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A Quantitative Comparison of Stochastic Mortality Models Using Data from England and Wales and the United States

Andrew J. G. Cairns,* David Blake,† Kevin Dowd,‡ Guy D. Coughlan,§ David Epstein,§ Alen Ong,§ and Igor Balevich¶

Abstract

We compare quantitatively eight stochastic models explaining improvements in mortality rates in England and Wales and in the United States. On the basis of the Bayes Information Criterion (BIC), we find that, for higher ages, an extension of the Cairns-Blake-Dowd (CBD) model that incorporates a cohort effect fits the England and Wales males data best, while for U.S. males data, the Renshaw and Haberman (RH) extension to the Lee and Carter model that also allows for a cohort effect provides the best fit. However, we identify problems with the robustness of parameter estimates under the RH model, calling into question its suitability for forecasting. A different extension to the CBD model that allows not only for a cohort effect, but also for a quadratic age effect, while ranking below the other models in terms of the BIC, exhibits parameter stability across different time periods for both datasets. This model also shows, for both datasets, that there have been approximately linear improvements over time in mortality rates at all ages, but that the improvements have been greater at lower ages than at higher ages, and that there are significant cohort effects.

1. Introduction

It has become increasingly clear that mortality improvements in countries where reliable data exist are driven by an underlying process that is stochastic. Since the early 1990s a number of stochastic models have been developed to analyze these mortality improvements. These include the Lee-Carter model and its extensions (Lee and Carter 1992; Brouhns, Denuit, and Vermunt 2002; Renshaw and Haberman 2003, 2006; Continuous Mortality Investigation Bureau [CMI] 2005, 2006); the P-splines model (Currie, Durban, and Eilers 2004; Currie 2006; CMI 2005, 2006), and the Cairns-Blake-Dowd (CBD 2006b) model (a stochastic version of the Perks 1932 model). A number of recent papers have sought to compare different mortality models, including Wong-Fupuy and Haberman (2004), Renshaw and Haberman (RH, 2006), and CMI (2005, 2006, 2007). Renshaw and Haberman (2006), for example, compare models in a quantitative fashion by analyzing the pattern of standardized residuals against age, year of observation, and year of birth. However, so far as we are aware, no studies have used formal

* Maxwell Institute for Mathematical Sciences and Department of Actuarial Mathematics and Statistics, Heriot-Watt University, Edinburgh, EH14 4AS, United Kingdom, A.Cairns@ma.hw.ac.uk.

† Pensions Institute, Cass Business School, City University, 106 Bunhill Row, London, EC1Y 8TZ, United Kingdom.

‡ Centre for Risk and Insurance Studies, Nottingham University Business School, Jubilee Campus, Nottingham, NG8 1BB, United Kingdom.

§ Pension ALM Group, JPMorgan Chase Bank, 125 London Wall, London, EC2Y 5AJ, United Kingdom.

¶ Pension Advisory Group, JPMorgan Securities Inc., 270 Park Avenue, New York, NY 10017-2070.

model selection criteria to compare and rank a variety of nested and non-nested models. This study undertakes such a comparison.

We consider a range of both existing and new models.[1] In the early part of the paper, we compare these on the basis of a set of desirable, *qualitative* properties: parsimony, transparency, ability to generate sample paths, incorporation of cohort effects (see Willets 1999, 2004; Richards, Kirkby, and Currie 2006), and ability to produce a nontrivial correlation structure. The study then pays considerable attention to two important *quantitative* criteria that can be evaluated only when each model is fitted to the data: consistency with historical data, and robustness of parameter estimates relative to the range of data employed.[2]

Our analysis focuses on mortality at higher ages (60–89), given our interest in pension-related applications where the risk associated with longer-term cash flows is primarily linked to uncertainty in future rates of mortality at higher ages. For models M5–M8, the focus on this higher age range allows us to exploit the relatively simple log-linear structure of the mortality curve resulting in a family of multifactor models that have parsimonious age effects.

We find that no single model dominates on the basis of all the above criteria. If we rank models using an objective model selection criterion based on the statistical quality of fit, then an extension of the CBD (2006b) model fits the England and Wales data best, while the RH (2006) model fits the U.S. data best. However, if we take the robustness of parameter estimates into account, then the preferred model is a different extension of the CBD model that allows for both a cohort effect and a period effect that is quadratic in age.

1.1 Notation

We consider eight models in this paper, and it is important that we use consistent and clear notation throughout:

- Calendar year t is defined as running from time t to time $t + 1$.
- We define $m_c(t, x)$ to be the *crude* (i.e., unsmoothed) death rate for age x in calendar year t. More specifically,

$$m_c(t, x) = \frac{\text{Number of deaths during calendar year } t \text{ aged } x \text{ last birthday}}{\text{Average population during calendar year } t \text{ aged } x \text{ last birthday}}.$$

  The average population is usually approximated by an estimate of the population aged x last birthday in the middle of the calendar year. The *underlying* death rate is then $m(t, x)$, which is equal to the *expected* deaths divided by the exposure.
- A second measure of mortality is the mortality rate $q(t, x)$. This is the *probability* that an individual aged exactly x at exact time t will die between t and $t + 1$.
- A third measure is the *force* of mortality, $\mu(t, x)$. This is interpreted as the instantaneous death rate at exact time t for individuals aged exactly x at time t. For these individuals, for small dt, the probability of death between t and $t + dt$ is approximately $\mu(t, x) \times dt$.
- For individuals who die aged x last birthday, in year t we use the convention that $t - x$ is the year of birth. However, the precise date of birth might be any time between January 1 in calendar year $t - x - 1$ and December 31 in calendar year $t - x$. For notational compactness we will sometimes use $c = t - x$.

[1] All models are described in the paper at the outset. However, the new models (labeled M6–M8) were developed in response to perceived problems with the original five models (M1–M5) as well as building on the strengths of these models.

[2] An earlier version of this paper, with the same title, looked in more detail at the underlying data and empirical illustrations of the cohort effect. See http://www.ma.hw.ac.uk/~andrewc/papers/.

1.2 Relationship between $m(t, x)$ and $q(t, x)$

The death rate, $m(t, x)$, and the mortality rate, $q(t, x)$, are typically very close to one another in value. With a simple assumption, we can formalize this relationship more precisely:

> Assumption 1: For integers t and x, and for all $0 \leq s, u < 1$, $\mu(t + s, x + u) = \mu(t, x)$: that is, the force of mortality remains constant over each year of integer age and over each calendar year.

This implies the following:

a. $m(t, x) = \mu(t, x)$
b. $q(t, x) = 1 - \exp[-\mu(t, x)] = 1 - \exp[-m(t, x)]$.

Relationship (a) is often used in the analysis of death rate data (see, e.g., Brouhns, Denuit, and Vermunt 2002). Relationship (b) is useful in the analysis of parametric models for mortality that are formulated in terms of $q(t, x)$.

Assumption 1 does not normally hold exactly, but the resulting relationship between $m(t, x)$ and $q(t, x)$ is generally felt to provide an accurate approximation.

2. DATA

We now discuss the general characteristics of both the England and Wales and the U.S. male data. The primary motivation for this study is to compare various mortality models and determine which are best suited to forecasting mortality at higher ages. This reflects a concern with longevity risk—the risk that realized survival rates might be higher than anticipated—to which pension plans and annuity providers are exposed. As a consequence, we use data at higher ages only (ages 60–89 inclusive) when we make our comparisons of the different models.

2.1 England and Wales: Crude Death Rates

In this paper we use crude mortality rates for England and Wales (EW) males between 1961 and 2004.[3] As ''stylized facts'' we can observe that over this period mortality rates have been declining at all ages, they have been declining at different rates at different ages, and they have been declining erratically (see, e.g., Cairns, Blake, and Dowd 2006a, Fig. 1.2).

A typical dataset consists of numbers of deaths, $D(t, x)$, and the corresponding exposures, $E(t, x)$, over a range of years t and ages x. Numbers of deaths are normally regarded as being reasonably accurate, although the recorded age at death is believed to be less accurate at very high ages. The exposure $E(t, x)$ represents the average, during calendar year t, of the number of people alive who were aged x last birthday. This quantity is normally not known with a high degree of accuracy, even in census years, and has to be estimated by the Office for National Statistics (ONS) (or its equivalent in other countries), taking account of recorded births and deaths and net immigration.

In the analysis that follows we shall exclude a number of seemingly unreliable data points (t, x):

- The 1886 cohort (i.e., $t - x = 1886$). Death rates for this cohort became markedly out of line with neighboring cohorts during the 1960s. This might be the result of poorly calculated exposures (i.e., estimates of average population size at each age).

[3] Data for this period were provided by the United Kingdom's Office for National Statistics. The Human Mortality Database (www.mortality.org) and LifeMetrics (www.lifemetrics.com) also provide useful sources of data. Both web sites include thorough technical documentation to support the data. In their analyses of the Lee and Carter (1992), Renshaw and Haberman (2006), and P-splines (Currie, Durban, and Eilers 2004) models, CMI (2005, 2006, 2007) look at females as well as males data, and life-office assured lives' data as well as national population data.

- Death rates at and above age 85 in the years up to and including 1970. Accurate exposures were not estimated by the ONS (or its predecessors) at these ages until 1971.

Additionally, because some of the models we are fitting are cohort models, we will exclude all cohorts that have fewer than five observations (after taking account of the exclusions above).[4] The rationale for including a ''cohort effect'' lies in an analysis of the rates at which mortality has been improving at different ages and in different years (see Willets 2004; Richards, Kirkby, and Kelly 2006). Cohorts born around 1930 experienced strong rates of improvement between ages 40 and 70 relative to, say, cohorts born 10 years earlier or 10 years later. For its part, the cohort born around 1950 seems to have experienced worse mortality than the immediately preceeding cohorts.

2.2 United States: Crude Death Rates

This paper also analyzes data for U.S. males aged 60 to 89 over the period 1968–2003.[5] We will focus on those aspects of the U.S. data that are different from the EW data.

With the EW data in a given year, t, we have identified (CBD 2006b) that logit $q(t, x)$ (i.e., $\log[q(t, x)/(1 - q(t, x))]$) is reasonably linear in x. Although this is approximately true for the U.S. data, we found that, in some years, there is a small degree of curvature in the plot of logit $q(t, x)$ against x. The curvature is not all that prominent, but it does turn out to be significant when we compare models with and without a quadratic term in logit $q(t, x)$, and it is an effect that changes over time. Although a cohort effect is evident in the U.S. data, the magnitude of the effect above age 60 is much smaller than the EW cohort effect.

Accurate exposures data are not available for the period 1968–1979 for ages above 84. Consequently we have used data for ages 85–89 only after 1979. Issues relating to the accuracy of mortality data at higher ages are explored further by Anderson (1999).

In contrast with the EW data, the U.S. data do not appear to have any individual cohorts that have identifiable problems. However, we found that the exposures data were, in general, less reliable as estimates of the underlying population sizes at specific ages in specific years.[6] This is most apparent if we follow the exposures data for a specific cohort over time.[7] Exposures data for cohorts born in 1928, 1918, 1908, and 1898 are plotted in Figure 1. We would normally expect to see a relatively smooth progression in the exposures data from one year to the next. The decrease in the exposure from one year to the next should reflect the numbers of deaths and net immigration from the cohort. If net immigration is zero, this should result in a fairly smooth, downwards progression of values in each plot. Instead, we see for each of the cohorts in Figure 1 that the pattern is somewhat erratic, especially for the top left plot. This might be explained by a volatile pattern of net immigration. However, it could also be explained by errors in the underlying data (particularly the exposures data). The corresponding plots for EW are much smoother.

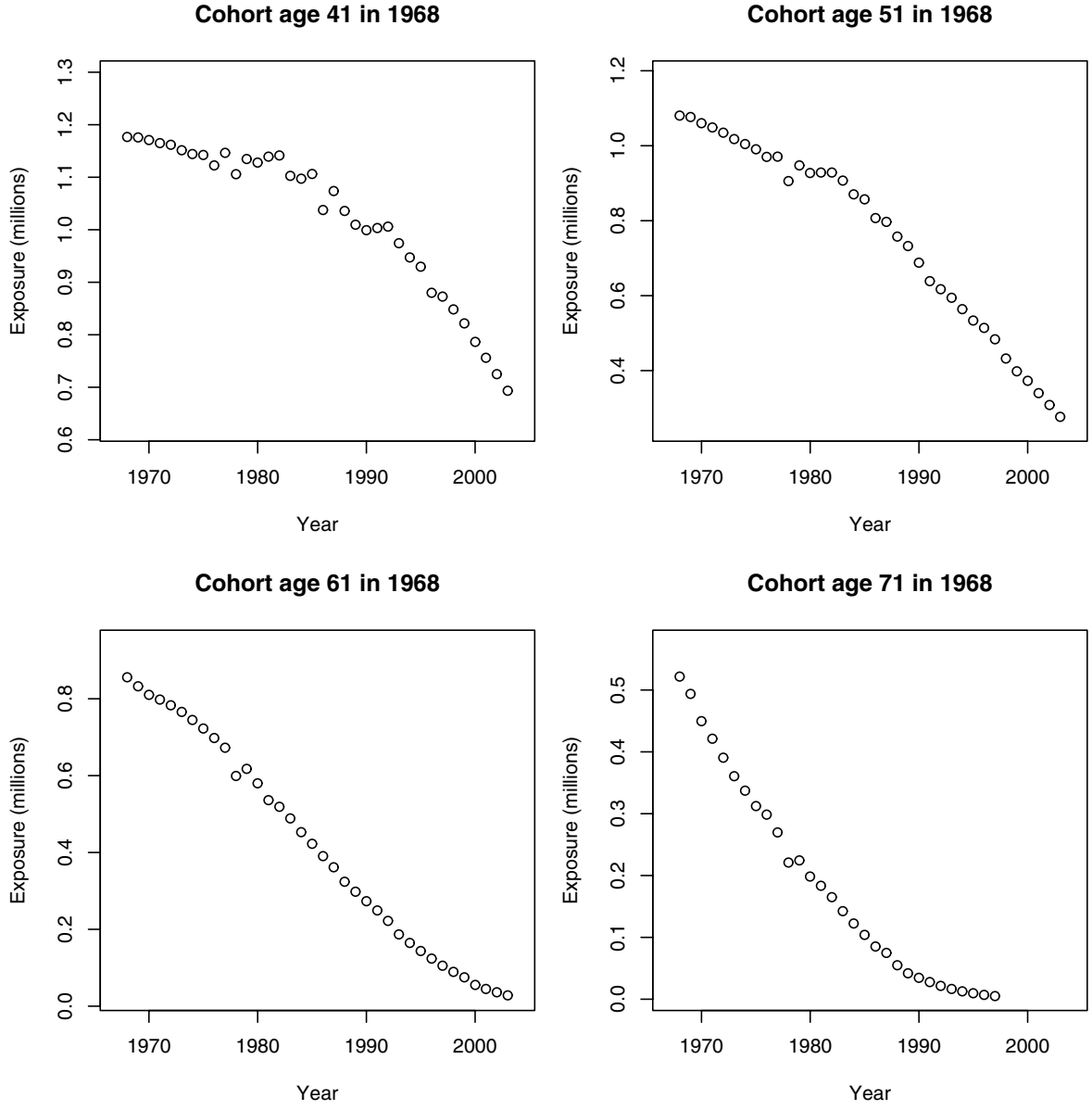
[4] The reliability of the estimates of the $\gamma_{t-x}^{(i)}$ cohort parameters (defined later) depends on the number of observations for each cohort. At one extreme, if we have just one observation, then the $\gamma_{t-x}^{(i)}$ parameter can be chosen so that the fitted death rate is exactly equal to the observed rate, a ''quality of fit'' that can be achieved without affecting any of the other estimated death rates. In effect, the single observation allows us to overfit the model, whereas the estimated $\gamma_{t-x}^{(i)}$ parameter is, in reality, subject to substantial parameter uncertainty. With more observations in a given cohort, the estimated $\gamma_{t-x}^{(i)}$ parameter becomes more reliable. Consequently we wish to exclude cohorts that have too few observations. However, if we exclude too many cohorts, then we are left with relatively little data. We therefore adopt a compromise and exclude cohorts with fewer than five observations.

[5] Data are available for higher ages, but, as previously discussed in the case of EW, age at death is often misreported at these high ages, resulting in unreliable estimated death rates at these ages.

[6] For further explanation, see Section 3.

[7] For example, if we follow exposures data for the ''1920'' cohort, then we look at the sequence $E(1920, 0)$, $E(1921, 1)$, . . . , $E(1980, 60)$, $E(1981, 61)$, ...

Figure 1
U.S. Exposures for Different Cohorts over the Period 1968–2003: $E(1967 + t, x + t)$ for $x = 40, 50, 60, 70$



3. ESTIMATION

We have data covering ages $x_1, \ldots, x_{n_a}$ and calendar years $t_1, \ldots, t_{n_y}$. For each age x and year t, we have an exposure of $E(t, x)$ (i.e., $E(t, x)$ is the average size of the population aged x last birthday during year t) and $D(t, x)$ deaths during year t recorded as age x last birthday at the date of death.

We will model the number of deaths using the Poisson model commonly employed in the literature on mortality modeling: we assume $D(t, x)$ has a Poisson distribution with mean $E(t, x) \times m(t, x)$ (or $D(t, x) \sim Po(E(t, x)m(t, x))$); see, for example, Brouhns et al. (2002).

Our analysis is complicated slightly by the fact that some of the models we consider model the death rate $m(t, x)$, whereas others model the mortality rate $q(t, x)$. To ensure a valid comparison between the different models, our analyses of the models for $q(t, x)$ involve an additional step. First, for a given

set of parameters, we calculate the $q(t, x)$. We then transform these into death rates using the identity $m(t, x) = -\log[1 - q(t, x)]$. We can now calculate the likelihood for all models consistently based on the $m(t, x)$ values.

For a given model we use ϕ to represent the full set of parameters, and the notation for $m(t, x)$ is augmented to read $m(t, x; \phi)$ to indicate its dependence on these parameters. Where we have a model for $q(t, x) = q(t, x; \phi)$ we define

$$m(t, x; \phi) = -\log[1 - q(t, x; \phi)].$$

For all models the log-likelihood is

$$l(\phi; D, E) = \sum_{t,x} D(t, x) \log[E(t, x)m(t, x; \phi)] - E(t, x)m(t, x; \phi) - \log[D(t, x)!],$$

and estimation is by maximum likelihood.[8]

4. THE MORTALITY MODELS

The data will cover the range $x_1, \ldots, x_{n_a}$ and $t_1, \ldots, t_{n_y}$ with unit increments in each case. Models will be labeled M1, M2, etc., and are listed in Table 1. Additionally we will use the following conventions:

- The $\beta_x^{(i)}$ functions will reflect age-related effects.
- The $\kappa_t^{(i)}$ functions will reflect period-related effects.
- The $\gamma_c^{(i)}$ functions will reflect cohort-related effects, with $c = t - x$.

All of the models that we examine, with the exception of the P-splines model, will be of the form log $m(t, x) = \Sigma_i \beta_x^{(i)} \kappa_t^{(i)} \gamma_{t-x}^{(i)}$ or logit $q(t, x) = \Sigma_i \beta_x^{(i)} \kappa_t^{(i)} \gamma_{t-x}^{(i)}$.

The method of recording the calendar year of death and the age last birthday at death means that the death count covers individuals born on January 1 in calendar year $t - x - 1$ through to December 31 $t - x$ (i.e., two years). The cohort index $c = t - x$ takes its values from the second of these years. To illustrate, the 1886 cohort discussed above in Section 2.1 covers individuals born between January 1, 1885 and December 31, 1886.

4.1 Model M1

Lee and Carter (1992) propose the following model for death rates:

$$\log m(t, x) = \beta_x^{(1)} + \beta_x^{(2)} \kappa_t^{(2)}.$$

Table 1
Formulae for the Mortality Models

Model	Formula
M1	$\log m(t, x) = \beta_x^{(1)} + \beta_x^{(2)} \kappa_t^{(2)}$
M2	$\log m(t, x) = \beta_x^{(1)} + \beta_x^{(2)} \kappa_t^{(2)} + \beta_x^{(3)} \gamma_{t-x}^{(3)}$
M3	$\log m(t, x) = \beta_x^{(1)} + n_a^{-1} \kappa_t^{(2)} + n_a^{-1} \gamma_{t-x}^{(3)}$
M4	$\log m(t, x) = \Sigma_{i,j} \theta_{ij} B_{ij}^{ay}(x, t)$
M5	$\text{logit } q(t, x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x})$
M6	$\text{logit } q(t, x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}$
M7	$\text{logit } q(t, x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \kappa_t^{(3)}((x - \bar{x})^2 - \hat{\sigma}_x^2) + \gamma_{t-x}^{(4)}$
M8	$\text{logit } q(t, x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}(x_c - x)$

Notes: The functions $\beta_x^{(i)}$, $\kappa_t^{(i)}$, and $\gamma_{t-x}^{(i)}$ are age, period, and cohort effects, respectively. The $B_{ij}^{ay}(x, t)$ are B-spline basis functions, and the θ_{ij} are weights attached to each basis function. n_a is the number of ages covered; $\bar{x}$ is the mean age over the range of ages being used in the analysis; $\hat{\sigma}_x^2$ is the mean value of $(x - \bar{x})^2$.

[8] Note that $D(t, x)!$ means "$D(t, x)$ factorial."

For this and some of the other models, there is an *identifiability* problem in parameter estimation. To see this, note that the revised parameterization

$$\log m(t, x) = \tilde{\beta}_x^{(1)} + \tilde{\beta}_x^{(2)}\tilde{\kappa}_t^{(2)},$$

where

$$\begin{aligned}\tilde{\beta}_x^{(1)} &= \beta_x^{(1)} + b\beta_x^{(2)},\\ \tilde{\beta}_x^{(2)} &= \beta_x^{(2)}/a,\\ \tilde{\kappa}_t^{(2)} &= a(\kappa_t^{(2)} - b)\end{aligned}$$

results in identical values for $\log m(t, x)$, and this means that we cannot distinguish between the two parameterizations. To circumvent this we need to impose two constraints on the parameters. To some extent the choice of constraints is a subjective one, although some choices are more natural than others. With the current model we use the following constraints:

$$\begin{aligned}\sum_t \kappa_t^{(2)} &= 0,\\ \sum_x \beta_x^{(2)} &= 1.\end{aligned}$$

The first is a natural constraint and implies that, for each x, the estimate for $\beta_x^{(1)}$ will be equal (at least approximately) to the mean over t of the $\log m(t, x)$. There also has to be a second constraint to pin down unique values of parameters a and b above. However, there is no natural choice for this, and, indeed, different choices can be seen in different applications of the Lee-Carter model in the academic literature. The important point to note, however, is that the choice of the second constraint has no impact on either the quality of the fit or on the forecasts of mortality.

4.2 Model M2

Renshaw and Haberman (2006) generalized the Lee-Carter model to include a cohort effect as follows:

$$\log m(t, x) = \beta_x^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\gamma_{t-x}^{(3)}.$$

Model M1 is then a special case where the $\beta_x^{(3)}$ and $\gamma_{t-x}^{(3)}$ are set to zero.

This model has similar identifiability problems as the previous model. We therefore impose the following constraints to ensure identifiability:

$$\begin{aligned}\sum_t \kappa_t^{(2)} &= 0,\\ \sum_x \beta_x^{(2)} &= 1,\\ \sum_{x,t} \gamma_{t-x}^{(3)} &= 0,\\ \sum_x \beta_x^{(3)} &= 1.\end{aligned} \tag{4.1}$$

The first and third constraints mean that the estimate for $\beta_x^{(1)}$ will be (at least approximately) equal to the mean over t of the $\log m(t, x)$. The second and fourth constraints are similar to the second constraint in model M1, in that there are no natural choices, although the actual choice makes no difference to the quality of fit.

The original RH (2006) study chose to fix estimates for $\beta_x^{(1)}$ at $n_y^{-1}\,\Sigma_t \log m(t, x)$; the remaining parameters were estimated using an iterative process. In contrast, we use RH's estimate for $\beta_x^{(1)}$ only as a starting value and include $\beta_x^{(1)}$ in the iterative scheme as well.

We found that parameter values converge very slowly to their maximum likelihood estimates. This suggests that an identifiability problem remains. It is not clear if this problem is an exact one or an approximate one. An exact identifiability problem means that the likelihood function will be absolutely flat in certain dimensions, whereas an approximate identifiability problem means that the likelihood function will be close to flat in certain dimensions.

4.3 Model M3

Currie (2006) introduces the simpler Age-Period-Cohort (APC) model

$$\log m(t, x) = \beta_x^{(1)} + \frac{1}{n_a}\kappa_t^{(2)} + \frac{1}{n_a}\gamma_{t-x}^{(3)},$$

where n_a is the number of ages in the dataset. The APC model has its origins in medical statistics and predates the Lee-Carter model (see, e.g., Osmond 1985; Jacobsen et al. 2002). The model is also a special case of model M2 with $\beta_x^{(2)} = 1/n_a$ and $\beta_x^{(3)} = 1/n_a$. Currie (2006) uses P-splines to fit $\beta_x^{(1)}$, $\kappa_t^{(2)}$, and $\gamma_{t-x}^{(3)}$ to ensure smoothness, although the method is approximate. In our analysis of M3, we do not impose any smoothness conditions.[9]

Without loss of generality, we impose the following constraints:

$$\sum_t \kappa_t^{(2)} = 0,$$

$$\sum_{x,t} \gamma_{t-x}^{(3)} = 0.$$

We need one further constraint, because we can otherwise add $\delta((t - \bar{t}) - (x - \bar{x}))$ to $\gamma_{t-x}^{(3)}$, subtract $\delta(t - \bar{t})$ from $\kappa_t^{(2)}$, and add $\delta(x - \bar{x})$ to $\beta_x^{(1)}$ with no impact on the two constraints above. We propose here that the tilting parameter, δ, be chosen within an iterative scheme to minimize

$$S(\delta) = \sum_x (\beta_x^{(1)} + \delta(x - \bar{x}) - \bar{\beta}_x^{(1)})^2,$$

where $\bar{\beta}_x^{(1)} = n_y^{-1}\Sigma_t \log m(t, x)$. This implies that

$$\delta = -\frac{\Sigma_x(x - \bar{x})(\beta_x^{(1)} - \bar{\beta}_x^{(1)})}{\Sigma_x(x - \bar{x})^2}.$$

Given that the $\kappa_t^{(2)}$ and $\gamma_{t-x}^{(3)}$ already satisfy the first two constraints, we revise our parameter estimates according to the following formulas:

$$\tilde{\kappa}_t^{(2)} = \kappa_t^{(2)} - n_a\delta(t - \bar{t}),$$

$$\tilde{\gamma}_{t-x}^{(3)} = \gamma_{t-x}^{(3)} + n_a\delta((t - \bar{t}) - x - \bar{x})),$$

$$\tilde{\beta}_x^{(1)} = \beta_x^{(1)} + \delta(x - \bar{x}).$$

Note that models M1 to M3 can be described as belonging to the family of generalized Lee-Carter models.

4.4 Model M4

Currie, Durban, and Eilers (2004) propose the use of B-splines and P-splines to fit the mortality surface:

$$\log m(t, x) = \sum_{i,j} \theta_{ij} B_{ij}^{ay}(x, t),$$

[9] Although we use the same model, Currie (2006) incorporates a penalty for lack of smoothness. As a result, estimates for the three functions will be different.

with smoothing of the θ_{ij} in the age and cohort directions. Currie also discuss the construction of B-splines and how they are fitted.

4.5 Model M5

M5 is the original CBD model. CBD (2006b) fitted the following model to mortality rates $q(t, x)$:

$$\left[\text{logit } q(t, x) = \log \frac{[q(t, x)}{(1 - q(t, x))}\right] = \beta_x^{(1)}\kappa_t^{(1)} + \beta_x^{(2)}\kappa_t^{(2)}.$$

For this model simple parametric forms were assumed for $\beta_x^{(1)}$ and $\beta_x^{(2)}$:

$$\beta_x^{(1)} = 1,$$

$$\beta_x^{(2)} = (x - \bar{x}),$$

where $\bar{x} = n_a^{-1}\Sigma_i x_i$ is the mean age in the sample range (in our analysis, therefore, $\bar{x} = 74.5$). Thus,

$$\text{logit } q(t, x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}).$$

This model has no identifiability problems.

4.6 Model M6

This model is the first generalization of the CBD model to include a cohort effect:

$$\text{logit } q(t, x) = \beta_x^{(1)}\kappa_t^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\gamma_{t-x}^{(3)}.$$

For this model simple parametric forms were assumed for $\beta_x^{(1)}$, $\beta_x^{(2)}$, and $\beta_x^{(3)}$:

$$\beta_x^{(1)} = 1,$$

$$\beta_x^{(2)} = (x - \bar{x}),$$

$$\beta_x^{(3)} = 1.$$

Thus,

$$\text{logit } q(t, x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}.$$

As with other models, we have an identifiability problem. Here we can switch from $\gamma_{t-x}^{(3)}$ to $\tilde{\gamma}_{t-x}^{(3)} = \gamma_{t-x}^{(3)} + \phi_1 + \phi_2(t - x - \bar{x})$, and, with corresponding adjustments to $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$, there is no impact on the fitted values of the $q(t, x)$. This requires two constraints to prevent arbitrary use of ϕ_1 and ϕ_2. The constraints we have used here are

$$\sum_{c \in C} \gamma_c^{(3)} = 0,$$

$$\sum_{c \in C} c\gamma_c^{(3)} = 0,$$

where the C is the set of cohort years of birth that have been included in the analysis (see Section 2.1). The reason for this choice is that if we use least squares to fit a linear function $\phi_1 + \phi_2 c$ to $\gamma_c^{(3)}$, the constraints ensure that $\hat{\phi}_1 = 0$ and $\hat{\phi}_2 = 0$. This ensures that the fitted $\gamma_c^{(3)}$ will fluctuate around 0 and will have no discernible linear trend.

4.7 Model M7

This model is a generalization of model M6 that adds a quadratic term to the age effect. The inclusion of the quadratic term is inspired by the possible curvature identified in the logit $q(t, x)$ plots in the U.S. data. Thus,

$$\text{logit } q(t, x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \kappa_t^{(3)}((x - \bar{x})^2 - \hat{\sigma}_x^2) + \gamma_{t-x}^{(4)}.$$

Here the constant $\hat{\sigma}_x^2 = n_a^{-1} \Sigma_i (x - \bar{x})^2$ is the mean of $(x - \bar{x})^2$.

As with model M6, we have an identifiability problem, and we can switch from $\gamma_{t-x}^{(4)}$ to $\tilde{\gamma}_{t-x}^{(4)} = \gamma_{t-x}^{(4)} + \phi_1 + \phi_2(t - x - \bar{x}) + \phi_3(t - x - \bar{x})^2$ and corresponding adjustments to $\kappa_t^{(1)}$, $\kappa_t^{(2)}$, and $\kappa_t^{(3)}$, without there being an impact on the fitted values of the $q(t, x)$. This requires three constraints to prevent arbitrary choices over ϕ_1, ϕ_2, and ϕ_3. The constraints we have used here are

$$\sum_{c \in C} \gamma_c^{(4)} = 0,$$

$$\sum_{c \in C} c\gamma_c^{(4)} = 0,$$

$$\sum_{c \in C} c^2\gamma_c^{(4)} = 0.$$

Thus, if we use least squares to fit a quadratic function $\phi_1 + \phi_2 c + \phi_3 c^2$ to $\gamma_c^{(4)}$, the constraints ensure that $\hat{\phi}_1 = 0$, $\hat{\phi}_2 = 0$ and $\hat{\phi}_3 = 0$, meaning that the fitted $\gamma_c^{(4)}$ will fluctuate around 0 and will have no discernible linear trend or quadratic curvature.

4.8 Model M8

Our third generalization of the CBD model builds on our experience from fitting model M2 (see the results in Section 6). This suggested that the impact of the cohort effect $\gamma_c^{(3)}$ for any specific cohort diminishes over time (i.e., $\beta_x^{(3)}$ is decreasing with x) instead of remaining constant (i.e., $\beta_x^{(3)}$ is constant). This leads to

$$\text{logit } q(t, x) = \beta_x^{(1)}\kappa_t^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\gamma_{t-x}^{(3)},$$

where

$$\beta_x^{(1)} = 1,$$

$$\beta_x^{(2)} = (x - \bar{x}),$$

$$\beta_x^{(3)} = (x_c - x)$$

for some constant parameter x_c to be estimated. This results in

$$\text{logit } q(t, x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}(x_c - x).$$

To avoid identifiability problems, we need to introduce one constraint:

$$\sum_{x,t} \gamma_{t-x}^{(3)} = 0.$$

Each of models M6–M8 is an extension of model M5 with some allowance for the cohort effect. Consequently models M5–M8 can be described as members of the family of generalized CBD-Perks models.

4.9 Philosophical Similarities and Differences between Models

We comment here briefly on the philosophical similarities and differences behind the structural assumptions of the models. The reader can use these to help form preferences over the models in the case where quantitative and qualitative criteria do not permit a clear-cut distinction between them.

All the models M1–M3 and M5–M8 share the same underlying assumption that the age, period, and cohort effects are qualitatively different in nature and hence need to be modeled in different ways. Specifically they recognize a randomness in mortality rates at each age from one year to the next, perhaps caused by local environmental factors (such as a winter influenza outbreak or a summer heat-

wave), which is not observed between adjacent ages. In contrast, the P-splines model, M4, assumes that there is smoothness in the underlying mortality surface in the period effects as well as in the age and cohort effects. Models M5–M8 differ from M1–M3 in that the former assume a functional relationship (and hence smoothness) between mortality rates over adjacent ages within the same year.

Depending on one's personal beliefs about the underlying randomness in the age, period, and cohort effects, one might attach greater weight to models that are aligned with these beliefs. For example, if one believes that there should be an underlying smoothness in mortality rates between adjacent ages within the same year, that there is randomness in mortality rates between cohorts, and randomness in mortality rates from one year to the next, then greater weight might be placed on models M6–M8.

5. ALTERNATIVE WAYS OF EVALUATING AND COMPARING MORTALITY MODELS

We can evaluate and compare the eight models using a range of criteria.

5.1 Desirable Properties of the Theoretical Models

First, we can compare the main features of the models against a set of desirable model properties as listed below and derived from CMI (2005, 2006) and Cairns, Blake, and Dowd (2006a). Table 2 assesses each model against these properties:

- Parsimony: Other things being equal, a model with fewer parameters is preferable to a model with more. Each of the models has a large number of parameters (see Table 3), so none could be described as parsimonious in any absolute sense. However, some models are more parsimonious than others, having fewer parameters.

Table 2
Desirable Model Properties

Model	M1	M2	M3	M4	M5	M6	M7	M8
Parsimony	?	?	?	?	?	?	?	?
Transparency	?	?	?	?	?	?	?	?
Ability to generate sample paths	Y	Y	Y	N	Y	Y	Y	Y
Incorporation of cohort effects	N	Y	Y	Y	N	Y	Y	Y
Nontrivial correlation structure	N	N?	N?	N	Y	Y	Y	Y

Notes: The table shows whether each model satisfies each of the stated criteria. Where a criterion cannot be answered with a simple Y(es) or N(o), the question mark indicates that the model lies somewhere in the middle, with further comments in the main text.

Table 3
England and Wales Males Aged 60–89 and Years 1961–2004

Model	Maximum Log-Likelihood	Effective Number of Parameters	BIC (Rank)
M1	−8,912.7	102	−9,275.8 (6)
M2	−7,735.6	203	−8,458.1 (3)
M3	−8,608.1	144	−9,120.6 (5)
M4	−9,245.9	74.2	−9,372.9 (7)
M5	−10,035.5	88	−10,348.8 (8)
M6	−7,922.3	159	−8,488.3 (4)
M7	−7,702.1	202	−8,421.1 (2)
M8	−7,823.7	161	−8,396.8 (1)

Notes: Maximum likelihood, effective number of parameters estimated, and Bayes Information Criterion (BIC) for each model. The effective number of parameters takes account of the constraints on parameters or the effect of the penalty functions in the case of model M4. For M4 with $dx = 4$ and $dt = 4$ interknot distances, the BIC is optimized over the penalty weights.

Of course, other things are not equal including the value of the maximum likelihood, and so a balance between statistical goodness of fit and parsimony is achieved through the use of the Bayes Information Criterion (see Section 6.1.1).

In Table 2 we have placed question marks next to all eight models in respect to parsimony. To record a ''N(o)'' would indicate that a model was unnecessarily complex, which we do not think is the case with any model, whereas a ''Y(es)'' would indicate that a model had a small number of parameters, which is also not the case with any model.

- Transparency: How much of the model and its output is treated as a ''black box''? It is important that the user of a given model understands the model and all of its workings, to avoid the danger that the model might be used inappropriately. At the same time, a model that is transparent to one person might not be transparent to someone else. Thus, rather than impose our own judgements here, we leave question marks next to all models.
- Whether the model has the ability to generate sample paths:[10] The process uncertainty that generates random sample paths is necessary for tasks such as pricing longevity-linked financial instruments and developing related hedging strategies (see, e.g., Blake et al. 2006). Only the P-splines model, M4, fails on this criterion. M4 assumes that there is an underlying smoothness to the mortality surface and that the only uncertainty in forecasts (which can be substantial) is due to model and parameter uncertainty.
- Incorporation of cohort effects: This is important if we believe that cohort effects are present and need to be allowed for (as discussed in Section 2.1).
- Ability to produce a nontrivial correlation structure between the year-on-year changes in mortality rates at different ages.[11] The correlation structure is described as trivial when there is perfect correlation between changes in mortality rates at different ages from one year to the next. This is the case for model M1, for example, where there is a single time series process $\kappa_t^{(2)}$. For models M2 and M3, we also have perfect correlation (for the same reason) at all ages except at the youngest age, where there is potentially additional randomness arising from the arrival of a new cohort with an unknown cohort effect. Models M5–M8 allow for a nontrivial correlation structure because they all have more than one underlying period risk factor.

5.2 Desirable Properties of the Fitted Models

Important additional properties can be evaluated only when we fit the model to the data:

- A good model should provide a good fit to the historical data, produce testable predictions that are consistent with the data, and rank well against other models by criteria such as the BIC.
- Parameter estimates should be robust relative to the range of data employed. For example, if we use EW data for 1981–2004, we would hope to see similar parameter estimates to those found using data for 1961–2004.
- Where a model is used for forecasting future rates of mortality, individual scenarios should exhibit ''biologically reasonable'' behavior. For example, forecast mortality rates should both be increasing in age in any given forecast year and change smoothly over time (see, e.g., Cairns, Blake, and Dowd 2006a). The quantitative and qualitative forecasting properties of models are considered only briefly in this paper (Sections 6.3 and 7.3). These issues are considered in detail elsewhere (Cairns et al. 2008; Dowd et al. 2008a,b).

[10] We refer here to sample paths for the underlying (and unobservable) death rates $m(t, x)$. A different type of sample path can be constructed when we look at crude (observable) death rates under the Poisson model: $m(t, x) = D(t, x)/E(t, x)$, where $D(t, x) \sim Po(m(t, x)E(t, x))$.

[11] Statistical analysis of mortality rates points to changes in the $m(t, x)$ at different ages being imperfectly correlated. The existence of a nontrivial correlation structure implies, for example, that hedging of longevity-linked liabilities, such as annuities, requires more than one hedging instrument. See also Cairns, Blake, and Dowd (2008) for further discussion of the evidence for imperfect correlation.

6. ANALYSIS OF MODELS USING ENGLAND AND WALES DATA

6.1 Model Selection Criteria

In this section we conduct formal model comparisons based on England and Wales data. For each model we estimate (as appropriate) the $\beta_x^{(i)}$, $\kappa_t^{(i)}$, and $\gamma_c^{(i)}$ for each factor, i, age, x, year, t, and cohort, $c = t - x$, by maximizing the log-likelihood function. Estimates of the $\beta_x^{(i)}$, $\kappa_t^{(i)}$, and $\gamma_c^{(i)}$ are plotted in Figures 3–9.

Values for the maximum likelihood, effective number of parameters (or degrees of freedom in estimation), and the Bayes Information Criterion (BIC) for each model are given in Table 3.

6.1.1 Bayes Information Criterion

If one simply compares the maximum likelihoods attained by each model, then it is natural for models with more parameters to fit the data "better." Such improvements are almost guaranteed if models are nested: if one model is a special case of another, then the model with more parameters will typically have a higher maximum likelihood, even if the true model is the model with fewer parameters.

To avoid this problem, we need to penalize models that are overparameterized. Specifically, for each parameter that we add to a model, we need to see a "significant" improvement in the maximum likelihood rather than just an increase of any size. A number of such penalties have been proposed. Here we focus on the Bayes Information Criterion (BIC; see, e.g., Hayashi 2000; Cairns 2000).

A key point about the use of the BIC is that it provides us with a mechanism for striking a balance between quality of fit (which can be improved by adding in more parameters) and parsimony. A second, and equally important point about the BIC, is that it allows us to compare models that are not necessarily nested. For example, M1 and M3 are nested within M2, but M1 is not nested within M3 and vice versa. A final point is that the BIC makes no assumptions about "prior" model rankings: that is, all models have equal status in terms of how we rank them. In contrast, hypothesis tests start from a null hypothesis that favors one specific model over the others.

The BIC for model r is defined as[12]

$$BIC_r = l(\hat{\phi}_r) - \frac{1}{2}\nu_r \log N,$$

where ϕ_r is the parameter vector for model r, $\hat{\phi}_r$ is its maximum likelihood estimate, $l(\hat{\phi}_r)$ is the maximum log likelihood, N is the number of observations (not counting those cells that have been excluded from the analysis), and ν_r is the effective number of parameters being estimated.[13]

The models can then be ranked, with the top model having the highest BIC. Values for the BIC are given in Table 3, and we see that model M8 comes out on top in the BIC rankings with M7 second.[14]

6.1.2 Standardized Residuals

A second model selection criterion relates to the standardized residuals:

$$Z(t, x) = \frac{D(t, x) - E(t, x)\hat{m}(t, x; \hat{\phi})}{\sqrt{E(t, x)\hat{m}(t, x; \hat{\phi})}}. \tag{6.1}$$

Embedded within our modeling hypothesis is an assumption that the death counts are independent Poisson random variables for each age and year (Section 3). If our hypothesis is true, therefore, the

[12] Some authors define the BIC as $-2l(\hat{\phi}_r) + \nu_r \log N$: that is, -2 times our definition. The two definitions are equivalent and have no impact on our analysis.

[13] For example, model M1 requires estimates for 30 values of $\beta_x^{(1)}$, 30 values of $\beta_x^{(2)}$, and 44 values of $\kappa_t^{(2)}$, totaling 104, but we then deduct 2 from this total to reflect the two constraints $\Sigma_t \kappa_t^{(2)} = 0$ and $\Sigma_x \beta_x^{(2)} = 1$. For the P-splines model the concept of the effective number of parameters is more abstract; see Currie, Durban, and Eilers (2004) and references therein for further details.

[14] This is no accident. We searched across a range of new functional forms to achieve this outcome (see also note 1).

Table 4
Sample Variances of Standardized Residuals for Models M1–M8

	Model							
	M1	M2	M3	M4	M5	M6	M7	M8
$Var[Z(t, x)]$	4.1	2.2	3.7	4.3	5.9	2.4	2.1	2.3

standardized residuals (eq. [6.1]) will be approximately i.i.d. standard normal random variables.

Models that have higher likelihood have a lower variance of the standardized residuals. However, for each model discussed in this paper, the variance of the standardized residuals is significantly greater than one (see Table 4). This ''overdispersion'' seems to be a general feature of mortality data in many countries. A possible source of this overdispersion lies in the fact that the exposures data are estimated. We conjecture that this overdispersion does not have a significant impact on our estimates of the future dynamics of the underlying mortality rates $q(t, x)$. Nevertheless, a Poisson model might underestimate the future variability of the actual death rates relative to the true underlying rates.

A simple means of considering the validity of the i.i.d. assumption of the $Z(t, x)$ is to look at the pattern of positive and negative standardized residuals (see Koissi, Shapiro, and Högnäs 2006): the pattern should be random under the i.i.d. hypothesis. For models M1, M3, and M5 (see Fig. 2), the plots of the residuals show a strong clustering of positives and negatives. Models M1 and M5 do not incorporate a cohort effect, and there are diagonal clusters of positive and negative residuals: this provides strong evidence for the existence of a cohort effect. M6 also shows some clustering, but much less than M1, M3, and M5. The standardized residuals in M4, at first glance, look reasonably random, but closer inspection reveals distinct vertical bands that suggest that there is a genuine random period effect that is being smoothed out too much under M4. M2, M7, and M8 all look reasonably random, and so all pass the test on a visual inspection.[15]

6.1.3 Comparison of Nested Models

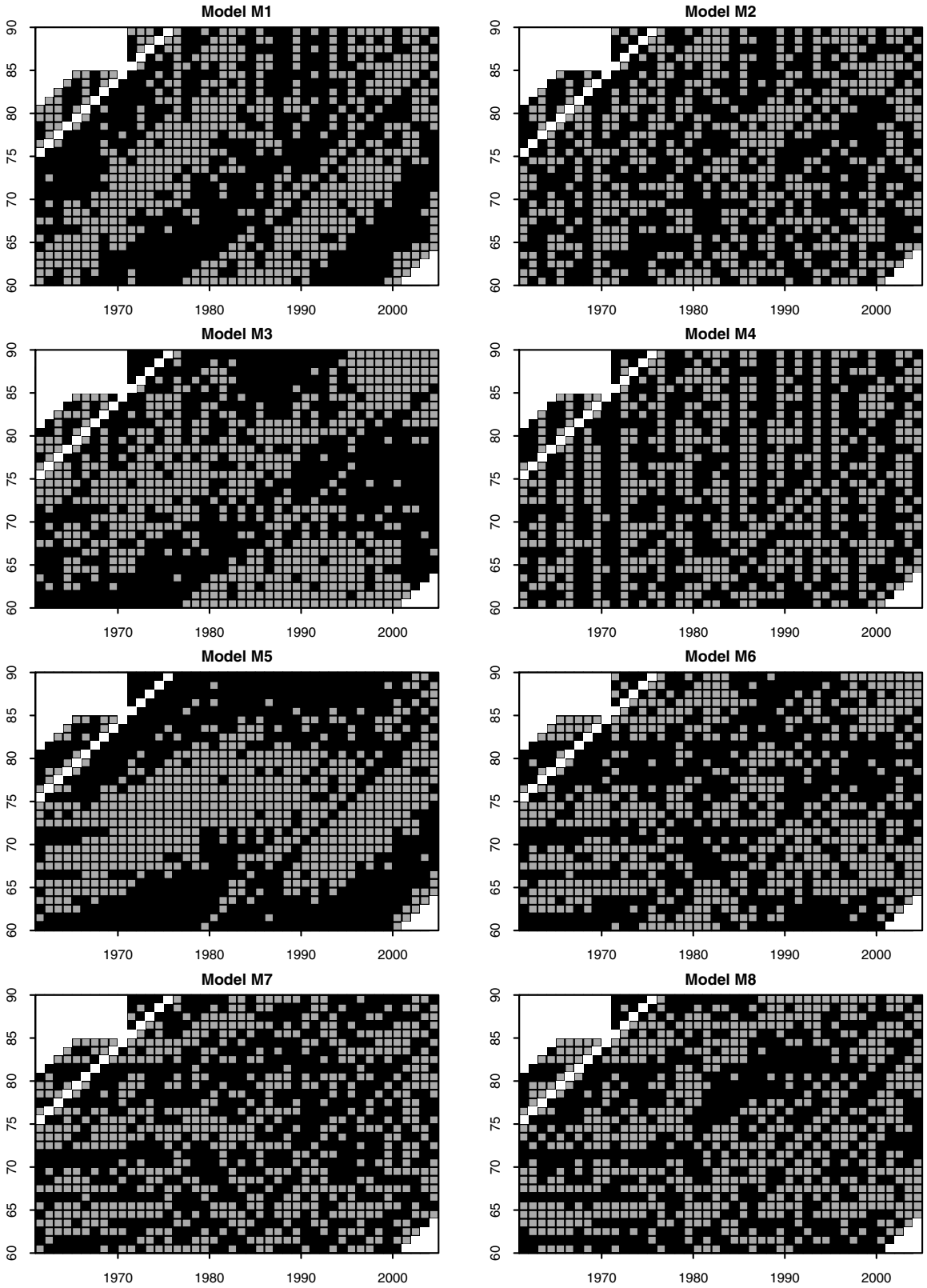
Some models are nested within one of the others: that is, they are special cases of more general models. For example, model M1 is nested within model M2, being a special case of M2 with $\beta_x^{(3)} = 0$ for all x, and $\gamma_c^{(3)} = 0$ for all $c = t - x$. In such circumstances we can use the likelihood ratio test to test the null hypothesis that the nested or restricted model is the correct model versus the alternative hypothesis that the more general model is correct. For the nested model, let $\hat{l}_1$ be the maximum log likelihood for model M1 and $\hat{l}_2$ be the maximum likelihood for model M2. Model M1 requires the estimation of $\nu_1 = 102$ parameters, while M2 requires $\nu_2 = 203$. The likelihood-ratio test statistic is $2(\hat{l}_2 - \hat{l}_1)$. If the null hypothesis is true, this should have approximately a chi-squared distribution with $\nu_2 - \nu_1$ degrees of freedom (d.f.). Thus we reject the null hypothesis in favor of the more general model if the test statistic is too large: specifically, if $2(\hat{l}_2 - \hat{l}_1) > \chi^2_{\nu_2-\nu_1,\alpha}$, where α is the significance level. Alternatively, we can calculate the p-value for this test as $p = 1 - \chi^2_{\nu_2-\nu_1}{}^{-1}(2(\hat{l}_2 - \hat{l}_1))$.

The eight models considered here include seven nested pairs. Each pair is considered in Table 5. In each case, the null hypothesis is rejected overwhelmingly in favor of the more general model. These results support our earlier findings based on the BIC. Additionally, the decisive rejection of models M1 and M5, in particular, gives a clear indication that the cohort effect is a key feature of EW males mortality data.

[15] Complementing these plots, one can plot standardized residuals against age, year of observation, and year of birth as in RH (2006). We do not include such plots here, but simply note that these plots reveal the same information, but in different ways, to Table 4 and Figure 2.

Figure 2

England and Wales Males: Standardized Residuals $Z(t, x)$ for Models M1–M8



Notes: Gray cells mean $Z(t, x) > 0$, black cells mean $Z(t, x) < 0$, and white cells mean the cell was excluded from the analysis.

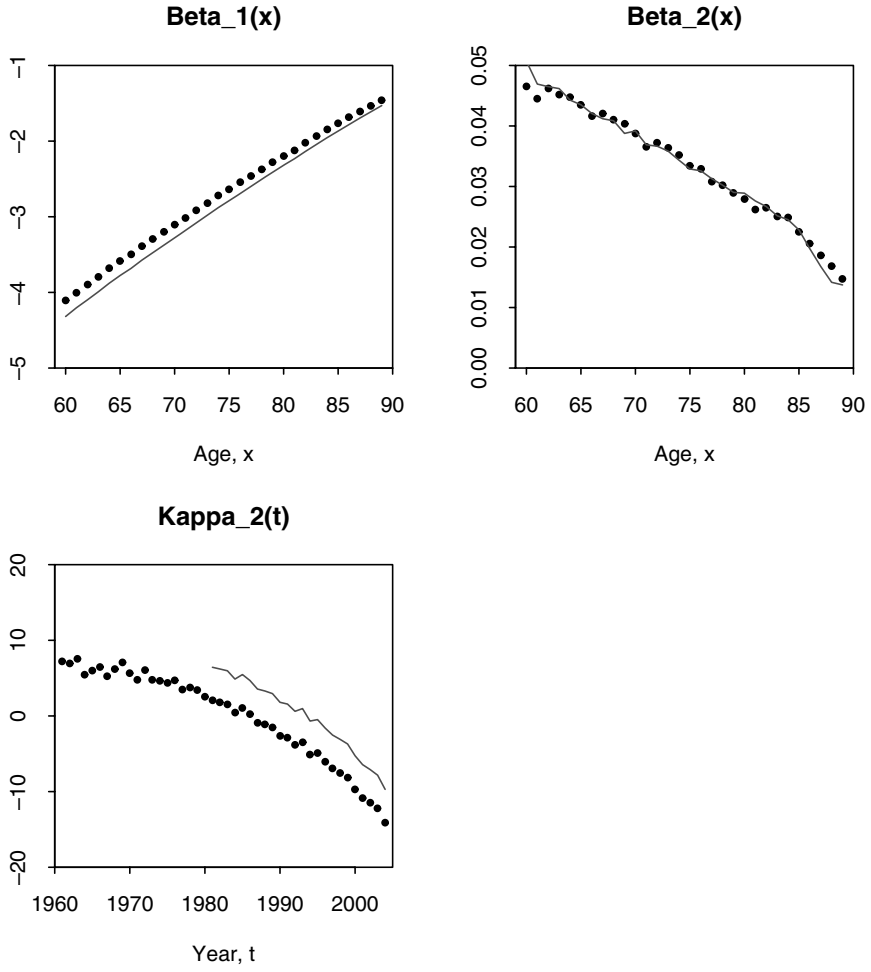
Table 5
England and Wales Data: Likelihood Ratio Test Results for Various Pairs of General and Nested Models

H_0: Restricted Model	H_1: General Model	LR Test Statistic	d.f.	p-Value
M1	M2	2,354.3	101	<0.000001
M3	M2	1,745.0	59	<0.000001
M5	M6	4,226.5	71	<0.000001
M5	M7	4,666.8	114	<0.000001
M6	M7	440.3	43	<0.000001
M5	M8	4,423.7	74	<0.000001
M6	M8	197.2	2	<0.000001

6.2 Parameter Estimates and Their Robustness

In Figures 3–9, we have plotted the maximum-likelihood estimates for the various parameters in all models, except M4, using EW males data, aged 60–89.[16] In this section we will focus on the parameter

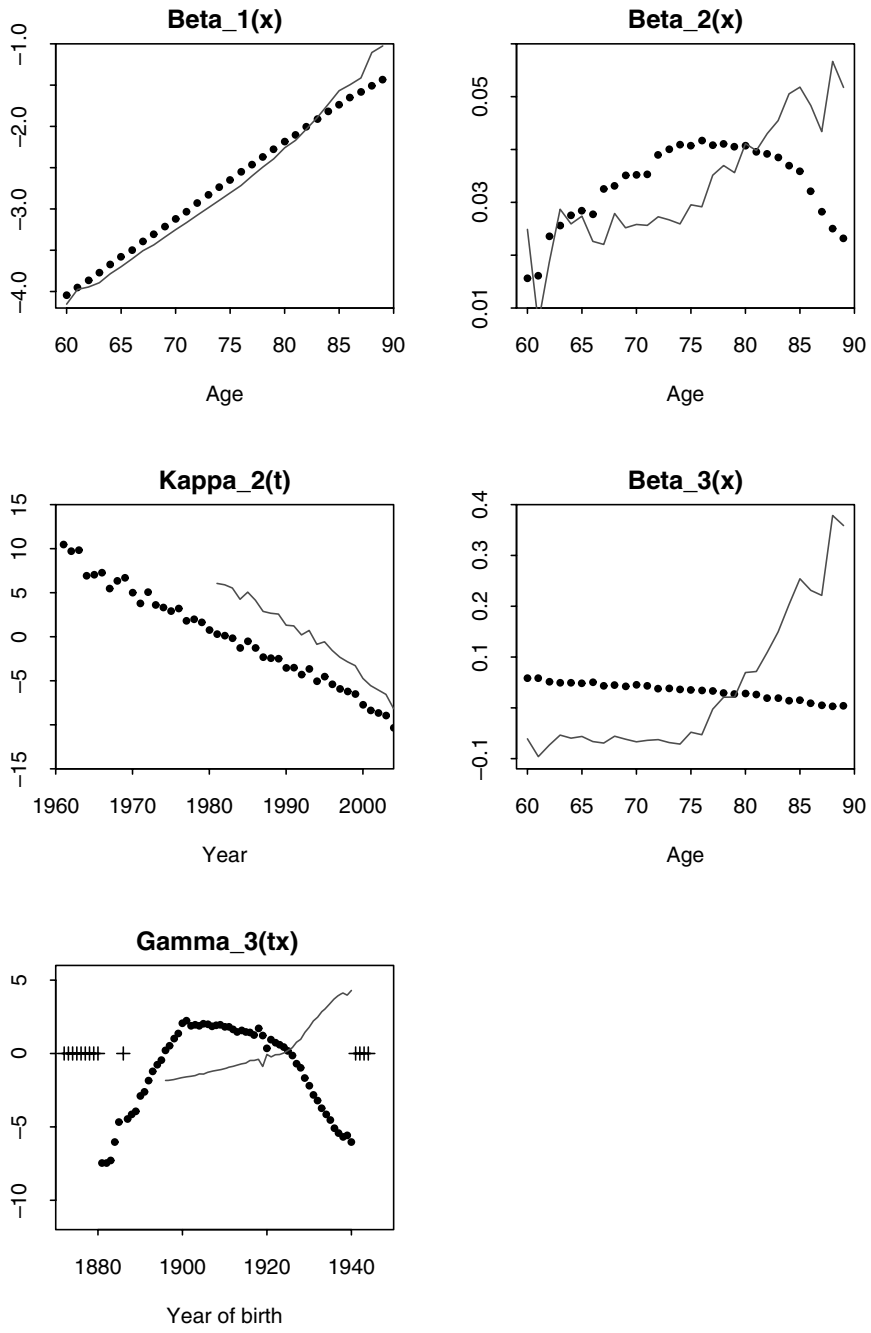
Figure 3
England and Wales Data: Parameter Estimates for Model M1



Notes: Derived from (*a*) data from 1961–2004 (*dots*) or (*b*) data from 1981–2004 (*solid lines*).

[16] See Section 2.1 for exclusions.

Figure 4
England and Wales Data: Parameter Estimates for Model M2

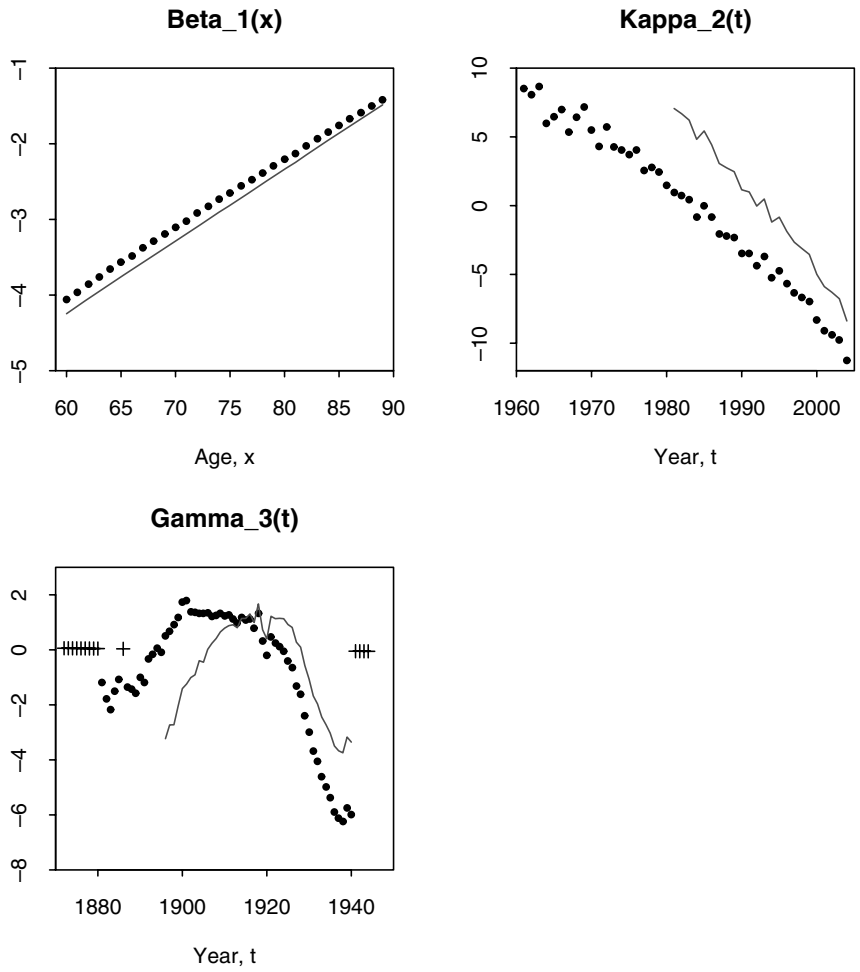


Notes: Derived from (*a*) data from 1961–2004 (*dots*) or (*b*) data from 1981–2004 (*solid lines*). Crosses in the bottom left plot correspond to excluded cohorts.

estimates based on data from the period 1961 to 2004, represented by dots in the figures.

For those models that incorporate a cohort parameter, we can see a distinctive cohort effect. In model M2, for example, we can see from the second kink in $\gamma^{(3)}_{t-x}$ that cohort mortality was falling at a faster rate for males born after 1920. The same feature can be seen in models M3 and M6.

Figure 5
England and Wales Data: Parameter Estimates for Model M3



Notes: Derived from (*a*) data from 1961–2004 (*dots*) or (*b*) data from 1981–2004 (*solid lines*). Crosses in the bottom left plot correspond to excluded cohorts. $\beta_x^{(2)} = \frac{1}{30}$ and $\beta_x^{(3)} = \frac{1}{30}$.

For models M7 and M8, the cohort effect follows a different pattern. In M7 part of the cohort effect has been substituted by the additional quadratic age effect. The M8 cohort effect seems to follow a similar pattern except for the fact that it has been tilted slightly.

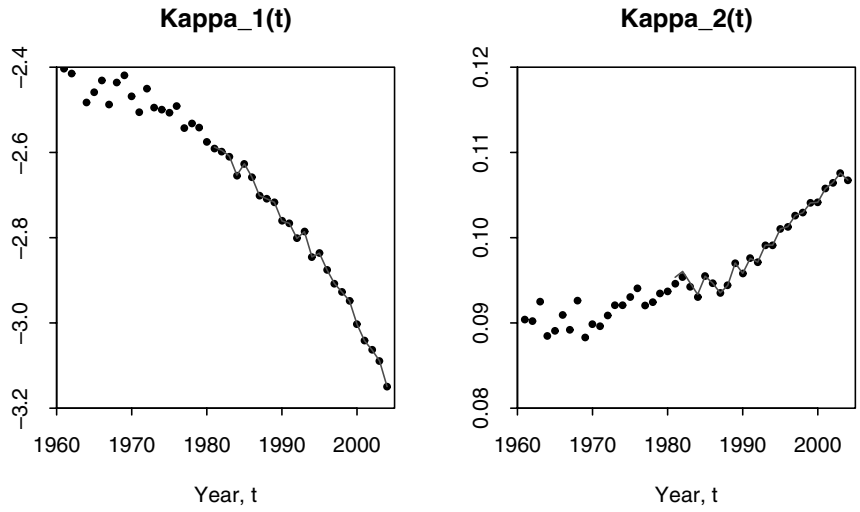
6.2.1 Robustness

An important property of a model is the robustness of its parameter estimates relative to changes in the period of data used to fit a given model.

For each model, except M4, we have plotted (Figs. 3–9) parameter estimates based on data from 1961 to 2004 (represented by dots) and from 1981 to 2004 (solid lines). We focus our comments here on the four highest BIC-ranked models M2, M6, M7, and M8. The plots reveal that, out of the four models, M7 seems to be the most robust relative to changes in the period of data used: that is, the parameter estimates hardly change even when we use a much shorter data period.

M2, on the other hand, seems to produce results that lack robustness, because the parameter estimates jump to a qualitatively quite different solution when we use less data. For example, consider the $\beta_x^{(3)}$ plot in Figure 4. Here $\beta_x^{(3)}$ is strictly positive and declining when we use data from 1961 to 2004. In contrast, for the 1981–2004 data, $\beta_x^{(3)}$ is flat and negative initially, but then becomes positive and

Figure 6
England and Wales Data: Parameter Estimates for Model M5



Notes: Derived from (*a*) data from 1961–2004 (*dots*) or (*b*) data from 1981–2004 (*solid lines*). $\beta_x^{(1)} = 1$ and $\beta_x^{(2)} = x - \bar{x}$.

increases steeply: a shape that cannot, qualitatively, be reconciled (e.g., by changing the identifiability constraint, eq. [4.1]) with the 1961–2004 shape for $\beta_x^{(3)}$. This lack of robustness brings into question the reliability of any projections made using M2. Reinforcing this concern, CMI (2007, Section 7) observed a similar lack of robustness in M2 when different age ranges are used.

M8 appears reasonably robust, and the differences that we do see are, in fact, consequences of the constraint that $\Sigma_{t,x}\ \gamma_{t-x}^{(3)} = 0$ when we are summing over different years. However, we did find that, for some datasets, the M8 fitting program was very slow to converge. We found a similar problem with M2 and put this down to the possible existence of multiple maxima in the likelihood function and the consequential risk of parameter instability. In our extensive testing, we found no such problems with M1, M3, M5, M6, or M7. M6 also appears reasonably robust, and again the bigger differences that we see are due to the identifiability constraints being applied over a different range of years.

The question of robustness is explored further and at length elsewhere (Cairns et al. 2008; Dowd et al. 2008a,b).

6.3 Model Forecasting Properties

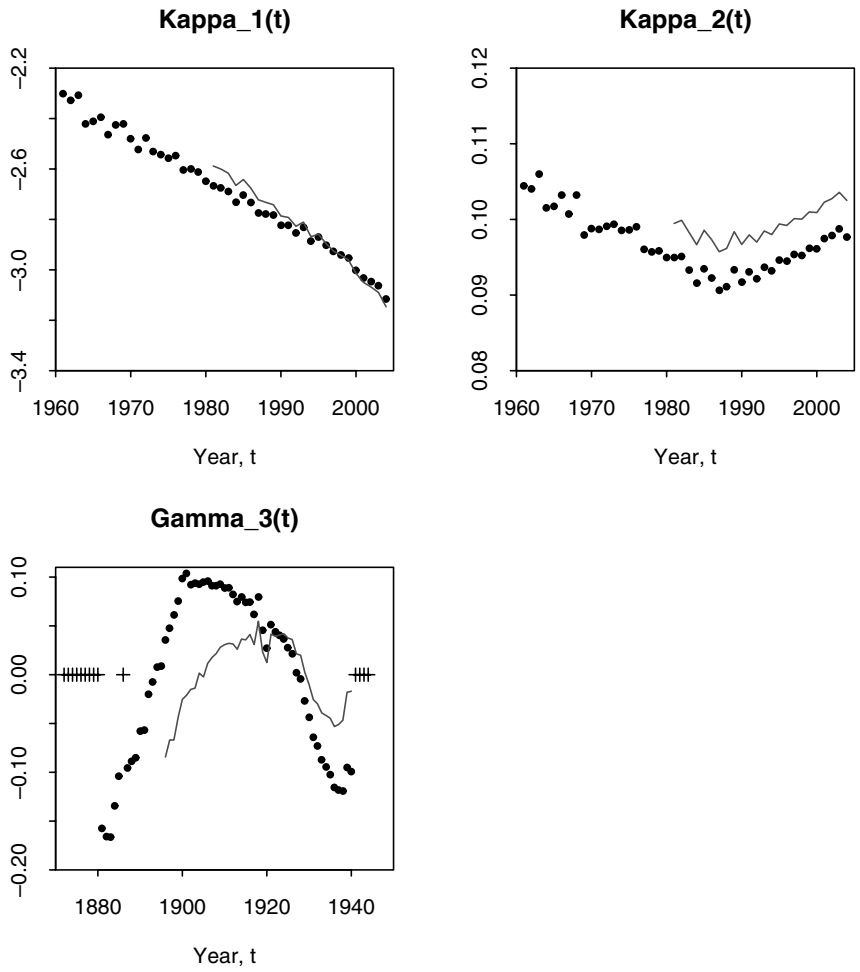
We will now carry out a number of tests to assess the impact of model choice on key outputs associated with projections of mortality rates into the future. We focus, for illustrative purposes, on models M2, M6, M7, and M8: these are the four top-ranked models for the EW males dataset (Table 3).

6.3.1 Survivor Index Projections

We first consider the impact of model choice on projected values of the survivor index $S(t, 65)$: the proportion out of the cohort aged 65 (and still alive) in 2004 who are still alive in year $2004 + t$.[17] In Figure 10 (top), we have plotted the mean and 90% prediction interval for the survivor index $S(t, 65)$ for a cohort aged 65 in 2004. It can be seen that these forecasts are little affected by the choice between models M7 and M8. M6 is slightly different, but consistent with M7 and M8. M2 is more out of line. This is connected to the qualitatively different shape of the parameter estimates under M2

[17] Projections are based on parameters fitted to data from 1981 to 2004 and use a multivariate random walk model in simulations based on the historical estimates of the $\kappa_t^{(j)}$. Further details are given in the Appendix.

Figure 7
England and Wales Data: Parameter Estimates for Model M6



Notes: Derived from (*a*) data from 1961–2004 (*dots*) or (*b*) data from 1981–2004 (*solid lines*). Crosses in the bottom left plot correspond to excluded cohorts. $\beta_x^{(1)} = 1$, $\beta_x^{(2)} = x - \bar{x}$, and $\beta_x^{(3)} = 1$.

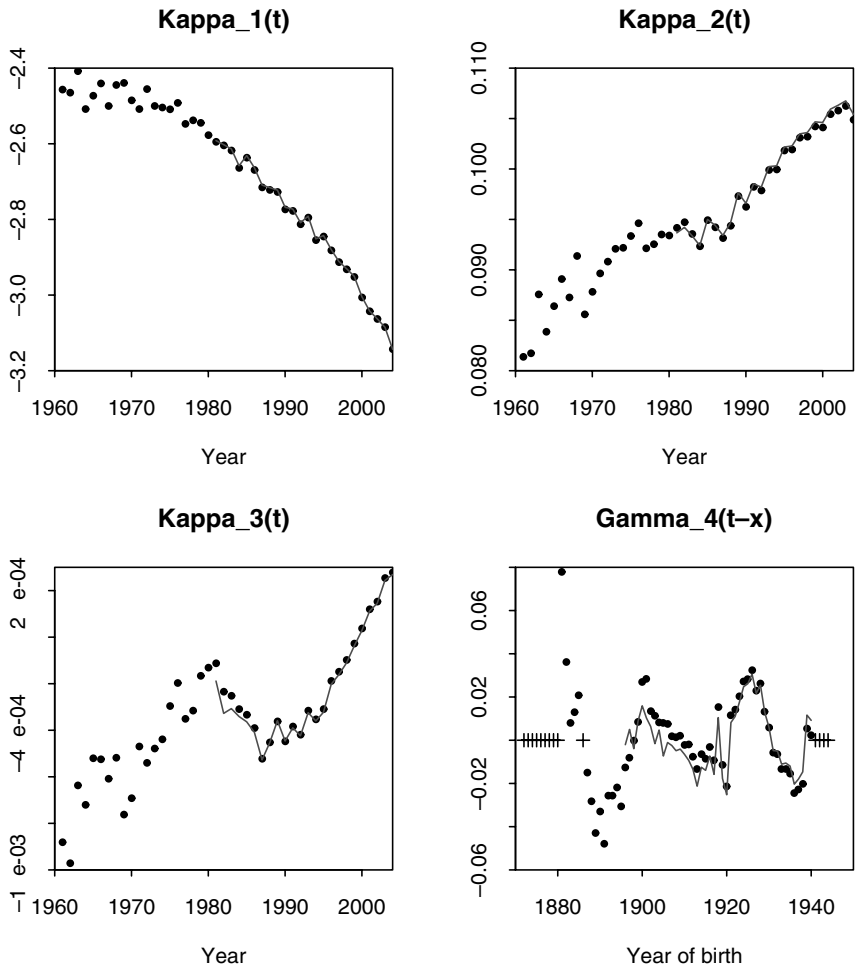
using data from 1981 to 2004 versus those using data from 1961 to 2004 (Fig. 4). Figure 11 compares projections of the survivor index for M2 using data from 1961 to 2004 (dashed lines)[18] and 1981 to 2004 (dotted lines). The former is much closer to the projections under M6, M7, and M8 in Figure 10, and the substantial differences reinforce the concern expressed in Section 6.2.1 that M2 is not robust.

In Figure 10 (bottom) we have plotted the variance of log $S(t, 65)$ over time. Again the differences are relatively small between M6, M7, and M8, although some differences emerge close to 25 years. M2 (1981–2004 data) stands out as having a much higher variance, suggesting that model risk might be an issue. A possible implication is that the choice of model might have a significant effect on quantities that rely, to some extent, on the variance of $S(t, x)$. For example, the price of a financial option that has $S(t, x)$ as its underlying quantity is strongly dependent on the variance of $S(t, x)$: everything else being equal, the higher the variance, the higher the value of the option.

We have concentrated here on the contribution of model risk to forecast uncertainty. However, it is appropriate to allow for parameter uncertainty to provide a more complete picture of the level of risk

[18] M2 is fitted to data from 1961 to 2004, but for greater consistency with the 1981–2004 data projections are based on the last 24 observations of $\kappa_t^{(2)}$ and the last 45 observations of $\gamma_c^{(3)}$.

Figure 8
England and Wales Data: Parameter Estimates for Model M7



Notes: Derived from (*a*) data from 1961–2004 (*dots*) or (*b*) data from 1981–2004 (*solid lines*). Crosses in the bottom right plot correspond to excluded cohorts. $\beta_x^{(1)} = 1$, $\beta_x^{(2)} = x - \bar{x}$, $\beta_x^{(3)} = (x - \bar{x})^2 - \hat{\sigma}_x^2$, and $\beta_x^{(4)} = 1$.

associated with, for example, future longevity-linked cash flows. The impact of parameter uncertainty is explored in more depth by Czado, Delwarde, and Denuit (2005), Cairns, Blake, and Dowd (2006b), and Dowd, Cairns, and Blake (2006), who take Bayesian approaches, and Koissi, Shapiro, and Högnäs (2006) and CMI (2005), who use a bootstrapping methodology. These analyses suggest that parameter uncertainty can significantly increase the overall level of measured uncertainty, particularly for more distant longevity-linked cash flows.

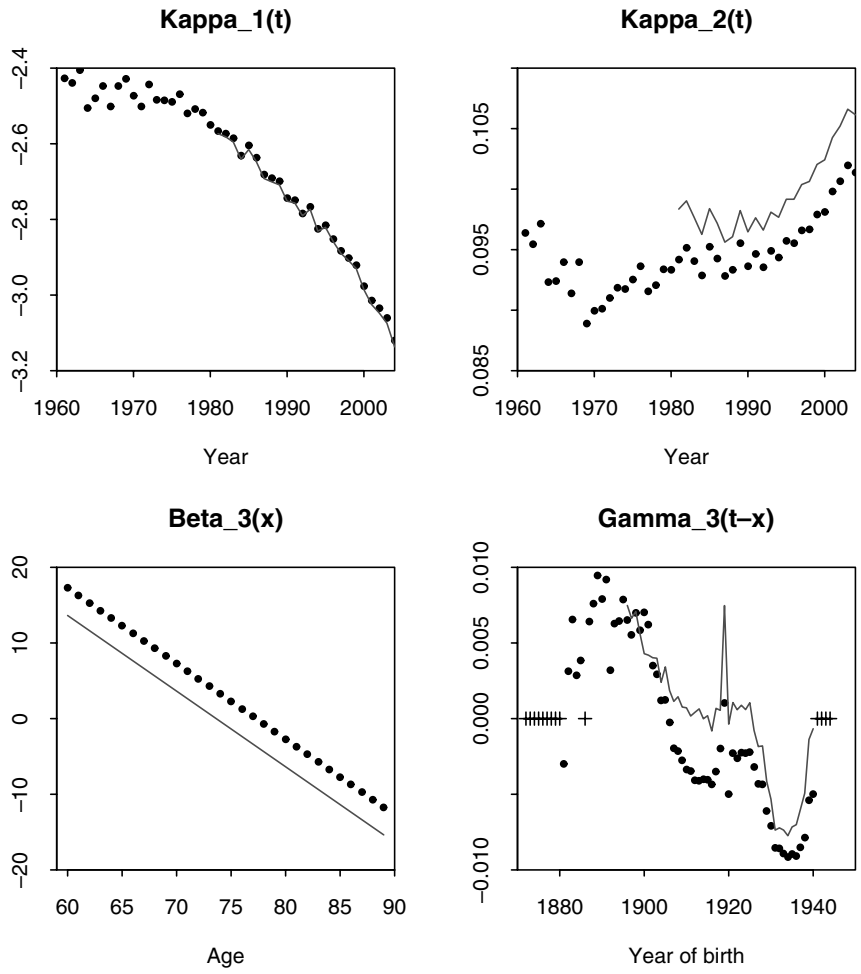
6.3.2 Projecting Annuity Values

As a second projection exercise, we calculated the value in 2004 of an annuity payable until age 90 (the maximum age in our projection model) for males aged 60, 65, 70, and 75 in 2004 at a constant interest rate of 4% p.a.:

$$a_x(2004) = \sum_{t=1}^{90-x} e^{-0.04t} E[S(t, x)].$$

Projections of $S(t, x)$ are based on EW males data from 1981 to 2004.

Figure 9
England and Wales Data: Parameter Estimates for Model M8



Notes: Derived from (*a*) data from 1961–2004 (*dots*) or (*b*) data from 1981–2004 (*solid lines*). Crosses in the bottom right plot correspond to excluded cohorts. $\beta_x^{(1)} = 1$ and $\beta_x^{(2)} = x - \bar{x}$.

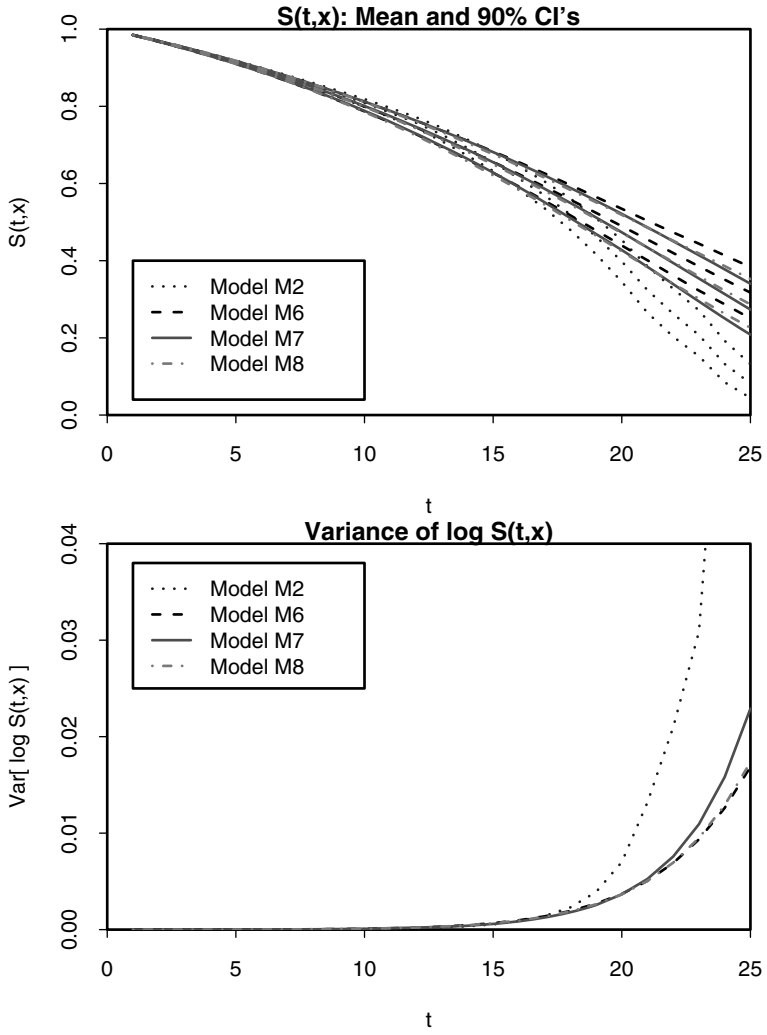
For ages 65, 70, and 75, we have already estimated the cohort effect for models M2, M6, M7, and M8 from the historical data. For age 60, $\gamma_{1944}^{(3/4)}$ is not known (i.e., the value of $\gamma_{t-x}^{(3)}$ or $\gamma_{t-x}^{(4)}$ for the cohort born in $t - x = 1944$). Because a single value is required, we adopt a subjective approach and consider two possible values for $\gamma_{1944}^{(3/4)}$ for each model with the aim of assessing the sensitivity of the results to this parameter. The two values that we consider lie at the upper and lower ends of the plausible range of outcomes for the 1944 cohort based on the historical estimates for earlier cohorts. In taking this subjective approach we are able, in a very simple, model-free way, to gauge how sensitive the value of an annuity might be to the value of the cohort effect. For M2, Figure 4 (solid line) suggests that a plausible range of values for $\gamma_{1944}^{(3)}$ is approximately 4.3 to 6.7. Similarly, under M6, Figure 7 (solid line) suggests a range of -0.14 to 0.11 for $\gamma_{1944}^{(3)}$; under M7 (Fig. 8, solid line) a range of -0.031 to 0.049 for $\gamma_{1944}^{(4)}$; and under M8 (Fig. 9, solid line) a range of -0.012 to 0.008 for $\gamma_{1944}^{(3)}$.

Values for ages 60, 65, 70, and 75 are given in Table 6. From this, we can observe the following:

- At age 60, differences are slightly larger, reflecting the inclusion of uncertainty in the value of $\gamma_{1944}^{(3)}$ in addition to model risk. The uncertainty in $\gamma_{1944}^{(3)}$ under M6 has the largest impact. From this we conclude that, although the cohort effect is statistically significant (in the sense of Table 3), it has an economically small effect (M6 excepted) on the pricing of annuities considered here.

Figure 10

England and Wales Data: *Top:* 5% and 95% Quantiles for Survivor Index *S*(*t*, 65) for Models M2, M6, M7, and M8, with Mean of *S*(*t*, 65) Running Down the Middle. *Bottom: Var*[log *S*(*t*, 65)] for Models M2, M6, M7, and M8



Notes: Projections based on parameters estimated from data from 1981–2004.

- Differences between models M6, M7, and M8 at ages 65, 70, and 75 are relatively modest.
- Values under M2 are consistently lower, reflecting the shape of the survivor curve in Figure 10.

7. Analysis of Models Using U.S. Data

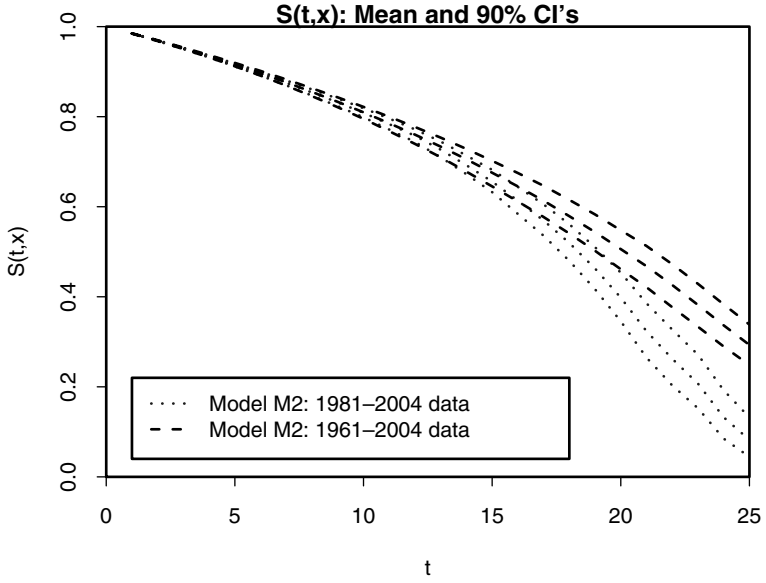
7.1 Model Selection Criteria

The U.S. males' mortality data were analyzed using models M1–M3 and M5–M8[19] using the data from 1968 to 2003 and for males aged 60–89. Over the period 1968–1979, ages 85–89 were excluded on the basis that the data at those ages in those years are not reliable.

[19] Model M4 is currently popular in the United Kingdom, but not in the United States. Our earlier analysis of the EW data did not find that M4 fitted very well in comparison with M2, M7, and M8. We have therefore not used it in our analysis of the U.S. data.

Figure 11

England and Wales Data: *Top:* 5% and 95% Quantiles for Survivor Index $S(t, 65)$ for Model M2 with Mean of $S(t, 65)$ Running Down the Middle



Notes: Projections based on parameters estimated from data from 1981–2004 (*dotted lines*) or from 1961–2004 (*dashed lines*).

For each model we estimated the $\beta_x^{(i)}$, $\kappa_t^{(i)}$, and $\gamma_c^{(i)}$ parameters by maximum likelihood using data covering the period 1968–2003. Maximum likelihoods are given in Table 7. Estimates of the parameters themselves are plotted in Figures 12–18. In these plots the dots are parameter estimates based on data from 1968 to 2003, and lines are based on data from 1980 to 2003.

7.1.1 Bayes Information Criterion

The BIC for each model is given in the final column in Table 7. In contrast with the results in Table 3 for the EW (EW) data, model M2 now comes out significantly better than the other models. However, in the subsections that follow, we will discuss graphical diagnostic tests that suggest M2 might be

Table 6

England and Wales Data: Annuity Values (Payable in Arrears until Death or Age 90) for Males of Various Ages Based on Data from 1981 to 2004

Model	$\gamma_{1944}^{(3/4)}$	Annuity Value			
		$x = 60$	$x = 65$	$x = 70$	$x = 75$
M2	$\gamma_{1944}^{(3)} = 6.7$	13.183	11.164	9.040	6.950
M2	$\gamma_{1944}^{(3)} = 4.3$	13.148	11.164	9.040	6.950
M6	$\gamma_{1944}^{(3)} = 0.11$	13.212	11.592	9.594	7.396
M6	$\gamma_{1944}^{(3)} = -0.14$	13.883	11.592	9.594	7.396
M7	$\gamma_{1944}^{(4)} = 0.049$	13.386	11.509	9.469	7.289
M7	$\gamma_{1944}^{(4)} = -0.031$	13.607	11.509	9.469	7.289
M8	$\gamma_{1944}^{(3)} = 0.008$	13.463	11.497	9.336	7.194
M8	$\gamma_{1944}^{(3)} = -0.012$	13.579	11.497	9.336	7.194

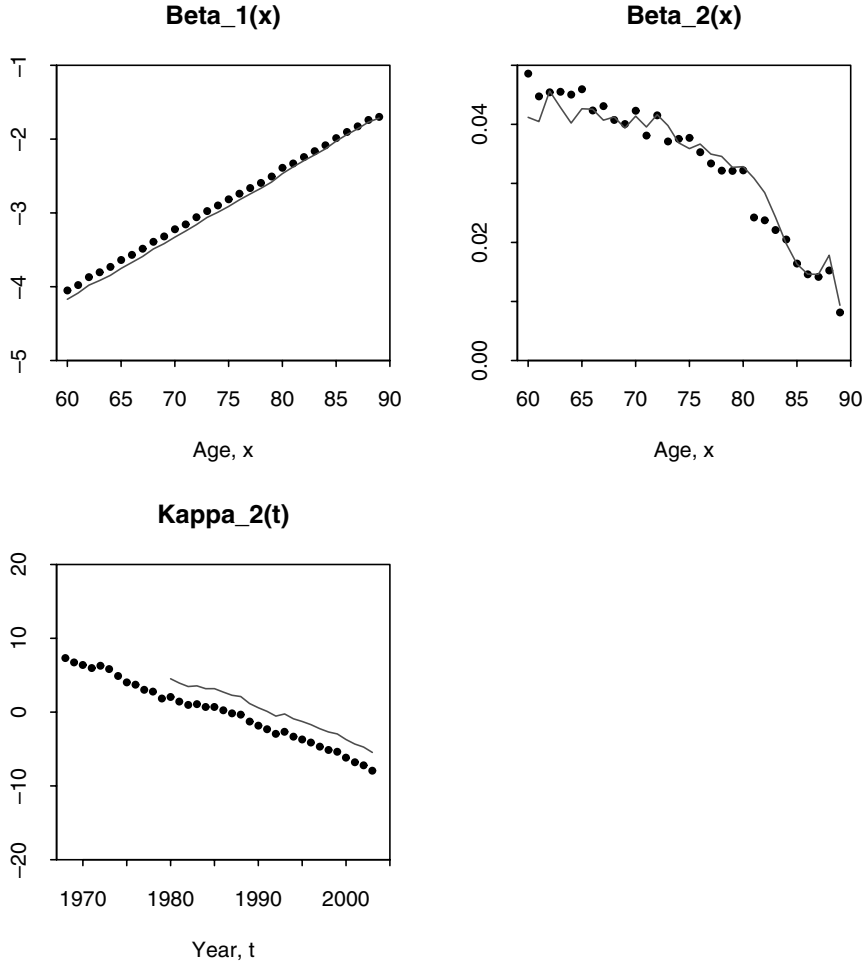
Note: Values for $\gamma_{1944}^{(3/4)}$ are estimated.

Table 7
U.S. Males Aged 60–89 and Years 1968–2003

Model	Maximum Log-Likelihood	Effective Number of Parameters	BIC (Rank)
M1	−12,265.4	94	−12,590.0 (6)
M2	−9,737.4	187	**−10,383.2 (1)**
M3	−11,854.2	128	−12,296.3 (3)
M5	−16,121.3	72	−16,370.0 (7)
M6	−11,948.4	135	−12,414.7 (5)
M7	−11,631.7	170	−12,218.9 (2)
M8	−11,841.1	137	−12,314.3 (4)

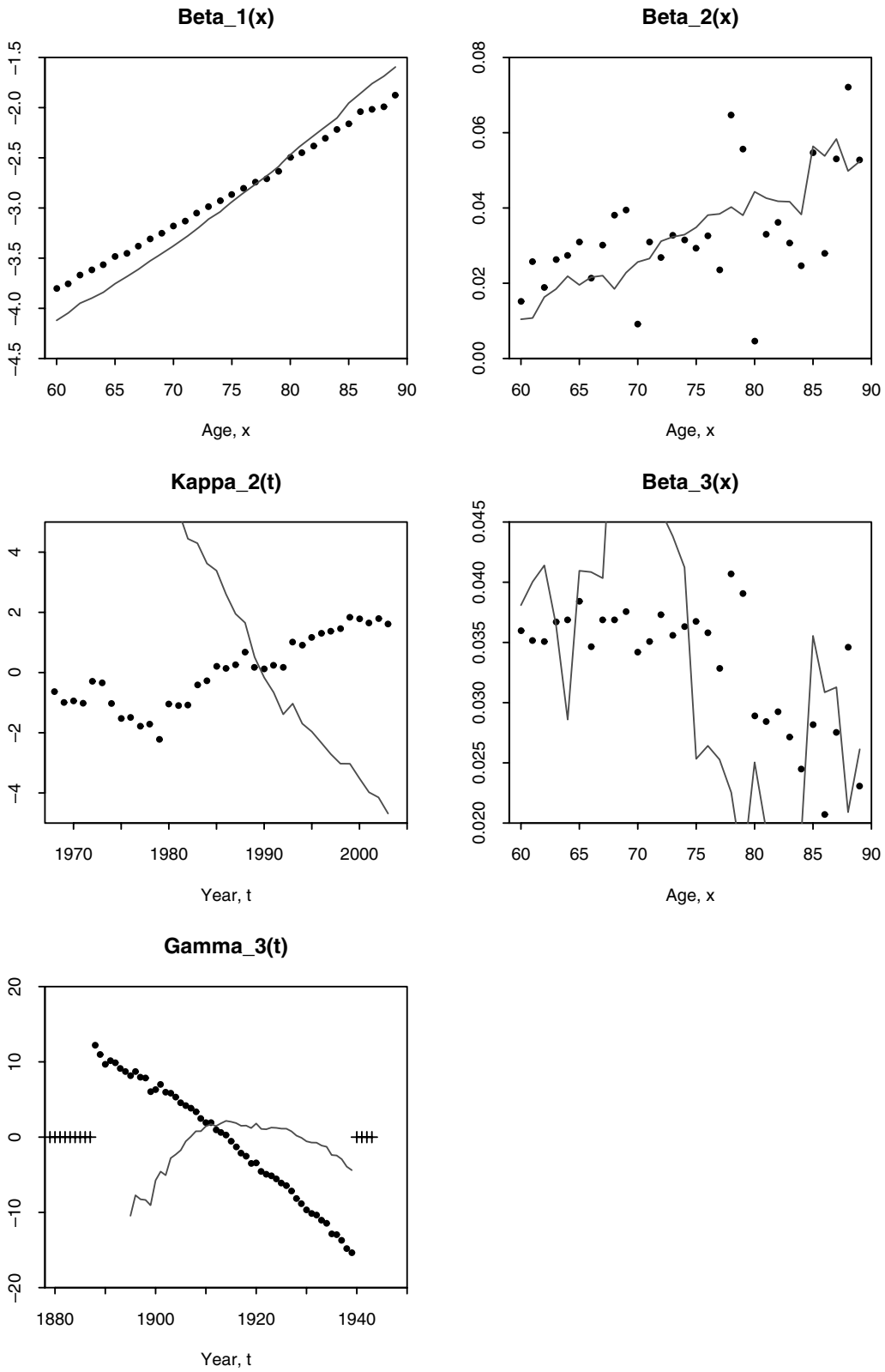
Notes: Maximum likelihood, effective number of parameters estimated, and Bayes Information Criterion (BIC) for each model. The effective number of parameters takes account of the constraints on parameters.

Figure 12
U.S. Data: Parameter Estimates for Model M1



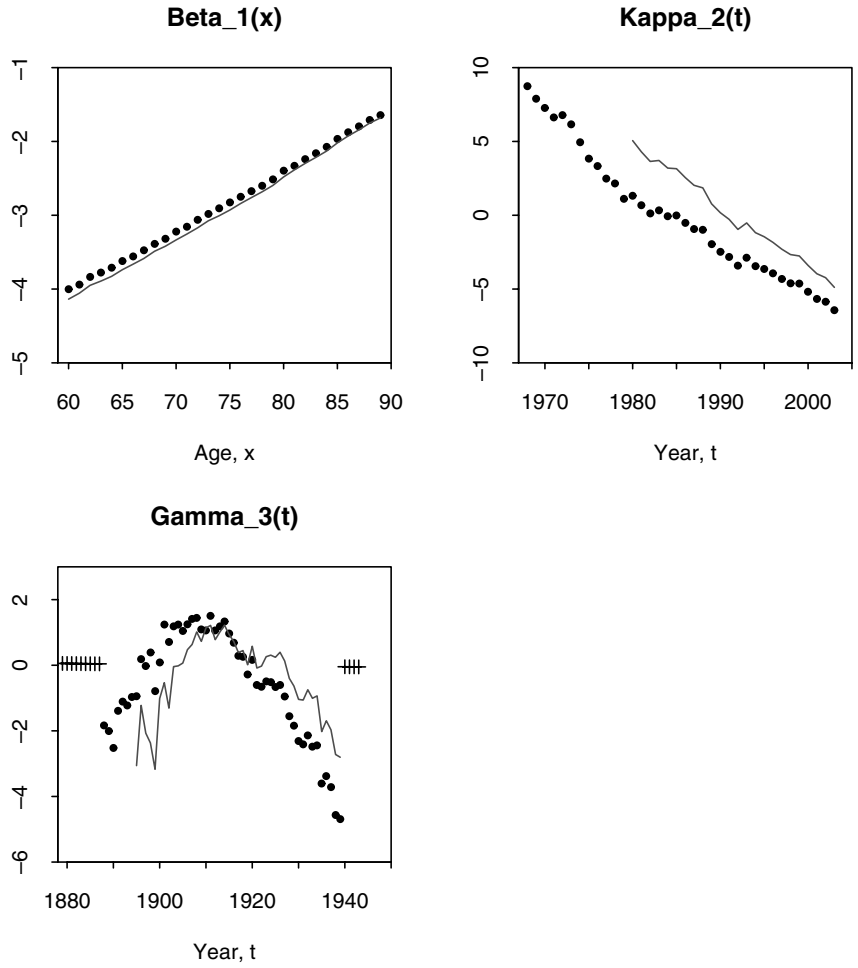
Notes: Derived from (*a*) data from 1968–2003 (*dots*) or (*b*) data from 1980–2003 (*solid lines*).

Figure 13
U.S. Data: Parameter Estimates for Model M2



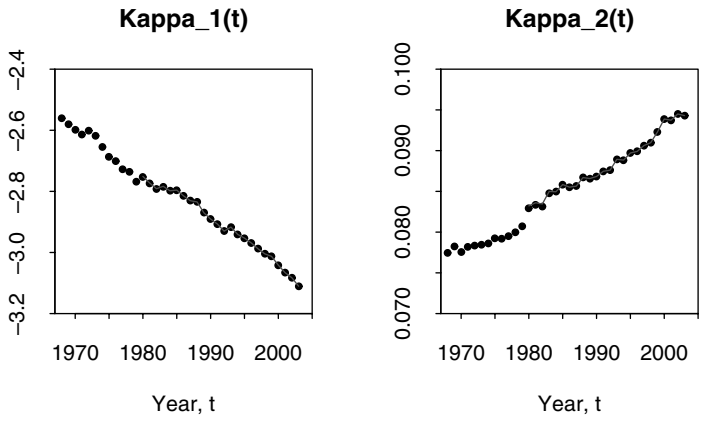
Notes: Derived from (*a*) Data from 1968–2003 (*dots*), (*b*) Data from 1980–2003 (*solid lines*). Crosses in the bottom left plot correspond to excluded cohorts.

Figure 14
U.S. Data: Parameter Estimates for Model M3



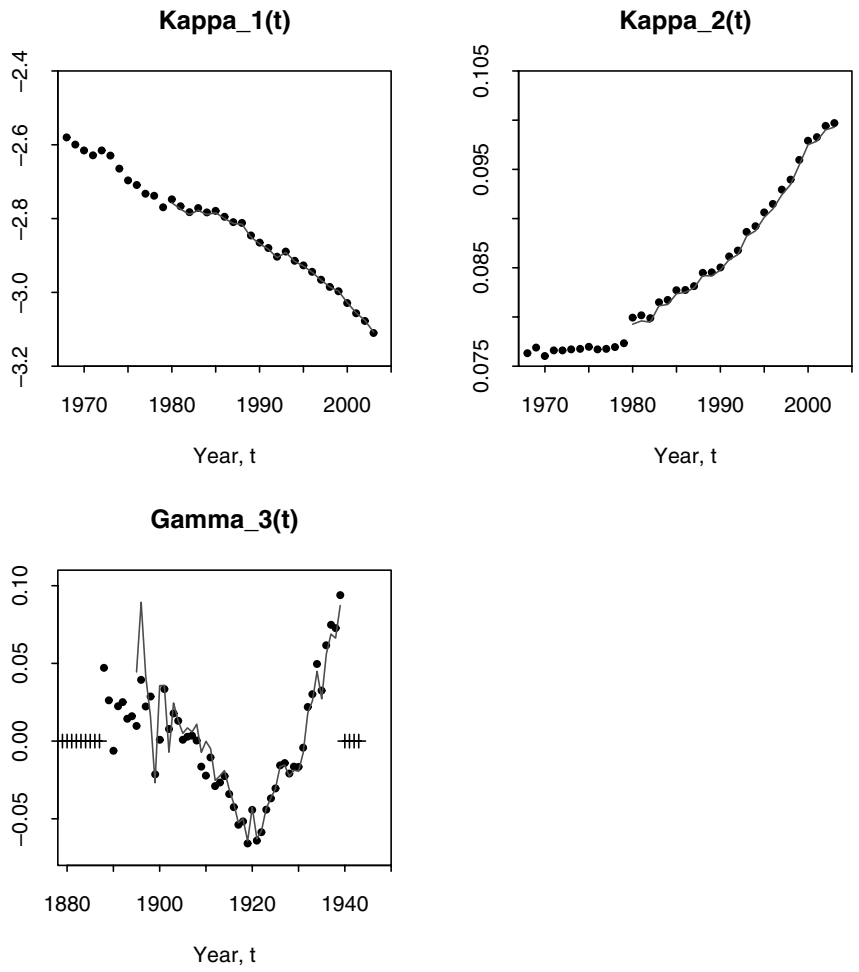
Notes: Derived from (*a*) Data from 1968–2003 (*dots*), (*b*) Data from 1980–2003 (*solid lines*). Crosses in the bottom left plot correspond to excluded cohorts.

Figure 15
U.S. Data: Parameter Estimates for Model M5



Notes: Derived from (*a*) Data from 1968–2003 (*dots*), (*b*) Data from 1980–2003 (*solid lines*).

Figure 16
U.S. Data: Parameter Estimates for Model M6



Notes: Derived from (*a*) Data from 1968–2003 (*dots*), (*b*) Data from 1980–2003 (*solid lines*). Crosses in the bottom left plot correspond to excluded cohorts.

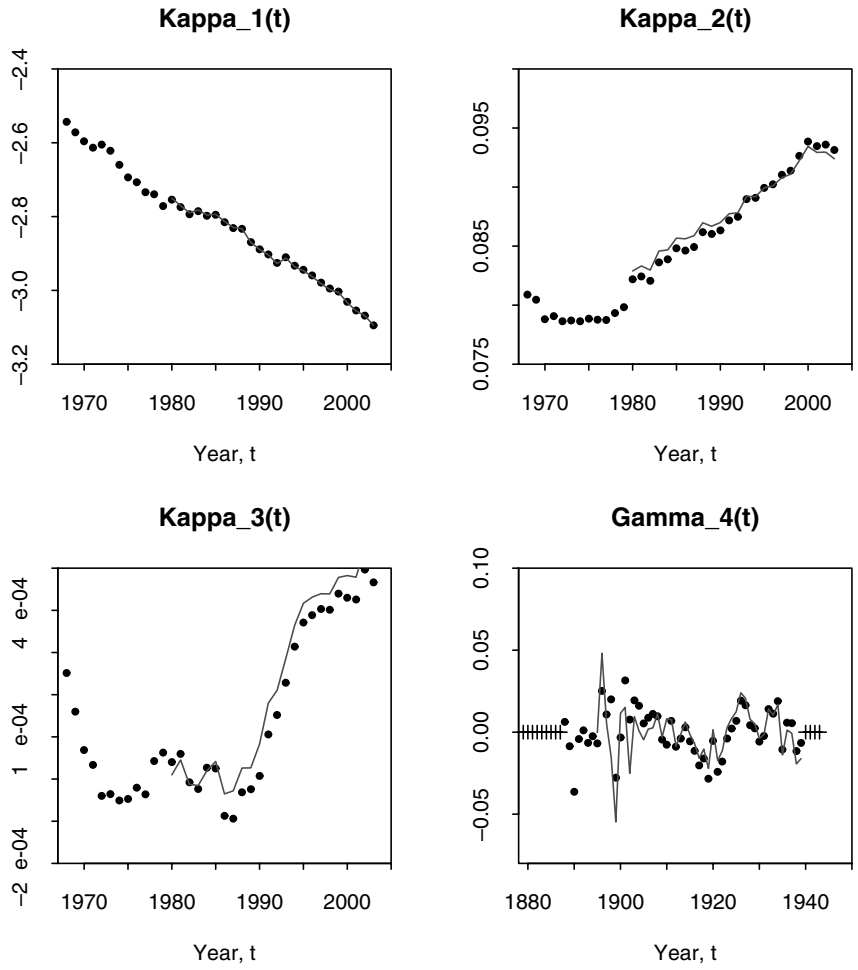
overfitting the data (especially when it is suspected—as discussed in Section 2.2—that the exposures data contain significant errors), and lead us to doubt its robustness.

7.1.2 Standardized Residuals

The variances of the standardized residuals for the U.S. data are very much higher than for EW (Table 4). Using 1968–2003 data, the variance is around 7.5 for model M2 and 11.5 for models M7 and M8. Using data from 1980 to 2003, these fall to about 3.3 and 7.5, respectively. As discussed before, if the data were wholly reliable, the Poisson assumption the right one, and the model the correct one, then this variance should be around 1. The high values we see here, therefore, lend weight to our earlier remarks concerning inaccuracies in the exposures data.

The plots of standardized residuals (not presented here) exhibit some degree of clustering. Out of these M2 looks the most random, but comparison of this with its EW counterpart suggests that M2 fits the U.S. data less well in terms of the i.i.d. assumption.

Figure 17
U.S. Data: Parameter Estimates for Model M7



Notes: Derived from (*a*) Data from 1968–2003 (*dots*), (*b*) Data from 1980–2003 (*solid lines*). Crosses in the bottom right plot correspond to excluded cohorts.

7.1.3 Comparison of Nested Models

We carried out likelihood ratio tests on models that are nested, as an alternative to model selection using the BIC.

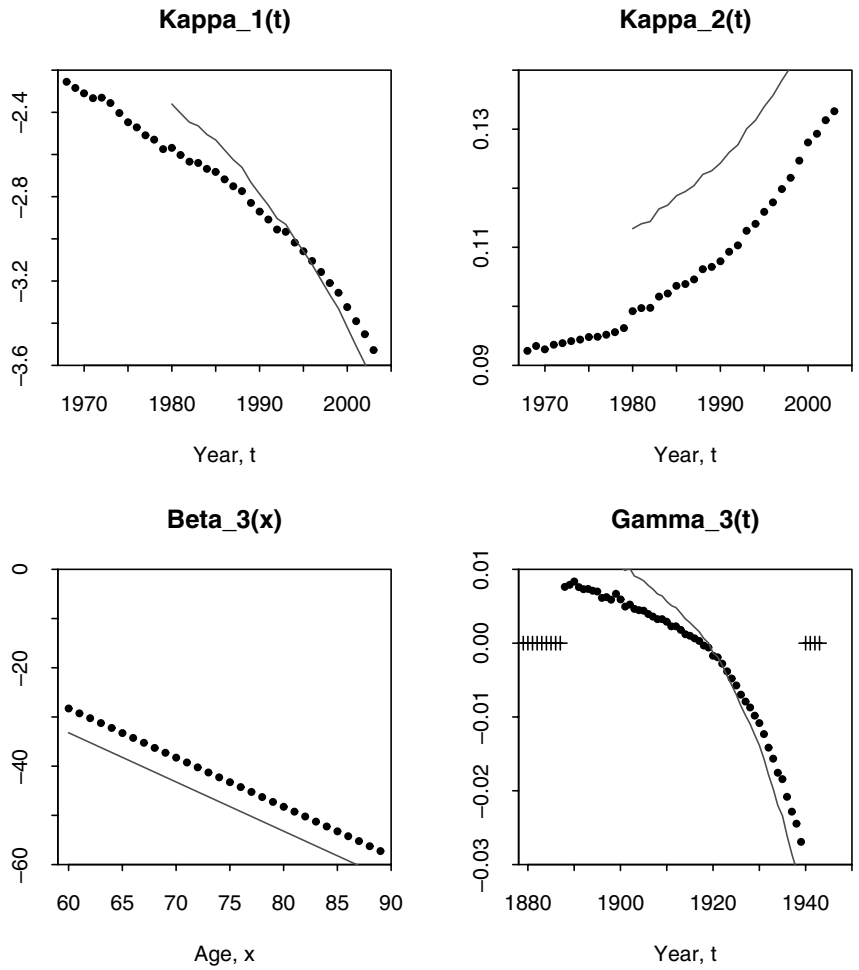
Test results for all seven nested pairs are presented in Table 8. These results support our earlier conclusions based on the BIC: namely, that the more complex models succeed in fitting the data better than the simpler models.

One can compare Tables 5 and 8 to investigate the relative importance of specific model features. For example, compare M6 with M7. With the U.S. data, the test statistic is larger than the EW test statistic with fewer degrees of freedom, and this indicates that the quadratic age effect is more prominent in the U.S. data.

7.2 Parameter Estimates and Their Robustness

Parameter estimates for the seven models are plotted in Figures 12–18. For those that incorporate a cohort effect, this effect is quite prominent. However, the form of the effect does seem to vary from

Figure 18
U.S. Data: Parameter Estimates for Model M8



Notes: Derived from (*a*) Data from 1968–2003 (*dots*), (*b*) Data from 1980–2003 (*solid lines*). Crosses in the bottom right plot correspond to excluded cohorts.

Table 8
U.S. Data: Likelihood Ratio Test Results for Various Pairs of General and Nested Models

H_0: Restricted Model	H_1: General Model	LR Test Statistic	d.f.	*p*-Value
M1	M2	5,056.0	93	<0.000001
M3	M2	4,233.8	59	<0.000001
M5	M6	8,345.7	63	<0.000001
M5	M7	8,979.2	98	<0.000001
M6	M7	633.4	35	<0.000001
M5	M8	8,560.5	66	<0.000001
M6	M8	214.7	2	<0.000001

one model to another (e.g., M3 versus M7). We will focus our remaining remarks on models M2, M7, and M8.

Comparing Figure 13 with its EW counterpart, Figure 4, the patterns for M2 are quite different. The strong, almost linear trend in $\gamma_c^{(3)}$ is rather indicative of a steady period effect that is independent of the $\kappa_t^{(2)}$ period effect.[20] Consequently, although M2 scores the highest BIC, Figure 13 suggests that a variation on M2 with an additional period factor might be better.

A further concern about the suitability of M2 arises when we look at estimates for $\beta_x^{(2)}$ and $\beta_x^{(3)}$. These display a much higher degree of randomness than we saw in the EW data. As we have discussed in Section 4.9, we would expect to see smoothness in each of the age effects: it is difficult to think of any biological or environmental factors that would result in this level of randomness in $\beta_x^{(2)}$ and $\beta_x^{(3)}$. Rather, the randomness suggests M2 might be overfitting the U.S. data.[21]

Now compare Figure 17 (model M7) with its EW counterpart Figure 8. The pattern of development of the various parameters in M7 is fairly consistent between the two countries. The main qualitative difference that we can identify under model M7 is that $\gamma_c^{(4)}$ has a less well-defined pattern in the U.S. results, and a greater degree of randomness. This suggests that cohort-related trends in mortality are less important in the United States than in EW. What remains of a cohort effect in the U.S. data might be the result of overfitting or perhaps due to genuine environmental factors that affect each cohort in their year of birth and that vary randomly from year to year (e.g., influenza epidemics).

For model M8 (Fig. 18), the trend in $\gamma_{t-x}^{(3)}$ is similar to the M2 cohort effect (Fig. 13). As with M2, therefore, it suggests that the model might be improved by the inclusion of an additional period effect.

7.2.1 Robustness

Figures 12–18 also include parameter estimates for each model based on data from 1980 to 2004 (solid lines in the plots). If we compare these with the original parameter estimates based on data from 1968 to 2003 (dots), we can make similar observations about each model as in the case of EW data. The simpler models, M1, M3, and M5, tend to show greater robustness.[22] M7 again seems to be the most robust out of M2, M7, and M8, while M2 again has problems, leading us to question its reliability as a means of projecting mortality rates.

7.3 Model Forecasting Properties

7.3.1 Survivor Index Projections

It is interesting and informative to consider the differences between projections using the four models with the top four BIC rankings (in order, M2, M7, M3, and M8). In Figure 19 (top) we have plotted the mean and 90% prediction intervals for the survivor index $S(t, 65)$: that is, the proportion out of those alive and aged 65 in 2003 surviving to year $2003 + t$.[23] Models M2, M3, and M7 produce relatively similar projections, although the prediction interval for M3 is narrower—a feature that is more obvious if we look at the variance of $\log S(t, 65)$ (Fig. 19, bottom).

M8 stands out as being substantially different. The steepening of $\gamma_c^{(3)}$ around 1920 in combination with the negative fitted values for $\beta_x^{(3)}$ (Fig. 18) implies that cohorts born after 1920 have increasingly poor mortality relative to the $\kappa_t^{(1)}$ improving trend. This form of cohort effect also appears in model M6, but not in any of the other models. As a consequence, M8 relative to M2, M3, and M7 has substantially lower survival rates in the 2003 age-65 cohort.

[20] That is, $\beta_x^{(2)}$ and $\beta_x^{(3)}$ are not identical.

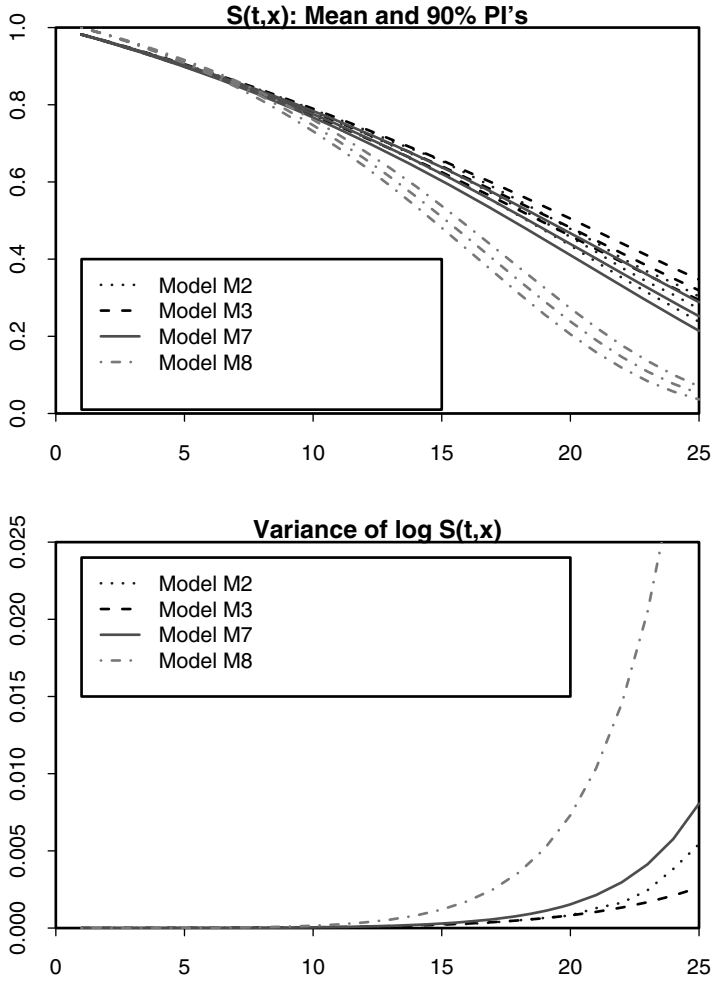
[21] A possible future refinement of M2, therefore, might be to replace the fully nonparametric $\beta_x^{(2)}$ with a smooth function of x by applying the method of P-splines.

[22] Much of the shifts we see in models M1 and M3 could be eliminated by adjusting the constraints.

[23] See the appendix for further details.

Figure 19

U.S. Data: *Top:* 90% Prediction Intervals for Survivor Index $S(t, 65)$ for Models M2, M3, M7, and M8, with Mean of $S(t, 65)$ Running Down the Middle. *Bottom: Var*[log $S(t, 65)$] for Models M2, M3, M7, and M8



7.3.2 Projecting Annuity Values

We calculated the value of a 25-year annuity payable to a male aged 65 in 2003. Values for models M2, M3, M7, and M8 based on 1968–2003 and 1980–2003 data are given in Table 9. Most noticeable in

Table 9

U.S. Data: 25-Year Annuity Values for a 65-Year-Old Male Calculated Using Different Models and Based on Different Periods of Data

	Model			
	M2	M3	M7	M8
1968–2003	11.298	11.417	11.175	10.165
1980–2003	11.656	11.351	11.198	9.981

this table are the relatively low values for M8, which reflects the unusual cohort effect discussed in the previous subsection. We can also see that the relative lack of robustness in parameter estimates under M2 and M8 means that values under these two models are more sensitive to changes in the period of data used than M3 and M7.

8. CONCLUSIONS

We have attempted to explain mortality improvements for males aged 60–89 in England and Wales (EW) and in the United States using a number of stochastic mortality models that decompose mortality improvements into one or more age-, period-, and cohort-related effects. No single model stands out as being best under all the selection criteria considered. However, different models have different strengths. For example, the Lee-Carter class of models allows for greater flexibility in the age effects, $\beta_x^{(i)}$, while one-dimensional P-splines can be exploited to smooth age effects if the roughness of the $\beta_x^{(i)}$ is seen as a drawback. For their part the CBD-Perks models impose smoothness in the age effects as an assumption, but allow for richer period effects than the Lee-Carter class. We therefore need to balance up the strengths and weaknesses of each model to form a conclusion, and to some extent it is up to potential users of the models to decide the weights they place on the different criteria.

If the reader looks only at the BIC ranking criterion, then model M8 for the EW data and model M2 for the U.S. data dominate. However, if the reader takes into account the robustness of the parameter estimates, then model M7 is preferred for both datasets. This model fits both datasets well, and the stability of the parameter estimates over time enables one to place some degree of trust in its projections of mortality rates. The lack of robustness in the other models means that we cannot wholly rely on projections produced by them.

Model M7 shows that mortality rates in both England and Wales and the United States have the following features in common (see Figs. 8 and 17):

- Mortality rates have been improving over time at all ages: the ''level'' period term ($\kappa_t^{(1)}$) has been declining over time, so that the upward-sloping plot of the logit of mortality rates against age has been shifting downwards over time.
- These improvements have been greater at lower ages than at higher ages: the ''slope'' period term ($\kappa_t^{(2)}$) has been increasing over time, so that the plot of the logit of mortality rates against age has been steepening as it shifts downwards over time. This phenomenon has been noted by the studies surveyed in Wong-Fupuy and Haberman (2004, Sections 5.2 and 5.3), for example.
- The changes over time in $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ have been approximately linear, and such linear improvements have also been noted in previous studies (e.g., Wong-Fupuy and Haberman 2004, Section 5.1).
- Mortality rates plotted on a logistic scale against age have a slight curvature over the 60–89 age range that can be modeled using a quadratic function of age. The inclusion of a component that combined a quadratic age effect with a stochastic period effect was found to be statistically significant.
- There is a significant cohort effect ($\gamma_c^{(4)}$) in mortality improvements, although this is more prominent and systematic in the EW than the U.S. data.

To a large extent, these commonalities are also reflected in the other models considered.

A good stochastic mortality model must take these features into account when forecasting mortality improvements and prediction intervals around these forecasts. This is important for quantifying longevity risk, for providing benchmarks for longevity-linked financial instruments (Blake and Burrows 2001; Blake, Cairns, and Dowd 2006; Blake et al. 2006; Dowd et al. 2006; Dawson et al. 2007), for pricing such instruments (Cairns, Blake, and Dowd 2006a,b), and for using these instruments for hedging (Dahl and Møller 2006; Dahl, Melchior, and Møller 2008).

As noted above, our analysis has focused on males aged 60–89 in England and Wales and the United States over the last 44 years. It is important to note that if the same models are applied to different

countries, to females rather than males, to a different age range, or to a different range of years, then the conclusions about which model is most suitable might be different.

In terms of using the models for forecasting, we have limited ourselves in this study to two brief illustrations, one looking at the survivor index and the other at annuity prices. In companion studies (Dowd et al. 2008a,b; Cairns et al. 2008), we investigate the goodness-of-fit, backtesting, and forecasting properties of the models in more detail.

APPENDIX

SIMULATION MODEL

For projection of the survivor index, $S(t, x)$, we need to take the fitted parameter values illustrated in Figures 3–9 and 12–18 and use these to develop a stochastic projection model.

For example, for model M5 we use the method described in Cairns, Blake, and Dowd (2006b) (CBD): thus we fitted a two-dimensional random-walk model to $(\kappa_t^{(1)}, \kappa_t^{(2)})$ using the the final 21 years of data (i.e., 20 observations of the change in $(\kappa_t^{(1)}, \kappa_t^{(2)})$. The form of $\beta_x^{(2)}$ in this paper is different from the original CBD paper, so parameter estimates are different.

In the main body of this paper, we report on simulation results for M2, M5, M7, and M8. For M2, M7, and M8, the $\beta_x^{(i)}$ age effects are fixed. For each of M2, M7, and M8, we adopt the same principles for simulation of the period $\kappa_t^{(i)}$ and cohort $\gamma_c^{(i)}$ effects. For model M7, for example, we take the following approach for EW data from 1961 to 2004:

- Fit the $\beta_x^{(i)}$, $\kappa_t^{(i)}$, and $\gamma_c^{(i)}$ to the full set of data from age 60 to 89.
- Then take $\kappa_t^{(1)}$, $\kappa_t^{(2)}$, and $\kappa_t^{(3)}$ for years 1984–2004 inclusive, and fit a three-dimensional random walk with drift.
- For the cohorts aged 65, 70, and 75 in 2004, we already have an estimate of the cohort effect, $\gamma_{1939}^{(4)}$, $\gamma_{1934}^{(4)}$, and $\gamma_{1929}^{(4)}$, so no model is required for these values.
- For the cohort aged 60 in 2004 (i.e., the 1944 birth cohort), we need to project the estimated $\gamma_c^{(4)}$ series. Our results clearly indicate that a random walk model is inappropriate, but the development of a more suitable model for $\gamma_c^{(4)}$ is beyond the scope of this study. If we wish to model the survivor index, $S(t, 60)$, for this cohort (which in turn will allow us to calculate annuity values), we need a single value (simulated or otherwise) for $\gamma_{1944}^{(4)}$ only, and not for any other years. Based on the historical development of $\gamma_c^{(4)}$, we try out two values for $\gamma_{1944}^{(4)}$ (one high and one low) to cover what we feel is the likely range of values that might be taken by $\gamma_{1944}^{(4)}$.

Once we have our simulation model for $S(t, x)$ we can calculate 4% term annuity values according to the formula

$$a_x(2004) = \sum_{t=1}^{90-x} e^{-1.04t} E[S(t, x)].$$

In this expression, the upper age limit of age 90 is imposed to avoid the requirement to extrapolate beyond the range of ages used in the model fitting process, thereby allowing us to focus on projection of the period and cohort effects.

REFERENCES

ANDERSON, R. N. 1999. Method for Constructing Complete Annual U.S. Life Tables. Vital and Health Statistics 2(129), National Center for Health Statistics.

BLAKE, D., AND W. BURROWS. 2001. Survivor Bonds: Helping to Hedge Mortality Risk. *Journal of Risk and Insurance* 68: 339–48.

BLAKE, D., A. J. G. CAIRNS, AND K. DOWD. 2006. Living with Mortality: Longevity Bonds and Other Mortality-Linked Securities. *British Actuarial Journal* 12: 153–97.

BLAKE, D., A. J. G. CAIRNS, K. DOWD, AND R. MACMINN. 2006. Longevity Bonds: Financial Engineering, Valuation and Hedging. *Journal of Risk and Insurance* 73: 647–72.

BROUHNS, N., M. DENUIT, AND J. K. VERMUNT. 2002. A Poisson Log-Bilinear Regression Approach to the Construction of Projected Life Tables. *Insurance: Mathematics and Economics* 31: 373–93.

CAIRNS, A. J. G. 2000. A Discussion of Parameter and Model Uncertainty in Insurance. *Insurance: Mathematics and Economics* 27: 313–30.

CAIRNS, A. J. G., D. BLAKE, AND K. DOWD. 2006a. Pricing Death: Frameworks for the Valuation and Securitization of Mortality Risk. *ASTIN Bulletin* 36: 79–120.

———. 2006b. A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration. *Journal of Risk and Insurance* 73: 687–718.

———. 2008. Measurement, Modelling and Management of Mortality Risk: A Review. *Scandinavian Actuarial Journal,* (2–3): 79–113.

CAIRNS, A. J. G., D. BLAKE, K. DOWD, G. D. COUGHLAN, D. EPSTEIN, AND M. KHALAF-ALLAH. 2008. Mortality Density Forecasts: An Analysis of Six Stochastic Mortality Models. Working paper, Heriot-Watt University, and Pensions Institute Discussion Paper PI-0801.

CONTINUOUS MORTALITY INVESTIGATION BUREAU (CMI). 2005. Projecting Future Mortality: Towards a Proposal for a Stochastic Methodology. Working paper 15.

———. 2006. Stochastic Projection Methodologies: Further Progress and P-Spline Model Features, Example Results and Implications. Working paper 20.

———. 2007. Stochastic Projection Methodologies: Lee-Carter Model Features, Example Results and Implications. Working paper 25.

CURRIE, I. D. 2006. Smoothing and Forecasting Mortality Rates with P-Splines. Paper given at the Institute of Actuaries, June 2006. http://www.ma.hw.ac.uk/~iain/research/talks.html.

CURRIE, I. D., M. DURBAN, AND P. H. C. EILERS. 2004. Smoothing and Forecasting Mortality Rates. *Statistical Modelling* 4: 279–98.

CZADO, C., A. DELWARDE, AND M. DENUIT. 2005. Bayesian Poisson Log-Linear Mortality Projections. *Insurance: Mathematics and Economics* 36: 260–84.

DAHL, M., M. MELCHIOR, AND T. MØLLER. 2008. On Systematic Mortality Risk and Risk Minimisation with Survivor Swaps. *Scandinavian Actuarial Journal* (2–3): 114–46.

DAHL, M., AND T. MØLLER. 2006. Valuation and Hedging of Life Insurance Risks with Systematic Mortality Risk. *Insurance: Mathematics and Economics* 39: 193–217.

DAWSON, P., D. BLAKE, A. J. G. CAIRNS, AND K. DOWD. 2007. Completing the Survivor Derivatives Market. Pensions Institute Discussion Paper PI-0712.

DOWD, K., D. BLAKE, A. J. G. CAIRNS, G. D. COUGHLAN, D. EPSTEIN, AND M. KHALAF-ALLAH. 2008a. Evaluating the Goodness of Fit of Stochastic Mortality Models. Pensions Institute Discussion Paper PI-0802.

———. 2008b. Backtesting Stochastic Mortality Models: An Ex-Post Evaluation of Multi-Period-Ahead Density Forecasts. Pensions Institute Discussion Paper PI-0803.

DOWD, K., D. BLAKE, A. J. G. CAIRNS, AND P. DAWSON. 2006. Survivor Swaps. *Journal of Risk and Insurance* 73: 1–17.

DOWD, K., A. J. G. CAIRNS, AND D. BLAKE. 2006. Mortality-Dependent Financial Risk Measures. *Insurance: Mathematics and Economics* 38: 427–40.

HAYASHI, F. 2000. *Econometrics*. Princeton: Princeton University Press.

JACOBSEN, R., N. KEIDING, AND E. LYNGE. 2002. Long-Term Mortality Trends behind Low Life Expectancy of Danish Women. *Journal of Epidemiology and Community Health* 56: 205–8.

KOISSI, M. C., A. F. SHAPIRO, AND G. HÖGNÄS. 2006. Evaluating and Extending the Lee-Carter Model for Mortality Forecasting: Bootstrap Confidence Intervals. *Insurance: Mathematics and Economics* 38: 1–20.

LEE, R. D., AND L. R. CARTER. 1992. Modeling and Forecasting U.S. Mortality. *Journal of the American Statistical Association* 87: 659–75.

OSMOND, C. 1985. Using Age, Period and Cohort Models to Estimate Future Mortality Rates. *International Journal of Epidemiology* 14: 124–29.

PERKS, W. 1932. On Some Experiments in the Graduation of Mortality Statistics. *Journal of the Institute of Actuaries* 63: 12–57.

RENSHAW, A. E., AND S. HABERMAN. 2003. Lee-Carter Mortality Forecasting with Age-Specific Enhancement. *Insurance: Mathematics and Economics* 33: 255–72.

———. 2006. A Cohort-Based Extension to the Lee-Carter Model for Mortality Reduction Factors. *Insurance: Mathematics and Economics* 38: 556–70.

RICHARDS, S. J., J. G. KIRKBY, AND I. D. CURRIE. 2006. The Importance of Year of Birth in Two-Dimensional Mortality Data. *British Actuarial Journal* 12: 5–38.

WILLETS, R. C. 1999. Mortality in the Next Millennium. Paper presented to the Staple Inn Actuarial Society.

———. 2004. The Cohort Effect: Insights and Explanations. *British Actuarial Journal* 10: 833–77.

WONG-FUPUY, C., AND S. HABERMAN. 2004. Projecting Mortality Trends: Recent Developments in the United Kingdom and the United States. *North American Actuarial Journal* 8(1): 56–83.

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