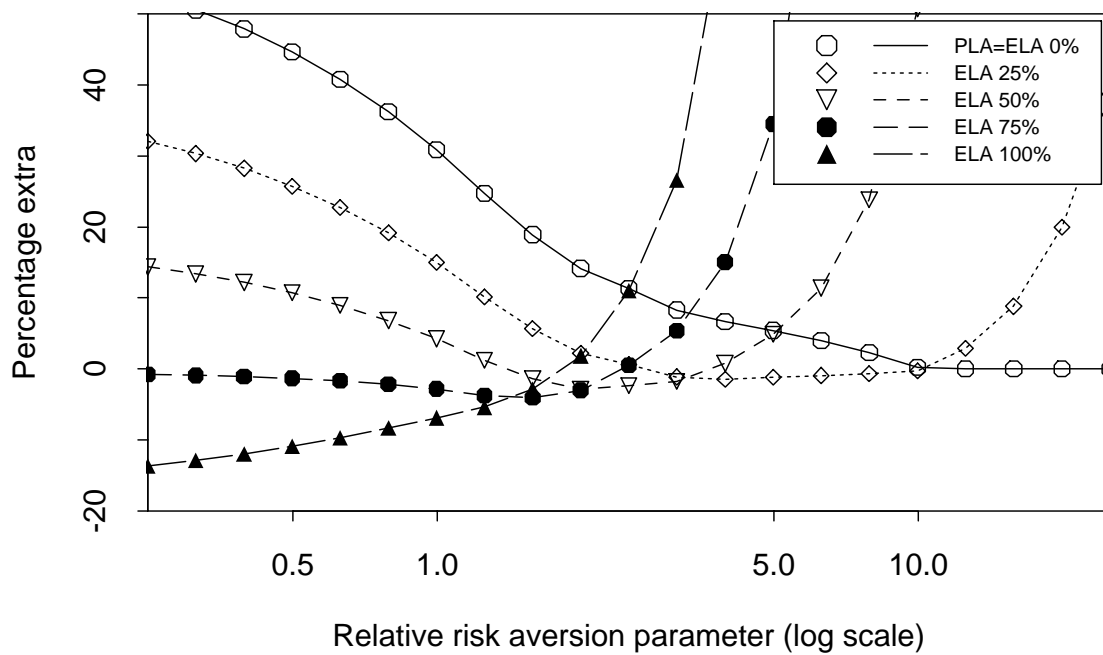
Figure 4.5.1: Extra cash required at time 0 for different annuity (PLA and ELA) programmes with compulsory annuitisation at age 85 to match the expected discounted utility of the optimal ELA programme with compulsory annuitisation at age 75 as a function of the relative risk aversion parameter. Standard mortality, $\beta = 0.0488$.
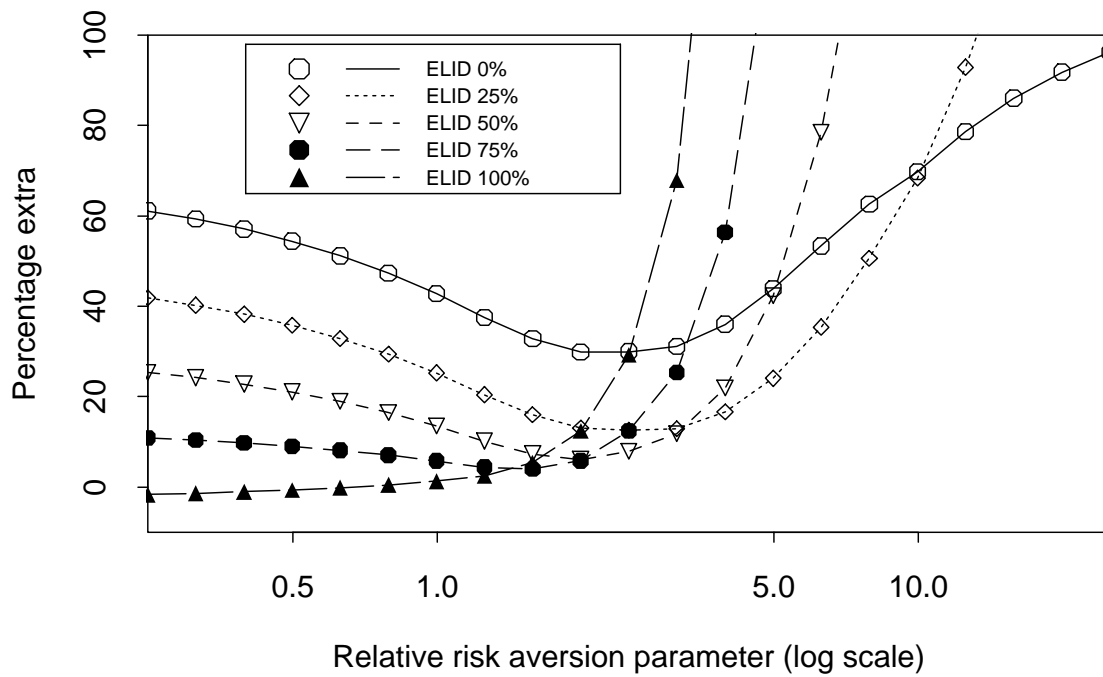
Figure 4.5.2: Extra cash required at time 0 for different ELID programmes with compulsory annuitisation at age 85 to match the expected discounted utility of the optimal ELID programme with compulsory annuitisation at age 75. Standard mortality, $k_2 = 5$, $d_2 = 10000$, $\beta = 0.0488$.

nificantly between ages 75 and 85. This means that it is better for the plan member to sacrifice the benefits of equity investment and the provision of a bequest in order to benefit from rising survival credits by switching to an annuity.

These observations lead us to ask the question: what is the optimal age to annuitise? To answer this question we compared the following three options:

- Annuitise immediately (that is, Programme 1).

- Employ the ELA or ELID programme with the optimal equity mix up to age 75 and then annuitise.

- Employ the ELA or ELID programme with the optimal equity mix and annuitise at the optimal age between age 65 and 85, with compulsory annuitisation at age 85 if voluntary annuitisation has not occurred beforehand. The age at which to annuitise is decided at age 65 and this decision is, for the moment, assumed to be irreversible.

We begin by comparing the ELA and PLA programmes. We find that it is optimal either to annuitise immediately or to wait until age 85, but never to annuitise at some intermediate age. This is consistent with our observations on Figure 4.5.1 and with the Merton (1990) result. It follows that the cost of compulsory annuitisation at age 75 therefore lies between 0% and 15% of the initial fund value depending on the level of risk aversion.

Next we compare the ELID and PLA programmes. Table 4.5.1 shows the outcome when the plan member has the right to invest freely up to age 85 and to annuitise at any age before age 85, but attaches no value to bequests ($k_2 = 0$). This means that the decision to defer annuitisation is driven purely by a comparison between the loss of future expected excess equity returns and the gain from future survival credits under the PLA. In the final column we report the cost of a regulation which compels plan members to annuitise at age 75. To illustrate, for a plan member with a very low RRA of 0.25, he would require an extra 1.6% (or £1,600) to compensate him for annuitising at 75 rather than 85. The table indicates that the optimal annuitisation age is very sensitive to the level of risk aversion, but that the overall cost of forced annuitisation at 75 (when there is no bequest motive) is relatively small and declines to zero with RRAs above unity.

At very low levels of RRA, the optimal annuitisation age of 79 is close to the age we would get (namely, 81) by applying Milevsky's (1998) rule: that is, switch at the point where the mortality drag matches the expected excess return on equities over bonds. Our analysis shows that this decision rule matches the one presented here *only* for a plan member who is *risk neutral*, that is, has RRA = 0. Our more general analysis demonstrates that decision making is much more complex than that suggested by Milevsky, with both the equity mix and the optimal time of annuitisation critically dependent on the level of risk aversion and the desire to make a bequest.

Table 4.5.2 shows the optimal age to switch from an ELID programme to the PLA when there is a positive bequest motive. The switching age is greater at RRAs

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RRA	Optimal equity mix	Optimal age to annuitise	Cost of annuitisation by age 75
0.25	100	79	1.6%
0.31	100	79	1.3%
0.40	100	78	1.1%
0.50	100	78	0.8%
0.63	100	77	0.5%
0.79	100	77	0.2%
1.00	100	76	0.0%
1.25	100	74	0.0%
1.58	75	72	0.0%
1.99	75	70	0.0%
2.50	50	69	0.0%
3.15	50	66	0.0%
3.96	0	65	0.0%
4.99	0	65	0.0%
6.28	0	65	0.0%
7.91	0	65	0.0%
9.95	0	65	0.0%
12.53	0	65	0.0%
15.77	0	65	0.0%
19.86	0	65	0.0%
25.00	0	65	0.0%

Table 4.5.1: Optimal decison rules for a plan member choosing the equity-linked income-drawdown programme when annuitisation can be taken at any time before a compulsory annuitisation age of 85 and there is no bequest motive. The final column shows the cost as a percentage of the initial fund of imposing a compulsory annuitisation age of 75. Standard mortality, $k_2 = 0$, $\beta = 0.0488$.

below 4.99 and above 12.53 than with the ELID programme without a bequest motive and is the same age, namely the retirement age of 65, for RRAs between 4.99 and 12.53 (cf Table 4.5.1). This finding is not entirely surprising. Under the PLA or ELA programmes the expected discounted utility is not affected by the inclusion of a bequest utility function since the utility attached to a zero bequest is zero. In comparison the expected discounted utility under the ELID programme immediately increases as a result of the inclusion of a bequest motive, thereby making the ELID programme relatively more attractive at each age and hence encouraging a further delay to annuitisation.[33] The cost of forcing annuitisation at 75 is around 50% higher for a plan member with a low RRA and a positive bequest motive compared

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[33]The U-shaped pattern of optimal annuitisation ages in Table 4.5.2 is an artefact of the way in which the functions $h_1(\gamma)$ and $h_2(\gamma)$ have been parameterised. As the RRA increases (γ decreases), plan members, besides investing more conservatively, place greater emphasis on the bequest, since $h_2(\gamma)$ is increasing in RRA.

RRA	Optimal equity mix	Optimal age to annuitise	Cost of annuitisation by age 75
0.25	100	80	2.4%
0.31	100	80	2.1%
0.40	100	80	1.9%
0.50	100	80	1.5%
0.63	100	79	1.2%
0.79	100	78	0.7%
1.00	100	77	0.3%
1.25	100	76	0.0%
1.58	75	74	0%
1.99	75	72	0%
2.50	50	70	0%
3.15	50	68	0%
3.96	25	67	0%
4.99	0	65	0%
6.28	0	65	0%
7.91	0	65	0%
9.95	0	65	0%
12.53	0	65	0%
15.77	0	69	0%
19.86	0	71	0%
25.00	0	72	0%

Table 4.5.2: Optimal decison rule for a plan member choosing the equity-linked income-drawdown programme when annuitisation can be taken at any time before a compulsory annuitisation age of 85 and there is a positive bequest motive. The final column shows the cost as a percentage of the initial fund of imposing a compulsory annuitisation age of 75. Standard mortality, $k_2 = 5$, $d_2 = 10000$, $\beta = 0.0488$.

with the case where there is no bequest motive, but it is still less than 2.5% of the initial fund value.

4.6 Dynamic stochastic optimisation

In the preceding discussion we assumed that the decision on the annuitisation age is made at age 65 and is not subject to change. For sophisticated plan members this is an unnecessary restriction. Suppose instead that a decision on whether or not to annuitise can be made at the end of each year taking into account the information available at that time. Within the present modelling framework this means that the decision at time t will depend on the current fund size, $F(t)$, and the current age of the plan member, $65 + t$. Since $F(t)$ is random, the age at which it is optimal for the plan member to annuitise might not be known in advance.

The optimisation process proceeds as follows:

- Let the optimal value function at time t be denoted $\hat{V}(t, F(t))$.

- Start at the age, $x = 65 + T$, at which annuitisation is compulsory. For each possible fund size at that time calculate the value function $V(T, F(T)) = \hat{V}(T, F(T))$.

- Next work backwards recursively:

  - Assume that the optimal value function $\hat{V}(t + 1, F(t + 1))$ is known for all $F(t + 1)$. Now consider the decision at time t when the fund size is $F(t)$. We need to compare the value function (a) assuming that the plan member annuitises immediately with (b) assuming that the plan member defers annuitisation until at least time $t + 1$ and then acts optimally thereafter. Under (b) we have several factors to take into account: the probability of survival, the pension payment at time t, and the possible bequest if the plan member dies before time $t + 1$. The optimal value function $\hat{V}(t, F(t))$ is the larger of the value functions under (a) and (b) and this indicates which decision (a) or (b) is taken.

  - This procedure is repeated over the full range of possible values for $F(t)$.

- Once this has been done, we can step backwards by one year, repeat the previous step and continue in this way until we reach the age of 65 at which point we stop[34].

One reason why the decision is likely to depend upon the fund size $F(t)$ is that the bequest utility does not exhibit constant RRA. Our results indicate that a plan member is more likely to prefer to delay annuitisation if his investments have been performing well. In contrast if the investments have been performing poorly, he might decide to annuitise sooner. To demonstrate this, suppose the fund size is almost zero and a plan member is considering a switch from the ELID programme to the PLA programme. On the one hand, because the fund size is very small, the negative impact on the bequest utility will be negligible. On the other hand, the payment of survival credits through the PLA will have a much more significant beneficial impact on the utility of consumption, since, as the fund size gets close to 0, the marginal utility of consumption, unlike the marginal utility of the bequest, tends to infinity. Both these factors encourage immediate annuitisation.

The dependence of the annuitisation decision on fund size is illustrated in Figure 4.6.1 which shows the outcome of the above optimisation process at selected RRAs. Consider a plan member with an RRA of 3.15 who is now aged 75 and who has not previously annuitised. If his current fund size is below about £90,000 then he should annuitise immediately. But if his current fund is above this level then it is optimal for him to defer annuitisation. We can see that the annuitisation region varies considerably with RRA. We also observe from these graphs that for any given age and RRA, annuitisation will either:

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[34]Throughout this exercise, we assume that the equity mix up to the time of annuitisation is held constant. Strictly this may, itself, not be optimal, but in the context of power utility on the pension component of the utility function this approximation will be reasonable.

- not be optimal for any fund size;

- be optimal for all fund sizes; or

- be optimal for low fund sizes but not for fund sizes above some threshold.

In each graph the dots show how the plan member's fund value would change over time if he had opted at age 65 for the PLA. This gives a useful reference point for projecting the stochastic fund size under the ELID programme at different ages. Thus with an RRA of 1.58 we can see that annuitisation is likely to occur some time between the ages of 72 (if equities perform poorly) and 80 (if equities perform moderately well). However, if equities perform sufficiently well then the fund-age trajectory will lie above the shaded region and annuitisation might only take place when it is compulsory at age 85. In the adjacent plot where the RRA is 3.15, the shaded annuitisation 'hill' is somewhat lower, implying that a relatively large proportion of the stochastic trajectories of $F(t)$ will avoid hitting the hill at ages below 85. On the other hand, if $F(t)$ is going to hit the hill it will probably hit it within the first 3 or 4 years. We can infer from these observations that in some cases (low RRA) the dynamic stochastic element in the annuitisation decision will not add much value (the plan member will choose to annuitise at around age 80 regardless). In other cases (high RRA) the shape and height of the annuitisation hill are such that the majority of stochastic fund-age trajectories cross over the hill without hitting it, suggesting that the dynamic stochastic element in the annuitisation is a valuable feature.[35]

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[35]This added value is somewhat similar to that of an at-the-money equity call option.
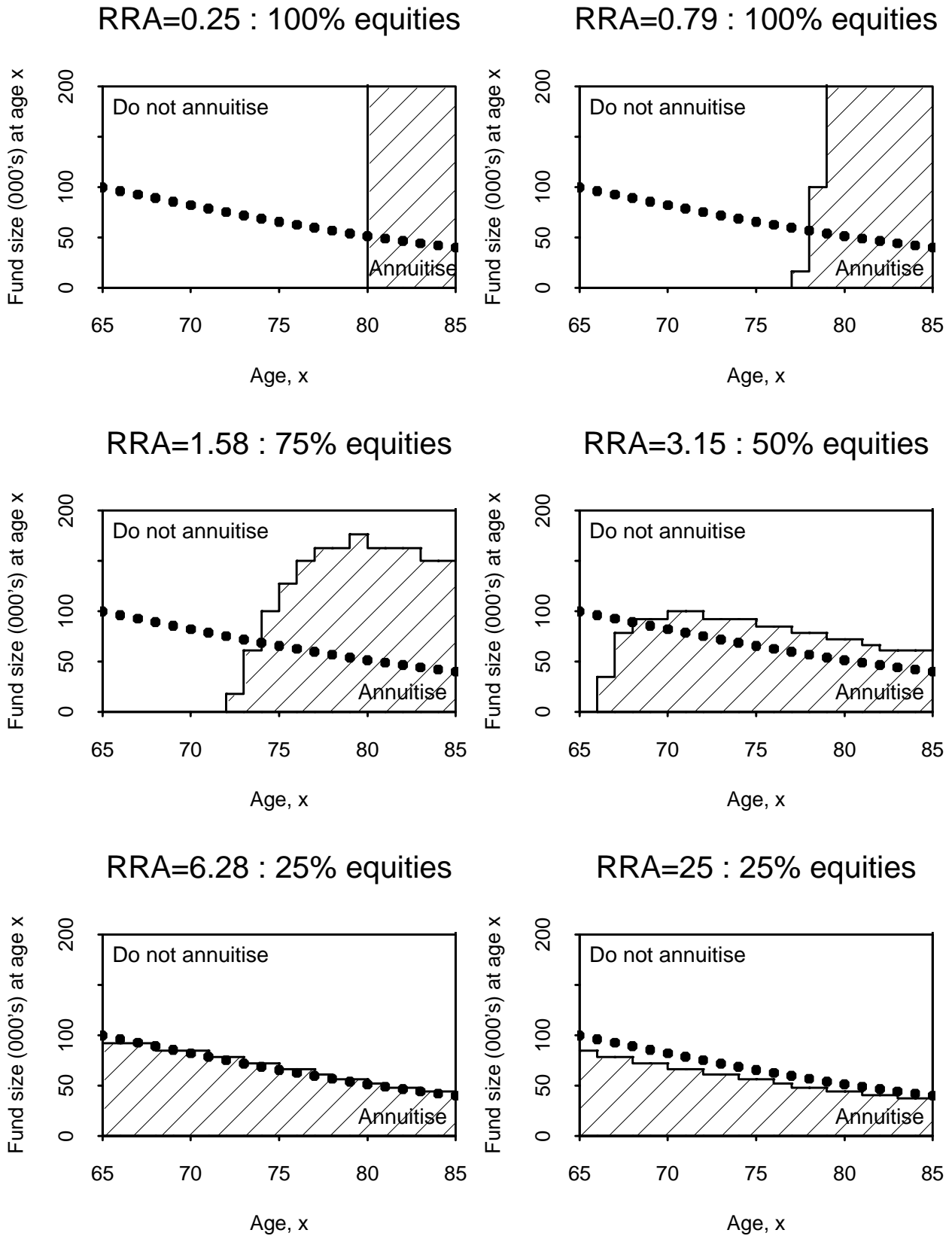
Figure 4.6.1: Relationship between the annuitisation decision and the plan member's age and fund size. If the fund-age trajectory enters the shaded area, then the plan member should annuitise immediately. The graphs are for different levels of risk aversion (RRA). Each graph assumes that before annuitisation the indicated, optimal equity mix has been used. The dotted line shows how the fund size would evolve with age if the plan member had opted for the PLA programme.

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4.7 Sensitivity analysis

The results discussed so far were tested for sensitivity to the parameter d_2 in the bequest utility function (equation (4.1.3)). In particular, we experimented with $d_2 = 50000$ instead of 10000. The results were indistinguishable from the earlier ones, indicating that our results are robust relative to large changes in this parameter.

We also considered the exponential utility function. This gives rise to constant *absolute* risk aversion and decreasing relative risk aversion as a function of the initial investment $F(0)$. The plots equivalent to Figures 4.2.1 and 4.2.3 were qualitatively different as one might expect. However, the optimal equity mixes were found to be essentially unaltered when we calculated the relative risk aversion parameter for the initial fund $F(0) = £100,000$. This suggests that it might be more important to determine the plan member's RRA parameter than the precise shape of his utility function.

5 Conclusions

Evaluating pension plan design was always important, but is becoming even more so as the move towards DC plans gathers pace. We hope that we have demonstrated that this remark is at least as valid for the distribution phase as it is for the accumulation phase of a DC plan.

Our results suggest that, for the central values chosen for the bequest utility function, the best distribution programme does not involve a bequest, but instead pays regular survival credits to the plan member in return for the residual fund reverting to the life office on the plan member's death. The best programme also depends on the plan member's attitude to risk: if he is highly risk averse (with a relative risk aversion exceeding 10), the appropriate programme is a conventional life annuity; on the other hand, if he has a stronger appetite for risk, the best programme involves a mixture of bonds and equities, with the optimal mix depending on the plan member's precise degree of risk aversion (plan members with a RRA below 1.25, for example, would be 100% invested in equities). Based upon the range of risk aversion parameters found for UK investors, the equity-linked annuity is likely to be chosen by relatively few plan members (only about 5% of the total), and very few of these would choose to invest 100% of their retirement fund in equities (at least in the UK).

There are a number of key implications arising from our findings:

1. The optimal choice of distribution programme appears to be fairly insensitive to the weight attached by the plan member to making a bequest. In particular, the weight would have to be substantially higher than that used in the paper to make programmes with a bequest optimal. This lack of sensitivity is consistent with the findings of Brown (2001). In other words, it would appear that members value a pension plan's ability to provide retirement income security for however long that they live over and above the possibility of being able to use the fund to make bequests to their children.

2. The equity proportion chosen for the distribution programme has a considerably more important effect on the plan member's welfare than the particular distribution programme chosen, and a poor choice can lead to substantially reduced expected discounted utility. However, restricting the equity proportion to a small number of evenly spread possibilities (0%, 25%, 50%, 75%, and 100%) does not have much effect on lowering expected discounted utility. This justifies pension providers having a simple product design.

3. Plan members in poor health relative to the average may, depending on the severity of their ill health, still prefer the ELA programme paying standard-rate survival credits to the ELID programme paying bequests. For the central values chosen for the bequest utility function, we found that impaired mortality rates would need to be four times standard rates before the ELID programme was preferred. Those in extremely poor health and attaching some weight to a bequest are rather more likely to prefer an income drawdown programme.

4. Forcing members of ELA programmes to annuitise at 75 rather than 85 can be expensive in terms of reduced expected discounted utility for those with low degrees of risk aversion: it is equivalent to 15% of the initial fund value for risk-neutral plan members. In contrast, forcing members of ELID programmes desiring to make a bequest to annuitise at 75 rather than 85 is rather less costly, at most 2% at low RRAs and nothing at all at RRAs above unity.

5. The optimal age at which to annuitise is very sensitive to the plan member's degree of risk aversion. Where no value is attached to bequests the optimal age ranges (Table 4.5.1) from 79 for a plan member with a very low RRA to immediate annuitisation at 65 if the RRA exceeds about 4. This suggests that any switching rule based solely on a comparison between the equity risk premium and the mortality drag is likely to be suboptimal and overestimate the optimal switching age if the member is risk averse. When value is attached to a bequest then the effect is to encourage plan members to defer anuitisation somewhat (Table 4.5.2).

6. The annuitisation-timing decision depends upon fund size. For the central values chosen for the bequest utility function, the larger the fund size, the more likely it is that the plan member will delay annuitisation. This lies in contrast to conclusion 1 above and it is a result of the introduction of the dynamic element into the annuitisation decision that depends on the state of the fund as well as age.

7. Once the plan member's degree of relative risk aversion (RRA) has been assessed, the optimal choice of programme is not overly sensitive to the form of utility function (for example, power or exponential). This again helps to simplify product design.

Three further extentions naturally suggest themselves:

- We have assumed here that the risk-free interest rate is fixed. Future work will investigate how the decision depends (if at all) on the risk-free rate of interest when this is stochastic.

- We are also aware that our analysis disregards the important issue of mortality improvements. This issue raises deeper questions than we can address here, and clearly warrants much further research. One promising avenue is to investigate flexible unit-linked programmes where the income received and the survival credits payable fall in response to mortality improvements.

- We have considered here the optimal choice for a plan member from the limited range of standard programmes that providers of retirement income products currently offer. Future work would investigate optimal solutions based on stochastic dynamic programming. The aim would be to measure any additional utility gains from full optimisation over the simple programmes considered here.

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