Mathematics for
Engineers and Scientists 3
F18XC1

Heriot-Watt University
Chapter 1

Introduction to Ordinary Differential Equations

A differential equation is an equation involving variables (say $x$ and $y$) and ordinary derivatives, i.e. $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.

When the derivative is of the form $\frac{dy}{dx}$ then $x$ is the independent variable and $y$ is the dependent variable.

To solve a differential equation of the form $\frac{dy}{dx} = f(x)$ means to determine $y$ as a function of $x$.

Example 1.1. Find a general solution of the differential equation

$$\frac{dy}{dx} = 1.$$ 

This can be solved by integrating both sides with respect to $x$, $\int \frac{dy}{dx} \, dx = \int 1 \, dx$, to get

$$y = x + A$$

where $A$ is a constant of integration.

Example 1.2. The following differential equation represents how the current $I$ in an electrical circuit changes with time $t$,

$$L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = 0,$$

where $L$, inductance, $R$, resistance and $C$, capacitance are constants.

Here, $I$ is the dependent variable and $t$ is the independent variable (we seek solutions for $I$ in terms of $t$). Note there is a second derivative $\frac{d^2I}{dx^2}$ and so this is called a second order differential equation.
1.1 Terminology

The order of a differential equation is determined by the highest order derivative in the differential equation.

Example 1.3. The general linear second order differential equation is represented by

\[ a(t) \frac{d^2 y}{dt^2} + b(t) \frac{dy}{dt} + c(t)y = f(t). \]

Here, \( y \) is the dependent variable and \( t \) is the independent variable, the highest order term is \( \frac{d^2 y}{dt^2} \) and so it is a second order differential equation.

It is linear since the coefficients of \( y \) and its derivatives are functions of \( t \) only (so there are no powers of \( y \) or its derivatives or products of these). If a differential equation is not linear it is said to be nonlinear.

If \( f(t) = 0 \) the the differential equation is homogeneous. Otherwise it is inhomogeneous.

1.2 1st Order Differential Equations: Simple Examples

Example 1.4. Find the general solution of the differential equation

\[ \frac{dy}{dx} = x. \]

By integrating both sides with respect to \( x \) we get the solution

\[ y = \frac{1}{2}x^2 + C \]

where \( C \) is an arbitrary constant. As the solution contains an arbitrary constant it is called a general solution.

If we are given further information, such as \( y(0) = 1 \), we can determine the value of \( C \) and find a particular solution.

Given that \( y = 1 \) when \( x = 0 \), we have \( 1 = 0 + C \), so that \( C = 1 \). Thus, the particular solution satisfying the condition \( y(0) = 1 \) is

\[ y = \frac{1}{2}x^2 + 1. \]
Example 1.5. Find the solution of the differential equation

\[ \frac{dy}{dx} = \cos x ; \quad y(\pi/2) = 2. \]

The general solution is

\[ y = \sin x + C \]

where \( C \) is an arbitrary constant.

Given that \( y = 2 \) when \( x = \frac{\pi}{2} \), we have \( 2 = 1 + C \), so that \( C = 1 \). Therefore the particular solution is

\[ y = \sin x + 1. \]

1.3 1st Order Differential Equation: Separation of Variables

Consider the generic separable differential equation.

\[ \frac{dx}{dt} = f(x)g(t) \]

This is called \textit{separable} as the differential equation to be written in the following form

\[ \frac{dx}{f(x)} = g(t) \, dt \]

where the functional dependence of \( x \) and \( t \) are on different sides of the equation. We can then integrate both sides as follows

\[ \int \frac{dx}{f(x)} = \int g(t) \, dt \]

to find a general solution to the differential equation.

Example 1.6. Find the solution of the differential equation

\[ \frac{dy}{dx} = \frac{x^2}{y}. \]

The differential equation can be written in the following form

\[ y \, dy = x^2 \, dx \quad \Rightarrow \quad \int y \, dy = \int x^2 \, dx. \]
By performing the integration, the general solution is

\[ \frac{1}{2}y^2 = \frac{1}{3}x^3 + C. \]

Note, we only need one arbitrary constant \( C \).

**Example 1.7.** Find the solution of the differential equation

\[ \frac{dy}{dx} = y^2e^{-x} : y(0) = 0.5. \]

The differential equation can be written in the following form

\[ \frac{1}{y^2} dy = e^{-x} \, dx \implies \int \frac{1}{y^2} \, dy = \int e^{-x} \, dx. \]

Integrating gives the general solution

\[ \frac{-1}{y} = -e^{-x} + C. \]

If \( y = 0.5 \) when \( x = 0 \) we get \( -2 = -1 + C \) and so \( C = -1 \). The particular solution is

\[ \frac{1}{y} = e^{-x} + 1 \implies y = \frac{1}{e^{-x} + 1}. \]

**Example 1.8.** Newton’s second law of motion (‘\( F = ma \)’) can be applied to a dragster that has deployed its parachute and is slowing down. The differential equation is

\[ F = m\frac{dv}{dt} \text{ where the drag force } \ F = -\frac{1}{2} C_D \rho A v^2 \]

where \( C_D \), the drag coefficient, \( \rho \), the density of air, and \( A \) the area of the parachute are constants. This looks complicated but we can combine all the constants as \( k = (C_D \rho A)/(2m) \) to write as a separable differential equation

\[ \frac{dv}{dt} = -k v^2. \]

This can be written as

\[ \int \frac{1}{v^2} \, dy = \int -k \, dt \quad \text{with general solution } \frac{1}{v} = kt + C. \]

If we are provided with the speed of the dragster when the parachute was deployed, say \( v(0) = 30 \) m s\(^{-1} \), we can find a particular solution.

If \( v = 30 \) when \( t = 0 \) then \( 1/30 = C \). Therefore the particular solution is

\[ \frac{1}{v} = kt + \frac{1}{30} \implies \frac{1}{v} = \frac{30kt + 1}{30} \implies v = \frac{30}{30kt + 1}. \]
1.4 1st Order Differential Equation: Transformations

Some differential equations are not separable but can be made separable by using a transformation (or substitution). The first transformation we will consider sets \( y = vx \) and considers homogeneous differential equations of the form

\[
\frac{dy}{dx} = f\left(\frac{y}{x}\right).
\]

To apply the transformation we substitute

\[
y = vx \quad \text{and} \quad \frac{dy}{dx} = v + x \frac{dv}{dx}
\]

into the original equations. (Note the substitution for \( \frac{dy}{dx} \) comes from using the product rule to give \( \frac{d(vx)}{dx} = v + x \frac{dv}{dx} \).) The differential equation is now separable as it can be written as

\[
v + x \frac{dv}{dx} = f(v) \quad \Rightarrow \quad \int \frac{1}{f(v) - v} \, dv = \int \frac{1}{x} \, dx
\]

**Example 1.9.** Find the solution of the differential equation

\[
\frac{dy}{dx} = \frac{y^2 + xy}{x^2}
\]

which can be written as \( \frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) \).

Substitute \( y = vx \) and \( \frac{dy}{dx} = v + x \frac{dv}{dx} \) to get

\[
v + x \frac{dv}{dx} = \frac{(vx)^2 + x(vx)}{x^2} \quad \Rightarrow \quad v + x \frac{dv}{dx} = v^2 + v \quad \Rightarrow \quad x \frac{dv}{dx} = v^2.
\]

Using the separable variables methods gives

\[
\int \frac{1}{v^2} \, dv = \int \frac{1}{x} \, dx \quad \Rightarrow \quad -\frac{1}{v} = \ln(x) + C.
\]

We substitute \( v = y/x \) to return to the original variables

\[
-\frac{x}{y} = \ln x + C \quad \Rightarrow \quad y = \frac{x}{\ln x + C}.
\]

Other transformations can be used. For example the transformation \( v = ax + by + c \) can be used for differential equations of the form

\[
\frac{dy}{dx} = f(ax + by + c).
\]

To apply the transformation we substitute

\[
v = ax + by + c \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{b} \left(\frac{dv}{dx} - a\right)
\]

into the original equations.
Example 1.10. Find the solution of the differential equation
\[
\frac{dy}{dx} = \frac{x - y + 2}{x - y + 3}.
\]

Let \( v = x - y + 3 \) and therefore substitute \( y = x - v + 3 \) and \( \frac{dy}{dx} = 1 - \frac{dv}{dx} \) into the original equation to get
\[
1 - \frac{dv}{dx} = \frac{v - 1}{v} \quad \Rightarrow \quad \frac{dv}{dx} = 1 - \frac{v - 1}{v} \quad \Rightarrow \quad \frac{dv}{dx} = \frac{1}{v}.
\]

Using the separable variables methods gives
\[
\int v \, dv = \int 1 \, dx \quad \Rightarrow \quad \frac{v^2}{2} = x + C \quad \Rightarrow \quad (x - y + 3)^2 = 2(x + C).
\]
We have chosen to leave the answer as an implicit solution for \( y \).

1.5 1st Order Linear Differential Equations: Integrating Factor Method

A linear first order differential equation is an equation of the form
\[
f(x) \frac{dy}{dx} + g(x) \, y = h(x)
\]
where \( f(x), g(x), h(x) \) are functions of \( x \).

In the special case that \( g(x) \) is the derivative of \( f(x) \), then the LHS can be simplified, using the product rule for differentiation, as illustrated by the following example.

Example 1.11. Solve the linear first order differential equation
\[
x^2 \frac{dy}{dx} + 2xy = e^x
\]

Since \( \frac{d}{dx}(x^2) = 2x \), the equation can be written
\[
\frac{d}{dx} (x^2 y) = e^x.
\]

Integrating gives the solution:
\[
x^2 y = e^x + C.
\]
If we do not have the special condition \( f'(x) = g(x) \), we can find a function \( I(x) \), called an integrating factor, such that on multiplying the differential equation by \( I(x) \) the resulting equation does satisfy the special condition.
Formula for integrating factor

By dividing throughout by \( f(x) \), the linear first order differential equation

\[
f(x) \frac{dy}{dx} + g(x) \ y = h(x)
\]

can be written in standard form

\[
\frac{dy}{dx} + p(x) \ y = q(x).
\]

If \( I = I(x) \) is an integrating factor for the standard form equation, then

\[
\frac{dI}{dx} = I(x)p(x)
\]

\[
\Rightarrow \frac{dI}{I} = p(x) \ dx \Rightarrow \ln I = \int p(x) \ dx.
\]

Thus:

\[
I(x) = e^{\int p(x) \ dx} \quad \text{i.e.} \quad I(x) = \exp \left( \int p(x) \ dx \right).
\]

In words: integrate \( p(x) \) (omit arbitrary constant \( C \)) and take \( e \) to this power. For example:

\[
p(x) = 2x \Rightarrow I(x) = e^{x^2}.
\]

Summary

1. Write the differential equation in standard form

\[
\frac{dy}{dx} + p(x) \ y = q(x).
\]

2. Determine the integrating factor

\[
I(x) = e^{\int p(x) \ dx}.
\]

3. Write

\[
I(x)y = \int I(x)q(x) \ dx
\]

and complete the integral on the RHS and solve for \( y \).

Example 1.12. Find the general solution of the linear first order differential equation

\[
\frac{dy}{dx} + 4y = e^{-2x}.
\]
The equation is already in standard form with \( p(x) = 4 \) and \( q(x) = e^{-2x} \). Therefore the integrating factor is

\[
I(x) = e^{\int p(x) \, dx} = e^{4 \, dx} = e^{4x}.
\]

Writing

\[
I(x)y = \int I(x)q(x) \, dx \Rightarrow e^{4x}y = \int e^{4x}e^{-2x} \, dx.
\]

Therefore

\[
e^{4x}y = \frac{1}{2} e^{2x} + C \Rightarrow y = \frac{1}{2} e^{-2x} + Ce^{-4x}.
\]

**Example 1.13.** Find the general solution of the linear first order differential equation

\[
x \frac{dy}{dx} + 2y = x \cos x^3.
\]

Divide the equation throughout by \( x \) to put in standard form:

\[
\frac{dy}{dx} + \frac{2}{x} y = \cos x^3.
\]

The integrating factor is

\[
I(x) = e^{\int \frac{2}{x} \, dx} = e^{\ln x} = x^2.
\]

Therefore

\[
x^2y = \int x^2 \cos x^3 \, dx = \int \frac{1}{3} \cos u \, du
\]

using the substitution \( u = x^3 \) so that \( du = 3x^2 \, dx \). Thus:

\[
x^2y = \frac{1}{3} \sin u + C = \frac{1}{3} \sin x^3 + C.
\]

**Example 1.14.** The lumped capacity differential equation model that represents the cooling of a block of hot steel with initial temperature \( T_0 \) can be solved using the integrating factor method.

\[
\frac{dT}{dt} = -k(T - T_{amb}) \Rightarrow \frac{dT}{dt} + kT = kT_{amb}
\]

where \( k \) and the ambient temperature of the environment, \( T_{amb} \), are constants. This is in standard form and so we determine the integrating factor:

\[
I(t) = e^{\int k \, dt} = e^{kt}.
\]

Therefore

\[
e^{kt}T = \int e^{kt}kT_{amb} \, dt \Rightarrow e^{kt}T = T_{amb}e^{kt} + C \Rightarrow T = T_{amb} + Ce^{-kt}.
\]

Using \( T = T_0 \) when \( t = 0 \) gives \( T_0 = T_{amb} + C \) and so \( C = T_0 - T_{amb} \). The particular solution is therefore

\[
T = T_{amb} + (T_0 - T_{amb}) e^{-kt}.
\]
1. Problems: 1st Order ODEs

Problem 1.1.

(a) Find the general solution of the differential equation
\[ \frac{dy}{dx} = e^{4x}. \]

(b) Find the solution of the differential equation
\[ \frac{dy}{dx} = \sin x + \cos x \]
satisfying the condition \( y = 2 \) when \( x = \pi/2 \).

Problem 1.2.

(a) Find the general solution of the differential equation
\[ \frac{dy}{dx} = x^2 e^y. \]

Find the solution to the following initial value problems

(b) \[ \frac{dx}{dt} = \frac{\sin t}{x^2} : x(0) = 0, \quad (c) \frac{dy}{dx} = xy : y(0) = 1. \]

Find the general solution of the differential equations

(d) \[ x \frac{dx}{dt} = \sin t, \quad (e) \sqrt{t} \frac{dx}{dt} = \sqrt{x}. \]

(f) A chemical reaction is governed by the differential equation
\[ \frac{dY}{dt} = K(5 - Y)^2 \]
where \( Y(t) \) is the concentration of a chemical at time \( t \). The initial concentration is zero and the concentration at \( t = 5s \) is 2 mol cm\(^{-3}\). Find the particular solution to the differential equation and find the value of \( K \). What value does the concentration approach in the long term?

Problem 1.3.

Find the general solution of the differential equations using the transformation \( y = vx \).

(a) \[ xy \frac{dy}{dx} = y^2 + x^2, \quad (b) x \frac{dy}{dx} = y^2 + xy \frac{y}{x}, \quad (c) x \frac{dy}{dx} = y + xe^{y/x}. \]

(d) Use the transformation \( v = y - x + 1 \) to find the general solution of
\[ \frac{dy}{dx} = \frac{y - x + 2}{y - x + 1}. \]
(e) Use a suitable transformation to find the general solution of
\[ \frac{dy}{dx} = 2x + y + 2. \]

(f) Use a suitable transformation to find the general solution of
\[ \frac{dy}{dx} = (2x + y)^2 - 2. \]

**Problem 1.4.**

Find the general solution of the linear first order differential equations

(a) \[ \frac{dy}{dt} - 2y = e^t, \]

(b) \[ t \frac{dy}{dt} + 4y = \frac{1}{t^2}, \]

(c) \[ x \frac{dy}{dx} - 2y = x^3 \sin 2x, \]

(d) \[ \frac{dx}{dt} + 3t^2x = t^2. \]

(e) Find the particular solution of the differential equation
\[ \frac{dy}{dx} = x + y : y(0) = 0. \]

(f) In an open storage vessel where the flow rates are governed by gravity, the flow rate $Q$ out of the vessel at time $t$ is represented by the following initial value problem
\[ \frac{dQ}{dt} + aQ = bh(t) : Q(0) = Q_0, \]

where $a$ and $b$ are constants representing chemical and frictional losses and $h(t)$ is the depth of fluid in the storage vessel. If $a = 5$, $b = 2$ and $h(t) = 10e^{-4t}$ solve the initial value problem.

**Answers**

1. (a) \(y = \frac{1}{4} e^{4x} + C\), (b) \(y = \sin x - \cos x + 1\).

2. (a) \(-e^{-y} = \frac{t^3}{3} + C\), (b) \(x = (3 - 3 \cos t)^{\frac{3}{2}}\), (c) \(y = e^{x^{2}/2}\), (d) \(x = \pm \sqrt{2C - 2 \cos t}\),

(e) \(x = (t^{1/2} + C/2)^{2}\), (f) \(Y = 5 - \frac{5}{5Kt + 1}, K = 2/75 \) and \(Y = 5\text{mol cm}^{-3}\).

3. (a) \(y = \pm \sqrt{2x^2 \ln(Cx)}\), (b) \(y = -\frac{x}{\ln C} + C\), (c) \(y = -x \ln(- \ln Cx))\),

(d) \(y = x - 1 \pm \sqrt{2x + 2C}\), (e) \(y = Ce^x - 2x - 4\), (f) \(y = -2x - \frac{1}{x+C}\).

4. (a) \(y = -e^t + Ce^{2t}\), (b) \(y = \frac{1}{2t^2} + \frac{C}{t}\), (c) \(y = -\frac{x^2}{2} \cos 2x + Cx^2\),

(d) \(x = \frac{1}{3} + Ce^{-t^3}\), (e) \(y = -x - 1 + e^x\), (f) \(Q = 20e^{-4t} + (Q_0 - 20)e^{-5t}\).
Chapter 2

Homogeneous, Linear, Constant-Coefficient, 2nd-Order ODEs

2.1 Introduction to Linear Second-Order Ordinary Differential Equations

We first remind ourselves of how a first-order linear ODE is solved:

**Example 2.1.** The general 1st-order problem,

\[ \frac{dy}{dt} + a(t)y = f(t), \]

can be solved by multiplying by the integrating factor, \( I(t) = \exp\left(\int a(t) \, dt\right), \) to get

\[ \frac{d}{dt} (I(t)y) = I(t) \frac{dy}{dt} + a(t)I(t)y = I(t)f(t) \]

and integrating:

\[ I(t)y = \int I(t)f(t) \, dt + C \quad \text{so} \quad y = \left(\int I(t)f(t) \, dt\right)/I(t) + C/I(t). \]

In particular, if \( a(t) = \text{constant} \) and \( f(t) = Ke^{kt}, \) with both \( k \) and \( K \) constant, and \( k \neq -a, \)

\[ I(t) = e^{at}, \quad \int I(t)f(t) \, dt = \frac{Ke^{(a+k)t}}{(a+k)}, \]

so

\[ y = \frac{Ke^{kt}}{(a+k)} + Ce^{-at}. \]

Note that the first term is of the same form of the right-hand side in the original ODE, and the second term is the general solution for a corresponding “homogeneous” problem, with \( f(t) \) replaced by 0.
Question 2.1. Taking $a$, $\alpha$ and $\beta$ to be constants, with $\beta \neq 0$, solve \[ \frac{dy}{dt} + ay = \alpha \sin \beta t. \]

Solution. As in the previous example, the integrating factor is $I(t) = e^{at}$, so we must now find $F(t) = \int \alpha e^{at} \sin \beta t \, dt$. This we do by integrating by parts twice:

\[ F(t) = \frac{\alpha}{a} e^{at} \sin \beta t - \frac{\alpha \beta}{a} \int e^{at} \cos \beta t \, dt = \frac{\alpha}{a} e^{at} \sin \beta t - \frac{\alpha \beta}{a^2} e^{at} \cos \beta t - \frac{\alpha \beta^2}{a^2} \int e^{at} \sin \beta t \, dt. \]

Thus

\[ \left(1 + \frac{\beta^2}{a^2}\right) F(t) = \frac{\alpha}{a} e^{at} \sin \beta t - \frac{\alpha \beta}{a^2} e^{at} \cos \beta t \]

so

\[ F(t) = \frac{\alpha(a \sin \beta t - \beta \cos \beta t)}{a^2 + \beta^2} \]

and

\[ y = \frac{\alpha(a \sin \beta t - \beta \cos \beta t)}{a^2 + \beta^2} + Ce^{-at}. \]

More generally, we can see use integrating factors for problems of this type (with $a$ a constant) to find that:

1. If $f(t) = Kt^n e^{kt}$, with $k \neq -a$ and $n$ a non-negative integer, then $y = (A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0) e^{kt} + Ce^{-at};$

2. If $f(t) = Kt^n e^{-at}$, with $n$ a non-negative integer, then $y = (A_{n+1} t^{n+1} + A_n t^n + \cdots + A_1 t + C) e^{-at};$

3. If $f(t) = t^n e^{kt}(K_1 \cos \beta t + K_2 \sin \beta t)$, with $\beta \neq 0$ and $n$ a non-negative integer, then $y = (A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0) e^{kt} \cos \beta t + (B_n t^n + B_{n-1} t^{n-1} + \cdots + B_1 t + B_0) e^{kt} \sin \beta t + Ce^{-at}.$

The constants $A_0$, $B_0$, $A_1$, etc. are not arbitrary but are fixed by the values of the constants in the ODE (as with Ex. 2.1 and Qu. 2.1).

### 2.2 Newton’s second law

We shall begin looking at second-order ODEs by stating Newton’s fundamental kinematic law relating the force, mass and acceleration of an object whose position is $y(t)$ at time $t$.

\[
\text{Newton’s second law states that the force } F \text{ applied to an object is equal to its mass } m \text{ times its acceleration } \frac{d^2 y}{dt^2}, \text{ i.e.}
\]

\[ F = m \frac{d^2 y}{dt^2}. \]
Question 2.2. Find the height \( y(t) \), at time \( t \), of a body falling freely under gravity (take the convention that we measure positive displacements upwards).

Solution. The equation of motion of a body falling freely under gravity, is, by Newton’s second law (and weight = mass \times \text{gravity}),

\[
\frac{d^2y}{dt^2} = -g .
\]

We can solve equation (2.1) by integrating with respect to \( t \), which yields an expression for the velocity of the body,

\[
\frac{dy}{dt} = -gt + v_0 ,
\]

where \( v_0 \) is the constant of integration which here also happens to be the initial velocity. Integrating again with respect to \( t \) gives

\[
y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0 ,
\]

where \( y_0 \) is the second constant of integration which also happens to be the initial height of the body.

Equation (2.1) is an example of a second-order differential equation (because the highest derivative that appears in the equation is second order):

- the solutions of the equation are a family of functions with two parameters (in this case \( v_0 \) and \( y_0 \));
- choosing values for the two parameters corresponds to choosing a particular function of the family.

2.3 Springs and Hooke’s Law

Consider a mass \( m \) kg on the end of a spring, as in Figure 2.1. With the initial condition that the mass is pulled to one side and then released, what do you expect to happen?

Hooke’s law implies that, provided \( y \) is not so large as to deform the spring, then the restoring force is

\[
F_{\text{spring}} = -ky ,
\]

where the constant \( k > 0 \) depends on the properties of the spring, for example its stiffness.
Combining Hooke’s law and Newton’s second law implies

\[ m \frac{d^2 y}{dt^2} = -ky \]

(assuming \( m \neq 0 \))

\[ \frac{d^2 y}{dt^2} = -\frac{k}{m} y \]

(setting \( \omega = +\sqrt{k/m} \))

\[ \frac{d^2 y}{dt^2} = -\omega^2 y \]

\[ \frac{d^2 y}{dt^2} + \omega^2 y = 0. \quad (2.2) \]

Eqn. (2.2) is an example of Simple Harmonic Motion (SHM).

Can we guess a solution of (2.2), i.e. a function that satisfies the relation (2.2)? When written in the form on the line just above (2.2), we see that we are essentially asking ourselves: what function, when you differentiate it twice, gives you minus \( \omega^2 \) times the original function you started with?

### 2.4 Simple Harmonic Motion

The general solution to

\[ y'' + \omega^2 y = 0 \]

is

\[ y(t) = C_1 \sin \omega t + C_2 \cos \omega t \]

where \( C_1 \) and \( C_2 \) are arbitrary constants. This is an oscillatory solution with frequency of oscillation \( \omega \). The period of the oscillations is the time taken for a cycle to be completed and is

\[ T = 2\pi/\omega. \]
Example 2.2. For the equation
\[ \frac{d^2y}{dt^2} + 36y = 0 \]
we have that \( \omega^2 = 36 \) and \( \omega = 6 \) so the general solution is \( y(t) = C_1 \sin 6t + C_2 \cos 6t \).

Example 2.3. For the equation \( 16 \frac{d^2y}{dt^2} + y = 0 \) we first rewrite it as \( \frac{d^2y}{dt^2} + \frac{y}{16} = 0 \). For this equation, \( \omega^2 = 1/16 \) and \( \omega = 1/4 \) so the general solution is \( y = C_1 \sin \frac{t}{4} + C_2 \cos \frac{t}{4} \).

Question 2.3. Find the solution of
\[ \frac{d^2y}{dt^2} + 9y = 0 \]
with \( y(0) = 2 \), \( \frac{dy}{dt}(0) = 0 \). Find also the frequency of oscillations, the period of oscillations, the amplitude of the solution, and the maximum velocity.

Solution. The general solution to \( \frac{d^2y}{dt^2} + 9y = 0 \) is
\[ y(t) = C_1 \sin 3t + C_2 \cos 3t . \quad (2.3) \]
Differentiating (2.3) we get
\[ \frac{dy}{dt} = 3C_1 \cos 3t - 3C_2 \sin 3t . \quad (2.4) \]
Using \( y(0) = 2 \) in (2.3) and \( \sin 0 = 0 \), \( \cos 0 = 1 \), we obtain \( C_2 = 2 \).
Using \( \frac{dy}{dt}(0) = 0 \) in (2.4) we obtain \( C_1 = 0 \). Hence \( y(t) = 2 \cos 3t \). The frequency of oscillations is 3 and the period is \( 2\pi/3 \).
The amplitude of the solution is the maximum displacement. From \( y(t) = 2 \cos 3t \) the amplitude is 2.
The velocity is \( \frac{dy}{dt} = -6 \sin 3t \) so the the maximum velocity is 6.

Question 2.4. Find the solution of
\[ \frac{d^2y}{dt^2} + 16y = 0 \]
with \( y(0) = 0 \), \( \frac{dy}{dt}(0) = 12 \).

Solution. The general solution to \( \frac{d^2y}{dt^2} + 16y = 0 \) is
\[ y(t) = C_1 \sin 4t + C_2 \cos 4t . \quad (2.5) \]
Differentiating (2.5) we get

\[ \frac{dy}{dt} = 4C_1 \cos 4t - 4C_2 \sin 4t. \quad (2.6) \]

Using \( y(0) = 0 \) in (2.5) and \( \sin 0 = 0, \cos 0 = 1 \) we obtain \( C_2 = 0 \).

Using \( \frac{dy}{dt}(0) = 12 \) in (2.6) we obtain \( 4C_1 = 12 \) so \( C_1 = 3 \). Hence \( y(t) = 3\sin 4t \).

2.5 Damped oscillations

Consider a more realistic spring which has resistance to motion.

In general, the frictional force or drag is proportional to velocity, \( i.e. \)

\[ F_{\text{friction}} = -C \frac{dy}{dt}, \]

where \( C \) is a constant known as the drag or friction coefficient. The frictional force acts in a direction opposite to that of the motion and so \( C > 0 \).

Newton’s Second Law implies (adding the restoring and frictional forces together)

\[ m \frac{d^2y}{dt^2} = F_{\text{spring}} + F_{\text{friction}}, \]

\( i.e. \)

\[ m \frac{d^2y}{dt^2} = -ky - C \frac{dy}{dt}. \]

Hence the damped oscillations of a spring are described by the differential equation

\[ m \frac{d^2y}{dt^2} + C \frac{dy}{dt} + ky = 0. \quad (2.7) \]

We show how to solve equation (2.7) in the next chapter.

2.6 ODE classification (revisited)

ODEs are classified according to order, linearity and homogeneity.
• **Order.** The order of a differential equation is the order of the highest derivative present in the equation.

• **Linear or nonlinear.** A second-order ODE is said to be linear if it can be written in the form

\[ a(t) \frac{d^2y}{dt^2} + b(t) \frac{dy}{dt} + c(t)y = f(t), \] (2.8)

where the coefficients \(a(t), b(t)\) and \(c(t)\) can, in general, be functions of \(t\). An equation that is not linear is said to be nonlinear. Note that linear ODEs are characterised by two properties:

1. The dependent variable and all its derivatives are of first degree, \(i.e.,\) the power of each term involving \(y\) is 1.
2. Each coefficient depends on the independent variable \(t\) only.

• **Homogeneous or inhomogeneous.** The linear differential equation (2.8) is said to be homogeneous if \(f(t) \equiv 0\); if \(f(t) \not\equiv 0\), the differential equation is said to be inhomogeneous.

---

**Example 2.4.** The differential equation

\[ \frac{d^2y}{dt^2} + 5 \left( \frac{dy}{dt} \right)^3 - 4y = e^t, \]

is second order because the highest derivative is second order, and nonlinear because the second term on the left-hand side is cubic in \(dy/dt\).

---

### 2.7 Homogeneous linear ODEs: The Principle of Superposition

Consider the linear, second-order, homogeneous, ordinary differential equation

\[ a(t) \frac{d^2y}{dt^2} + b(t) \frac{dy}{dt} + c(t)y = 0, \] (2.9)

where \(a(t), b(t)\) and \(c(t)\) are known functions.
• If $y_1(t)$ and $y_2(t)$ satisfy (2.9), then for any two constants $C_1$ and $C_2$, the Principle of Superposition says that

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

(2.10)

is a solution also.

• If $y_1(t)$ is not a constant multiple of $y_2(t)$, then the general solution of (2.9) takes the form (2.10).

2.8 Solving linear second-order constant-coefficient homogeneous ODEs

2.8.1 Exponential solutions

We restrict ourselves here to the case when the coefficients $a$, $b$ and $c$ in (2.9) are constants, i.e. (2.9) is

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$  

(2.11)

Let us try to find a solution to (2.11) of the form

$$y = e^{\lambda t}.$$ 

(2.12)

The reason for choosing the exponential function is that we know that solutions to linear first-order constant-coefficient ODEs always have this form for a specific value of $\lambda$ that depends on the coefficients. So we’ll try to look for a solution to a linear second-order constant-coefficient ODE of the same form, where at the moment we will not specify what $\lambda$ is—-with hindsight we will see that this is a good choice.

Substituting (2.12) into (2.11) implies

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = a\lambda^2e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t}$$

$$= e^{\lambda t}(a\lambda^2 + b\lambda + c)$$

which must = 0.

Since the exponential function is never zero, i.e. $e^{\lambda t} \neq 0$, then we see that $\lambda$ has to satisfy the auxiliary equation:

$$a\lambda^2 + b\lambda + c = 0,$$

then (2.12) will be a solution of (2.11). There are three cases we need to consider.

2.8.2 Case I: $b^2 - 4ac > 0$

There are two real and distinct solutions to the auxiliary equation,

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$
and so two functions,

\[ e^{\lambda_1 t} \quad \text{and} \quad e^{\lambda_2 t}, \]

satisfy the ordinary differential equation (2.11). The Principle of Superposition implies that the general solution is

\[ y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \]

**Question 2.5.** Find the general solution to the ODE \( \frac{d^2 y}{dt^2} + 2\frac{dy}{dt} - 3y = 0. \)

**Solution.** The auxiliary equation is \( \lambda^2 + 2\lambda - 3 = 0. \)

\( b^2 - 4ac = 4 + 12 = 16 \)

so

\[ \lambda = \frac{-2 \pm 4}{2} \]

hence \( \lambda = 1 \) or \( \lambda = -3. \)

The general solution is

\[ y = C_1 e^t + C_2 e^{-3t}. \]

**2.8.3  Case II: \( b^2 - 4ac = 0 \)**

In this case there is one real repeated root to the auxiliary equation, namely

\[ \lambda_1 = \lambda_2 = -\frac{b}{2a}. \]

Hence we have one solution, which is

\[ y(t) = e^{\lambda_1 t} = e^{-\frac{b}{2a}t}. \]

However, there should be another independent solution (for a second-order differential equation we should be able to impose two items of initial data to fix two constants, namely the coefficients of two independent solutions). It’s not obvious what it might be, but let’s make the educated guess

\[ y = te^{\lambda_1 t} \]

where \( \lambda_1 \) is the same as above, i.e. \( \lambda_1 = -\frac{b}{2a}. \) Substituting this guess for the second solution into our second-order differential equation,

\[ a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = a (\lambda_1^2 te^{\lambda_1 t} + 2\lambda_1 te^{\lambda_1 t}) + b (e^{\lambda_1 t} + \lambda_1 te^{\lambda_1 t}) + c te^{\lambda_1 t} \]

which in fact \( = 0, \)

since we note that \( a\lambda_1^2 + b\lambda_1 + c = 0 \) and \( 2a\lambda_1 + b = 0 \) because \( \lambda_1 = -b/2a. \) Thus \( te^{-\frac{b}{2a}t} \) is another solution (which is clearly not a constant multiple of the first solution). The Principle of Superposition implies that the general solution is

\[ y = (C_1 + C_2 t)e^{-\frac{b}{2a}t}. \]
**Example 2.5.** Find the general solution to the ODE \( \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 4y = 0 \).

**Solution.** The auxiliary equation is \( \lambda^2 + 4\lambda + 4 = 0 \).

\[
 b^2 - 4ac = 16 - 16 = 0
\]

so \( \lambda = -2 \) and the general solution is

\[
 y = (C_1 + C_2t) e^{-2t}.
\]

### 2.8.4 Case III: \( b^2 - 4ac < 0 \)

In this case, there are two complex roots to the auxiliary equation, namely

\[
 \lambda_1 = p + iq, \quad \lambda_2 = p - iq, \quad \text{where} \quad p = -\frac{b}{2a} \quad \text{and} \quad q = \frac{\sqrt{b^2 - 4ac}}{2a}. \tag{2.13}
\]

Hence the Principle of Superposition implies that the general solution takes the form

\[
 y(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}
 = A_1 e^{(p+iq)t} + A_2 e^{(p-iq)t}
 = A_1 e^{pt} e^{iqt} + A_2 e^{pt} e^{-iqt}
 = e^{pt} \left( A_1 e^{iqt} + A_2 e^{-iqt} \right)
\]

(using Euler’s formula)

\[
 = e^{pt} \left( A_1 (\cos qt + i \sin qt) + A_2 (\cos qt - i \sin qt) \right)
 = e^{pt} \left( (A_1 + A_2) \cos qt + i(A_1 - A_2) \sin qt \right), \tag{2.14}
\]

where:

1. We have used Euler’s formula

\[
 e^{iz} \equiv \cos z + i \sin z,
\]

first with \( z = qt \) and then secondly with \( z = -qt \), *i.e.* we have used that

\[
 e^{iqt} = \cos qt + i \sin qt \quad \text{and} \quad e^{-iqt} = \cos qt - i \sin qt
\]

since \( \cos(-qt) = \cos qt \) and \( \sin(-qt) \equiv -\sin qt \);

2. \( A_1 \) and \( A_2 \) are arbitrary (and in general complex) constants—at this stage this means we appear to have a total of four constants because \( A_1 \) and \( A_2 \) both have real and imaginary parts. However we expect the solution \( y(t) \) to be real—the coefficients are real and we shall pose real initial data.

The solution \( y(t) \) in (2.14) will be real if and only if

\[
 A_1 + A_2 = C_1, \quad i(A_1 - A_2) = C_2,
\]

where \( C_1 \) and \( C_2 \) are real constants. Hence the general solution in this case has the form

\[
 y(t) = e^{pt} (C_1 \cos qt + C_2 \sin qt).
\]
Question 2.6. Find the solution to \( \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 9y = 0 \).

**Solution.** The auxiliary equation is \( \lambda^2 + 4\lambda + 9 = 0 \).

\[ b^2 - 4ac = 16 - 36 = -20 = 20i^2 \] so \( \lambda = (-4 \pm \sqrt{20}i)/2 = -2 \pm \sqrt{5}i \).

The general solution is \( y = e^{-2t} (C_1 \cos \sqrt{5}t + C_2 \sin \sqrt{5}t) \).

---

Question 2.7. Find the solution to the initial-value problem \( \frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 0 \), with \( y(0) = 5 \), \( \frac{dy}{dt}(0) = 10 \).

**Solution.** The auxiliary equation is \( \lambda^2 - \lambda - 6 = 0 \).

\( b^2 - 4ac = 1 + 24 = 25 \) so that \( \lambda = 3 \) or \( \lambda = -2 \).

The general solution is \( y = C_1 e^{3t} + C_2 e^{-2t} \).

Now \( \frac{dy}{dt} = 3C_1 e^{3t} - 2C_2 e^{-2t} \).

Using \( y(0) = 5 \), \( \frac{dy}{dt}(0) = 10 \) and \( e^0 = 1 \),

\[ C_1 + C_2 = 5, \quad 3C_1 - 2C_2 = 10. \]

Solving these gives \( C_1 = 4 \) and \( C_2 = 1 \) so

\[ y = 4e^{3t} + e^{-2t}. \]

---

Question 2.8. Find the solution to the initial-value problem \( \frac{d^2y}{dt^2} - 6 \frac{dy}{dt} + 9y = 0 \), with \( y(0) = 2 \), \( \frac{dy}{dt}(0) = 1 \).

**Solution.** The auxiliary equation is \( \lambda^2 - 6\lambda + 9 = 0 \).

Then \( b^2 - 4ac = 36 - 36 = 0 \) so \( \lambda = 3 \). The general solution is

\[ y = (C_1 + C_2t) e^{3t}. \]

Then \( \frac{dy}{dt} = 3 (C_1 + C_2t) e^{3t} + C_2 e^{3t} \).

Using \( y(0) = 2 \) and \( \frac{dy}{dt}(0) = 1 \), \( C_1 = 2 \) and \( 3C_1 + C_2 = 1 \) so

\[ y = (2 - 5t) e^{3t}. \]
<table>
<thead>
<tr>
<th>Case</th>
<th>Roots of auxiliary equation</th>
<th>General solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b^2 - 4ac &gt; 0$</td>
<td>$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$</td>
<td>$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$</td>
</tr>
<tr>
<td>$b^2 - 4ac = 0$</td>
<td>$\lambda_{1,2} = -\frac{b}{2a}$</td>
<td>$y = (C_1 + C_2 t) e^{\lambda_1 t}$</td>
</tr>
<tr>
<td>$b^2 - 4ac &lt; 0$</td>
<td>$\lambda_{1,2} = p \pm iq$</td>
<td>$y = e^{pt} (C_1 \cos qt + C_2 \sin qt)$</td>
</tr>
</tbody>
</table>

$p = \frac{-b}{2a}$, $q = \frac{\sqrt{|b^2 - 4ac|}}{2a}$

Table 2.1: Solutions to the linear second-order, constant-coefficient, homogeneous ODE $a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = 0$.

**Question 2.9.** Find the solution to the initial-value problem $\frac{d^2 y}{dt^2} + 6 \frac{dy}{dt} + 13y = 0$, with $y(0) = -1$, $\frac{dy}{dt}(0) = 11$.

**Solution.** The auxiliary equation is $\lambda^2 + 6 \lambda + 13 = 0$.

$b^2 - 4ac = 36 - 52 = -16 = 16i^2$ so

$$\lambda = \frac{-6 \pm 4i}{2} = -3 \pm 2i.$$ 

The general solution is

$$y = e^{-3t} (C_1 \cos 2t + C_2 \sin 2t)$$

so

$$\frac{dy}{dt} = -3e^{-3t} (C_1 \cos 2t + C_2 \sin 2t) y + e^{-3t} (-2C_1 \sin 2t + 2C_2 \cos 2t).$$

Using $y(0) = -1$, $\frac{dy}{dt}(0) = 11$, $C_1 = -1$, $-3C_1 + 2C_2 = 11$ so

$$y = e^{-3t} (4 \sin 2t - \cos 2t).$$

**2.9 Practical example: damped springs**

For the case of the damped spring note that in terms of the physical parameters $a = m > 0$, $b = C > 0$ and $c = k > 0$. Hence

$$b^2 - 4ac = C^2 - 4mk.$$
2.9.1 **Overdamping:** $C^2 - 4mk > 0$

Since the physical parameters $m$, $k$ and $C$ are all positive, we have that

$$\sqrt{C^2 - 4mk} < |C|,$$

and so $\lambda_1$ and $\lambda_2$ are both negative. Thus for large times ($t \to \infty$) the solution $y(t) \to 0$ exponentially fast. For example, the mass might be immersed in thick oil. Two possible solutions, starting from two different initial conditions, are shown in Figure 2.2(a). Whatever initial conditions you choose, there is at most one oscillation. At some point for all practical purposes the spring is in the equilibrium position.
2.9.2 **Critical damping:** \( C^2 - 4mk = 0 \)

In appearance—see Figure 2.2(b)—the solutions for the critically damped case look very much like those in Figure 2.2(a) for the overdamped case.

2.9.3 **Underdamping:** \( C^2 - 4mk < 0 \)

Since for the spring

\[
p = -\frac{b}{2a} = -\frac{C}{2m} < 0,
\]

the mass will oscillate about the equilibrium position with the amplitude of the oscillations decaying exponentially in time; in fact the solution oscillates between the exponential envelopes which are the two dashed curves \( Ae^{pt} \) and \(-Ae^{pt}\), where \( A = +\sqrt{C_1^2 + C_2^2} \)—see Figure 2.2(c). In this case, for example, the mass might be immersed in light oil or air.
2. Problems: 2nd Order Homogeneous ODEs

Problem 2.1. Find the general solutions to the following differential equations:

(a) \( \frac{d^2y}{dt^2} - 9y = 0 \); (b) \( \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0 \); (c) \( \frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 16y = 0 \).

Problem 2.2. Find the general solutions to the following differential equations:

(a) \( \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0 \); (b) \( \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 0 \); (c) \( \frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 13y = 0 \).

Problem 2.3. Find the solutions of:

(a) \( \frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0 \), with \( y(0) = 0, \frac{dy}{dt}(0) = 3 \);
(b) \( \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0 \), with \( y(0) = 1, \frac{dy}{dt}(0) = 2 \);
(c) \( \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 0 \), with \( y(0) = 3, \frac{dy}{dt}(0) = 1 \).

Problem 2.4. Find the solution of \( \frac{d^2y}{dt^2} + \omega^2 y = 0 \) with \( y(0) = a, \frac{dy}{dt}(0) = 0 \). Write down the frequency, period and amplitude of oscillation.

Problem 2.5. The following represent the motion of oscillating springs.

1. \( \frac{d^2y}{dt^2} + 4y = 0 \), \( y(0) = 5, \frac{dy}{dt}(0) = 0 \),
2. \( 4\frac{d^2y}{dt^2} + y = 0 \), \( y(0) = 10, \frac{dy}{dt}(0) = 0 \),
3. \( \frac{d^2y}{dt^2} + 6y = 0 \), \( y(0) = 4, \frac{dy}{dt}(0) = 0 \),
4. \( 6\frac{d^2y}{dt^2} + y = 0 \), \( y(0) = 20, \frac{dy}{dt}(0) = 0 \).

Use the previous problem to determine which differential equation represents

(a) the spring oscillating most quickly (with the shortest period)?
(b) the spring oscillating with the largest amplitude?
(c) the spring oscillating most slowly (with the longest period)?
(d) the spring oscillating with the largest maximum velocity?
Answers

1.(a) $y = C_1 e^{3t} + C_2 e^{-3t}$, 1.(b) $y = C_1 e^{2t} + C_2 e^{-3t}$, 1.(c) $y = (C_1 + C_2 t) e^{-3t}$.

2.(a) $y = e^{-t} (C_1 \cos 2t + C_2 \sin 2t)$, 2.(b) $y = e^{2t} (C_1 \cos t + C_2 \sin t)$, 2.(c) $y = e^{3t} (C_1 \cos 2t + C_2 \sin 2t)$.

3.(a) $y = e^{-2t} - e^{-5t}$, 3.(b) $y = (1 + 5t) e^{-3t}$, 3.(c) $y = e^{-2t} (3 \cos t + 7 \sin t)$.

4. $y(t) = a \cos \omega t$. Frequency $= \omega$, period $= 2\pi/\omega$, amplitude $= a$.

5.(a) 3, 5.(b) 4, 5.(c) 4, 5.(d) 1.
Chapter 3

Inhomogeneous Linear ODEs

3.1 Examples of applications

**Example 3.1. Forced spring systems** What happens if our spring system (damped or undamped) is forced externally? For example, consider the following initial-value problem for a forced harmonic oscillator (which models a mass on the end of a spring which is forced externally)

\[
m \frac{d^2 y}{dt^2} + C \frac{dy}{dt} + ky = f(t), \quad y(0) = \frac{dy}{dt}(0) = 0.
\]

Here \( y(t) \) is the displacement of the mass, \( m \), from equilibrium at time \( t \). The external forcing \( f(t) \) could be oscillatory, say

\[
f(t) = A \sin \omega t,
\]

where \( A \) and \( \omega \) are also (given) positive constants. We will see in this chapter how solutions to such problems can behave quite dramatically when the frequency of the external force \( \omega \) matches that of the natural oscillations \( \omega_0 = +\sqrt{k/m} \) of the undamped \( (C \equiv 0) \) system—undamped resonance! We will also discuss the phenomenon of resonance in the presence of damping \( (C > 0) \).

3.2 Linear operators

Consider the general inhomogeneous second-order linear ODE

\[
a(t) \frac{d^2 y}{dt^2} + b(t) \frac{dy}{dt} + c(t)y = f(t).
\]

We can abbreviate the ODE (3.1) to

\[
Ly(t) = f(t),
\]

where \( L \) is the differential operator

\[
L = a(t) \frac{d^2}{dt^2} + b(t) \frac{d}{dt} + c(t).
\]

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We can re-interpret our general linear second-order ODE as follows. When we operate on a function \( y(t) \) by the differential operator \( L \), we generate a new function of \( t \):

\[
Ly(t) = a(t) \frac{d^2y}{dt^2}(t) + b(t) \frac{dy}{dt}(t) + c(t)y(t).
\]

To solve (3.2), we want the most general expression, \( y \) as a function of \( t \), which is such that \( L \) operated on \( y \) gives \( f(t) \).

An operator \( L \) is said to be linear if

\[
L(\alpha y_1 + \beta y_2) = \alpha Ly_1 + \beta Ly_2,
\]

for every \( y_1 \) and \( y_2 \), and all constants \( \alpha \) and \( \beta \).

As an example, the operator \( L \) in (3.3) is linear.

### 3.3 Solving inhomogeneous linear ODEs

Consider the linear second-order ODE

\[
Ly = f.
\] (3.4)

To solve this problem we first consider the solution to the associated homogeneous ODE:

\[
Ly_{CF} = 0.
\] (3.5)

This solution is called a Complementary Function (CF). Since the ODE (3.5) is linear, second-order and homogeneous, we can always find an expression for the solution. In the constant-coefficient case the solution has one of the forms given in Table 2.1. Now suppose that we can find a particular solution—often called the particular integral (PI)—of (3.4), i.e. some function, \( y_{PI} \), which satisfies (3.4):

\[
Ly_{PI} = f.
\]

Then the complete general solution of (3.4) is

\[
y = y_{CF} + y_{PI}.
\]

This must be the general solution because it contains two arbitrary constants (in the \( y_{CF} \) part) and satisfies the ODE, since, using that \( L \) is a linear operator,

\[
L(y_{CF} + y_{PI}) = Ly_{CF} + Ly_{PI} = f.
\]

Hence to summarise:
To solve an inhomogenous linear ODE of the form

\[ Ly = f : \]

1. Find the general solution—the complementary function \( y_{CF} \)—to the corresponding homogeneous linear ODE

\[ Ly_{CF} = 0. \]

2. Find any solution—the particular integral \( y_{PI} \)—to the full inhomogeneous linear ODE

\[ Ly_{PI} = f. \]

Then the general solution to the inhomogeneous linear ODE is

\[ y = y_{CF} + y_{PI}. \]

### 3.4 Method of undetermined coefficients

We now need to know how to obtain a particular integral. For special cases of the inhomogeneity \( f(t) \) we use the method of undetermined coefficients. In the method of undetermined coefficients we make an initial assumption about the form of the particular integral \( y_{PI} \), but with the coefficients left unspecified. We substitute our guess for \( y_{PI} \) into the linear ODE, \( Ly = f \), and attempt to determine the coefficients so that \( y_{PI} \) satisfies the equation.

---

**Question 3.1.** Find the general solution of the linear ODE

\[ \frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 3e^{2t}. \]

**Solution.**

**Step 1: Find the complementary function**

Looking for a solution of the form \( e^{\lambda t} \), the auxiliary equation is \( \lambda^2 - 3\lambda - 4 = 0 \) which has two real distinct roots \( \lambda_1 = 4 \) and \( \lambda_2 = -1 \) so that

\[ y_{CF}(t) = C_1e^{4t} + C_2e^{-t}. \]
Table 3.1: Method of undetermined coefficients. When the inhomogeneity $f(t)$ has the form (or is any constant multiplied by this form) shown in the left-hand column, then you might try a $y_{p1}(t)$ of the form shown in the right-hand column. We can also make the obvious extensions for combinations of the inhomogeneities $f(t)$ shown.
Step 2: Find the particular integral

Assume that the particular integral has the form (using Table 3.1)

\[ y_{PI}(t) = Ae^{2t}, \]

where the coefficient \( A \) is yet to be determined. Substituting this form for \( y_{PI} \) into the ODE, we get

\[ (4A - 6A - 4A)e^{2t} = 3e^{2t} \quad \iff \quad -6Ae^{2t} = 3e^{2t}. \]

Hence \( A \) must be \(-\frac{1}{2}\) and a particular solution is

\[ y_{PI}(t) = -\frac{1}{2}e^{2t}. \]

Hence the general solution to the differential equation is

\[ y(t) = C_1e^{4t} + C_2e^{-t} \begin{array}{c} -\frac{1}{2}e^{2t} \end{array} + y_{CF}. \]

---

**Question 3.2.** Find the general solution of the linear ODE

\[ \frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 2\sin t. \]

**Solution.**

Step 1: Find the complementary function

In this case, the complementary function is clearly the same as in the last example—the corresponding homogeneous equation is the same—hence

\[ y_{CF}(t) = C_1e^{4t} + C_2e^{-t}. \]

Step 2: Find the particular integral

Assume that \( y_{PI} \) has the form (using Table 3.1)

\[ y_{PI}(t) = A\sin t + B\cos t, \]

where the coefficients \( A \) and \( B \) are yet to be determined. Substituting this form for \( y_{PI} \) into the ODE implies

\[ (-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t \]
\[ \iff (-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t. \]

Equating coefficients of \( \sin t \) and also \( \cos t \), we see that

\[ -5A + 3B = 2 \quad \text{and} \quad -5B - 3A = 0. \]
Hence \( A = -\frac{5}{17} \) and \( B = \frac{3}{17} \) and so
\[
y_{PI}(t) = -\frac{5}{17} \sin t + \frac{3}{17} \cos t.
\]

Thus the general solution is
\[
y(t) = \left( C_1 e^{4t} + C_2 e^{-t} \right) y_{CF} - \frac{5}{17} \sin t + \frac{3}{17} \cos t \cdot y_{PI}.
\]

**Question 3.3.** Find the solution to the initial value problem
\[
\frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 5y = 20, \quad y(0) = 2, \quad \frac{dy}{dt}(0) = 7.
\]

**Solution.**

**Step 1** Solve \( \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 5y = 0 \).

Auxiliary equation: \( \lambda^2 + 4\lambda + 5 = 0 \).

Then \( b^2 - 4ac = 16 - 20 = -4 = 4i^2 \) so
\[
\lambda = \frac{-4 \pm 2i}{2} = -2 \pm i.
\]

The general solution for the homogeneous equation is \( y_{CF} = e^{-2t} \left( C_1 \cos t + C_2 \sin t \right) \).

**Step 2** Find a particular integral of \( \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 5y = 20 \).

Set \( y = C \).

Then \( \frac{dy}{dt} = \frac{d^2y}{dt^2} = 0 \) so \( 5C = 20 \) and \( C = 4 \). Hence a particular integral is \( y_{PI} = 4 \).

**Step 3** The general solution of \( \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 5y = 20 \) is
\[
y = e^{-2t} \left( C_1 \cos t + C_2 \sin t \right) + 4.
\]

**Step 4** Use the initial conditions for the general solution obtained in Step 3. Now
\[
\frac{dy}{dt} = -2e^{-2t} \left( C_1 \cos t + C_2 \sin t \right) + e^{-2t} \left( -C_1 \sin t + C_2 \cos t \right).
\]

Using \( y(0) = 2, \frac{dy}{dt}(0) = 7 \),
\[
C_1 + 4 = 2, \quad -2C_1 + C_2 = 7,
\]
so \( C_1 = -2 \) and \( C_2 = 3 \). Hence
\[
y = e^{-2t} \left( 3 \sin t - 2 \cos t \right) + 4.
\]
3.5 Degenerate inhomogeneities

Question 3.4. Find the general solution of the degenerate linear ODE

\[ \frac{d^2y}{dt^2} + 4y = 3 \cos 2t. \]

Solution.

Step 1: Find the complementary function

First we solve the corresponding homogeneous equation

\[ \frac{d^2y}{dt^2} + 4y = 0, \tag{3.6} \]

to find the complementary function. Two solutions to this equation are \( \sin 2t \) and \( \cos 2t \), and so the complementary function is

\[ y_{CF}(t) = C_1 \sin 2t + C_2 \cos 2t, \]

where \( C_1 \) and \( C_2 \) are arbitrary constants.

Step 2: Find the particular integral

Assume that \( y_{PI} \) has the form

\[ y_{PI}(t) = A \sin 2t + B \cos 2t, \]

where the coefficients \( A \) and \( B \) are yet to be determined. Substituting this form for \( y_{PI} \) into the ODE implies

\[
\begin{align*}
(-4A \sin 2t - 4B \cos 2t) + 4(A \sin 2t + B \cos 2t) &= 3 \cos 2t \\
\Leftrightarrow (4B - 4B) \sin 2t &+ (4A - 4A) \cos 2t = 3 \cos 2t.
\end{align*}
\]

Since the left-hand side is zero, there is no choice of \( A \) and \( B \) that satisfies this equation. Hence for some reason we made a poor initial choice for our particular solution \( y_{PI}(t) \). This becomes apparent when we recall the solutions to the homogeneous equation (3.6) are \( \sin 2t \) and \( \cos 2t \). These are solutions to the homogeneous equation and cannot possibly be solutions to the inhomogeneous case we’re considering. We must therefore try a slightly different choice for \( y_{PI}(t) \), for example,

\[ y_{PI}(t) = At \cos 2t + Bt \sin 2t. \]

Substituting this form for \( y_{PI} \) into the ODE and cancelling terms imply

\[ -4A \sin 2t + 4B \cos 2t = 3 \cos 2t. \]

Therefore, equating coefficients of \( \sin 2t \) and \( \cos 2t \), we see that \( A = 0 \) and \( B = \frac{\frac{3}{4}}{4} \) and so

\[ y_{PI}(t) = \frac{3}{4} t \sin 2t. \]
Hence the general solution is

\[ y(t) = C_1 \sin 2t + C_2 \cos 2t + \frac{3}{4} t \sin 2t. \]

Occasionally such a modification will be insufficient to remove all duplications of the solutions of the homogeneous equation, in which case it is necessary to multiply by \( t \) a second time. For a second-order equation though, it will never be necessary to carry the process further than two modifications.

---

**Question 3.5.** Consider the following equation

\[ \frac{d^2y}{dt^2} + 16y = f(t). \]  

(a) Find the general solution of equation (3.7) when \( f(t) = 0 \).

(b) Now suppose that \( f(t) = \cos 2t \). Write down what form for the particular integral you would try.

(c) Now suppose that \( f(t) = \sin 4t \). Write down what form for the particular integral you would try.

**Solution.**

(a) \( y = C_1 \cos 4t + C_2 \sin 4t \).

(b) \( y = A \sin 2t + B \cos 2t \).

(c) \( y = A t \sin 4t + B t \cos 4t \).

---

**Question 3.6.** Consider the following equation

\[ \frac{d^2y}{dt^2} - 36y = f(t). \]  

(a) Find the general solution of equation (3.8) when \( f(t) = 0 \).

(b) Now suppose that \( f(t) = e^{4t} \). Write down what form for the particular integral you would try.

(c) Now suppose that \( f(t) = e^{-6t} \). Write down what form for the particular integral you would try.

**Solution.** For part (a), the auxiliary equation is \( \lambda^2 - 36 = 0 \) so \( \lambda = \pm 6 \) and the general solution is

\[ y = C_1 e^{6t} + C_2 e^{-6t}. \]

(b) \( y = Ae^{4t}. \)

(c) \( y = Ate^{-6t}. \)
Resonance

Example 3.2. Undamped resonance Consider the following initial value problem for a forced harmonic oscillator, which, for example, models a mass on the end of a spring which is forced externally,

\[ \frac{dy}{dt} + \omega_0^2 y = \frac{1}{m} f(t), \quad y(0) = \frac{dy}{dt}(0) = 0. \]

Here \( y(t) \) is the displacement of the mass \( m \) from equilibrium at time \( t \), and \( \omega_0 = \sqrt{k/m} \) is a positive constant representing the natural frequency of oscillation when no forcing is present. Suppose \( f(t) = A \sin \omega t \) is the external oscillatory forcing, where \( A \) and \( \omega \) are also positive constants.

The solution depends on if \( \omega \neq \omega_0 \) or if \( \omega = \omega_0 \).

If \( \omega \neq \omega_0 \), a calculation shows that the solution is

\[ y(t) = \frac{A \omega}{m(\omega^2 - \omega_0^2)} \cdot \left( \frac{1}{\omega_0} \sin \omega_0 t - \frac{1}{\omega} \sin \omega t \right), \tag{3.9} \]

where the first oscillatory term represents the natural oscillations, and the second, the forced mode of vibration.

What happens when \( \omega \rightarrow \omega_0 \)? If we naively take the limit \( \omega \rightarrow \omega_0 \) in (3.9) we see that the two oscillatory terms combine to give zero, but the denominator in the multiplicative term

\[ \frac{A \omega}{m(\omega^2 - \omega_0^2)} \]

also goes to zero. This implies we should be much more careful.

If \( \omega = \omega_0 \) the solution to the initial-value problem is

\[ y(t) = \frac{A}{2m\omega_0} \cdot \left( \frac{1}{\omega_0} \sin \omega_0 t - t \cos \omega_0 t \right). \tag{3.10} \]

The important aspect to notice is that when \( \omega = \omega_0 \), the second term \('t \cos \omega_0 t'\) grows without bound (the amplitude of these oscillations grows like \( t \)) and this is the “signature of undamped resonance”.

Damped resonance

Suppose we introduce damping into our simple spring system so that the coefficient of friction \( C > 0 \). In the overdamped, critically damped or underdamped cases the complementary function is always exponentially decaying in time. We call this part of the
solution the transient solution—it will be significant initially, but it decays to zero ex-
ponentially fast. The contribution to the solution from the particular integral, which
arises from the external forcing, cannot generate unbounded resonant behaviour for any
bounded driving oscillatory force. However the amplitude of the forced oscillations of the
solution (from the particular integral, which is potentially the only significant component
for large times) does have a global maximum at a given practical resonance frequency.
3. Problems: 2nd Order Inhomogeneous ODEs

**Problem 3.1.** For the following inhomogeneous differential equations, write down what form for the particular integral you would try (just write it down, there’s no need to do any calculations):

(a) \( \frac{d^2y}{dt^2} + 16y = \sin 3t \);  
(b) \( \frac{d^2y}{dt^2} + 16y = \cos 4t \);  
(c) \( \frac{d^2y}{dt^2} - 25y = e^{4t} \);  
(d) \( \frac{d^2y}{dt^2} - 25y = e^{-5t} \).

**Problem 3.2.** Find the general solution to the inhomogeneous differential equations:

(a) \( \frac{d^2y}{dt^2} + 4y = \sin 3t \);  
(b) \( \frac{d^2y}{dt^2} - 9y = 16e^{-t} \);  
(c) \( \frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 2\sin 4t \);  
(d) \( \frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = e^{-2t} \);  
(e) \( \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = t \);  
(f) \( \frac{d^2y}{dt^2} + 4y = \sin 2t \).

**Problem 3.3.** Solve \( \frac{d^2y}{dt^2} - y = 12e^{2t} \), with \( y(0) = 4, \frac{dy}{dt}(0) = 10 \).

**Problem 3.4.** Solve \( \frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = \cos 3t \), with \( y(0) = 0, \frac{dy}{dt}(0) = 5 \).

**Problem 3.5.** The charge \( Q(t) \) in a simple electrical circuit, consisting of a coil with inductance \( L \), a capacitance \( C \) and resistance \( R \), satisfies

\[
L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = V(t)
\]

where \( V(t) \) is an imposed voltage. Find \( Q(t) \), given that \( L = 1, R = 2, C = 1/5 \), \( Q(0) = Q_0, \frac{dQ}{dt}(0) = 0 \), and the voltage is \( V(t) = e^{-t} \sin 3t \).

**Problem 3.6.** The landing gear of a plane, consisting of a spring and a damper, is tested using a drop test, where the response of a subsystem of the landing gear is subjected to an instantaneous force representative of a plane landing. The spring force \( F_{spring} \), modelled using Hooke’s law, and the damper force, \( F_{damper} \) are as follows:

\[
F_{spring} = kx, \quad F_{damper} = B \frac{dx}{dt},
\]

where \( x \) is the height and \( t \) is time.

Applying Newton’s 2nd law, ‘\( F = ma \)’, (and the force due to gravity) we get

\[
mg - B \frac{dx}{dt} - kx = m \frac{d^2x}{dt^2} \quad \Rightarrow \quad \frac{d^2x}{dt^2} + \frac{B}{m} \frac{dx}{dt} + \frac{k}{m} x = g.
\]

Given the initial conditions \( x(0) = 0 \) and \( \frac{dx}{dt} = V \) when \( t = 0 \) and that \( m = 50, k = 312.5, B = 150 \) and we approximate \( g = 10 \) find the solution to the initial value problem (your answer will be in terms of \( V \)).
Answers

1. (a) $y = A \sin 3t + B \cos 3t$; 1. (b) $y = At \sin 4t + Bt \cos 4t$; 1. (c) $y = Ae^{4t}$;
1. (d) $y = Ae^{-5t}$.

2. (a) $y = C_1 \sin 2t + C_2 \cos 2t - \frac{1}{5} \sin 3t$; 2. (b) $y = C_1 e^{3t} + C_2 e^{-3t} - 2e^{-t}$.
2. (c) $y = C_1 e^{3t} + C_2 e^{2t} + \frac{2}{25} \cos 4t - \frac{1}{25} \sin 4t$;
2. (d) $y = C_1 e^{3t} + C_2 e^{2t} + \frac{1}{25} e^{-2t}$.
2. (e) $y = (C_1 + C_2 t) e^{-t} + t - 2$; 2. (f) $y = C_1 \sin 2t + C_2 \cos 2t - \frac{t}{4} \cos 2t$.

3. $y = e^t - e^{-t} + 4e^{2t}$.

4. $y = \frac{31}{6} e^{3t} - \frac{67}{13} e^{2t} - \frac{5}{78} \sin 3t - \frac{1}{78} \cos 3t$.

5. $Q = e^{-t} \left( Q_0 \cos 2t + \left( \frac{5Q_0 + 3}{10} \right) \sin 2t \right) - \frac{1}{5} e^{-t} \sin 3t$.

6. $x = e^{-1.5t} \left( -1.6 \cos 2t + (0.5V - 1.2) \sin 2t \right) + 1.6$.
Chapter 4

Partial Differentiation

To date most functions have had only one variable; for example,

- “$y$ is a function of $x$”,
- “position is a function of time”.

However functions can have two or more variables — they often do in physical situations! Partial differentiation gives us a way to generalise differentiation for functions of several variables.

4.1 Reminder of derivatives of functions of a single variable

Remember that the derivative of a function, say $f$ of a single variable, say $x$, might be thought of as the slope of the graph $y = f(x)$ This derivative, $df/dx$ or $f'(x)$, can be defined by taking $h \to 0$ in

\[
\frac{f(x + h) - f(x)}{h}
\]

Alternatively, if $z = f(t)$ is position as a function of time $t$, its rate of change is the velocity $dz/dt = f'(t)$.

We should remember rules for differentiation including those for products and quotients as well as the chain rule.
Question 4.1. Let $y = x \ln x$. Find $dy/dx$.

Solution. Use the product rule: if $y(x) = u(x)v(x)$ then $\frac{dy}{dx} = \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$. Here we let $u = x$ and $v = \ln x$ so $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 1/x$. Then

$$\frac{dy}{dx} = \frac{d}{dx}(x \ln x) = x \cdot (1/x) + (\ln x) \cdot 1 = 1 + \ln x.$$ 

Question 4.2. Find $dy/dx$ where $y = x^2 \cos x$.

Solution. Let $u = x^2$ and $v = \cos x$ so $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = -\sin x$. Thus the product rule gives

$$\frac{dy}{dx} = -x^2 \sin x + 2x \cos x.$$ 

Question 4.3. Find $\frac{d}{dx} \left( \frac{x}{e^x} \right)$.

Solution. This time we use the quotient rule: $\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$. Letting $u = x$ and $v = e^x$ we have $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = e^x$. Thus the quotient rule gives

$$\frac{d}{dx} \left( \frac{x}{e^x} \right) = \frac{e^x - xe^x}{(e^x)^2} = (1 - x)e^{-x}.$$ 

Note that $x/e^x$ is the same as $xe^{-x}$ which can be differentiated using the product rule.

Question 4.4. Let $y = 2 \cos^2 x$. Find $dy/dx$.

Solution. Now use the chain rule: $\frac{dy}{dx} = \frac{du}{dx} \frac{dy}{du}$. Writing $y = 2(\cos x)^2 = 2u^2$ with $u = \cos x$ we have $\frac{du}{dx} = -\sin x$ and $\frac{dy}{du} = 4u$. Thus the chain rule gives

$$\frac{dy}{dx} = \frac{du}{dx} \frac{dy}{du} = (-\sin x)4u = -4 \sin x \cos x.$$ 

Question 4.5. A particle moves along a line $y = 6x - 3$ in such a way that its $x$ coordinate at time $t$ is $x = 3t + 5$. Find the instantaneous velocity in the $y$ direction.

Solution. It is possible to use the chain rule: $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = 6 \times 3 = 18$.

(Alternatively the expression for $x$ can be substituted into that for $y$: $y = 6(3t+5) - 3 = 18t + 27$. Then $dy/dt = 18$.)

### 4.2 Partial derivatives of functions of two variables

Example 4.1. If we define

$$f(x, y) = xy^2 + y^3$$

then $f$ is a function of two variables, namely $x$ and $y$. 


Example 4.2. Consider the volume $V$ of box with sides $x$, $y$ and $z$:

![Diagram of a box with sides labeled x, y, and z.]

Then $V = V(x, y, z) = xyz$ is a function of three variables.

Example 4.3. Consider the function $f$ of two variables given by

$$f(x, y) = x^2 + 3xy - y^2.$$  

We might be interested in how rapidly this varies as either $x$ or $y$ changes. In particular, fixing $y$ and regarding $f$ purely as depending upon $x$ we can then differentiate $f$ with respect to $x$. Similarly, we can regard $x$ as fixed to determine a derivative of $f$ with respect to $y$. We then get the two partial derivatives

$$\frac{\partial f}{\partial x} = 2x + 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x - 2y.$$  

Note the use of the different style “d”, $\partial$, when we take partial derivatives. These partial derivatives can be got from:

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}; \quad (4.1)$$

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}. \quad (4.2)$$

It can be seen from these that

\[
\frac{\partial f}{\partial x} \text{ is calculated by regarding } y \text{ as a constant and using the normal rules of differentiation to differentiate with respect to } x;
\]

\[
\frac{\partial f}{\partial y} \text{ is calculated by regarding } x \text{ as a constant and using the normal rules of differentiation to differentiate with respect to } y.
\]
Example 4.4. Consider the function $f$ of two variables given by

$$f(x, y) = x \sin y + ye^{2x}.$$ 

Then

$$\frac{\partial f}{\partial x} = \sin y + 2ye^{2x} \quad \text{and} \quad \frac{\partial f}{\partial y} = x \cos y + e^{2x}.$$ 

For $f$ a function of $x$ and $y$, we might visualise its graph as a surface in three-dimensional space. $\partial f/\partial x$ is then the slope of this surface in the $x$ direction (broken line in the figure below) while $\partial f/\partial y$ is the slope in the $y$ direction (dotted line).

Alternatively, still for a function of two variables $f(x, y)$, we might plot curves giving constant values of $f$ in the $x$ - $y$ plane. (Like contour lines on a map.)

Example 4.5. Taking $f(x, y) = x^2 + y^2$, $f = \text{constant}$, say $c$, if $x^2 + y^2 = r^2 = c \geq 0$,

with $r$ the distance of the point $(x, y)$ from the origin.
**Question 4.6.** Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x, y) = 3y/(y + \cos x)$.

**Solution.** For $\frac{\partial f}{\partial x}$ we regard $y$ as a constant and use the quotient rule:

$$
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left( \frac{3y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial x} (3y) - 3y \frac{\partial}{\partial x} (y + \cos x)}{(y + \cos x)^2}
$$

$$
= \frac{(y + \cos x) \times 0 - 3y \times (-\sin x)}{(y + \cos x)^2} = \frac{3y \sin x}{(y + \cos x)^2}.
$$

For $\frac{\partial f}{\partial y}$ we regard $x$ as a constant and use the quotient rule:

$$
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left( \frac{3y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial y} (3y) - 3y \frac{\partial}{\partial y} (y + \cos x)}{(y + \cos x)^2}
$$

$$
= \frac{(y + \cos x) \times 3 - 3y \times 1}{(y + \cos x)^2} = \frac{3 \cos x}{(y + \cos x)^2}.
$$

----

### 4.3 Higher-order partial derivatives

So far we have only seen first-order partial derivatives – taking a single derivative at a time. As with ordinary differentiation we can do repeated differentiation, taking derivatives of derivatives, to get higher-order derivatives.

**Example 4.6.** For the function

$$f(x, y) = e^{xy} + x^2 y$$

the first-order partial derivatives are

$$\frac{\partial f}{\partial x} = ye^{xy} + 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = xe^{xy} + x^2.$$  

We can now compute the second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( ye^{xy} + 2xy \right) = y^2 e^{xy} + 2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left( xe^{xy} + x^2 \right) = x^2 e^{xy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} \left( ye^{xy} + 2xy \right) = y \cdot xe^{xy} + e^{xy} + 2x = (xy + 1)e^{xy} + 2x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left( xe^{xy} + x^2 \right) = x \cdot ye^{xy} + e^{xy} + 2x = (xy + 1)e^{xy} + 2x.$$
It is a general rule that for “reasonable” functions,

\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}. \] (4.3)

For “mixed” derivatives of functions which you meet in this course, the order in which the derivatives are taken does not matter.

**Question 4.7.** Let \( f(x, y) = \ln(x^2 + y^2) \).

(i) Verify that \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \).

(ii) Show that \( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \).

**Solution.** Using the chain rule (regarding the variable we’re not differentiating with respect to as being constant), we have that

\[ \frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2} \]

and

\[ \frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}. \]

The quotient rule now gives

\[ \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{2x}{x^2 + y^2} \right) = \frac{(x^2 + y^2)2 - 2x \cdot 2x}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}, \]

\[ \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{2x}{x^2 + y^2} \right) = \frac{(x^2 + y^2)2 - 2y \cdot 2y}{(x^2 + y^2)^2} = \frac{-4xy}{(x^2 + y^2)^2}, \]

\[ \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{2y}{x^2 + y^2} \right) = \frac{(x^2 + y^2)2 - 2y \cdot 2y}{(x^2 + y^2)^2} = \frac{-4xy}{(x^2 + y^2)^2}, \]

\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{2y}{x^2 + y^2} \right) = \frac{(x^2 + y^2)2 - 2x \cdot 2y}{(x^2 + y^2)^2} = \frac{-4xy}{(x^2 + y^2)^2}. \]

Hence it can be seen that (i) \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \) and that (ii) \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \).

**Question 4.8.** Find \( \frac{\partial^2 w}{\partial x \partial y} \) given that \( w(x, y) = xy + \frac{e^y}{y^2 + 1} \).

**Solution.** This is made simpler by reversing the order of differentiation:

\[ \frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \left( \frac{e^y}{y^2 + 1} \right) = \frac{e^y}{y^2 + 1}, \]

\[ \frac{\partial w}{\partial y} = \frac{\partial}{\partial y} \left( \frac{e^y}{y^2 + 1} \right) = \frac{e^y \cdot y}{y^2 + 1} = \frac{e^y}{y^2 + 1}. \]

\[ \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial y} \right) = \frac{e^y}{y^2 + 1}. \]

\[ \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} \right) = \frac{e^y}{y^2 + 1}. \]

\[ \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x} = \frac{e^y}{y^2 + 1} = \frac{e^y}{y^2 + 1}. \]

**4.4 Directional Derivatives**

The value of \( \frac{\partial f}{\partial x} \) at a point \((a, b)\) gives the gradient of \( f(x, y) \) in the \( x \)-direction at \((a, b)\).

Similarly, the value of \( \frac{\partial f}{\partial y} \) at a point \((a, b)\) gives the gradient of \( f(x, y) \) in the \( y \)-direction at \((a, b)\).
If we require the gradient $m_\alpha$ of $f(x,y)$ at a point $(x,y)$ in a direction $\alpha$ to the $x$-axis (measured in the anti-clockwise direction) we can use the expression

$$m_\alpha = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \sin \alpha$$

**Example 4.7.** Consider the Rosenbrock’s function

$$f(x,y) = 100(y - x^2)^2 + (1 - x)^2$$

Evaluate the gradient of $f(x,y)$ in the (a) $x$-direction and (b) the $y$-direction at $(0,0)$, (c) at $(1,1)$ at an angle of $60^\circ$ to the $x$-axis, and (d) at $(1,2)$ at an angle of $30^\circ$ to the $x$-axis in the clockwise direction.

(a) $\frac{\partial f}{\partial x} = -400x(y - x^2) - 2(1 - x)$ therefore at $(0,0)$ we get $\frac{\partial f}{\partial x} = -2$.  

(b) $\frac{\partial f}{\partial y} = 200(y - x^2)$ therefore at $(0,0)$ we get $\frac{\partial f}{\partial y} = 0$.  

(c) At $(1,1)$ we get $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ so $m_\alpha = 0$.  

(d) At $(1,2)$ we get $\frac{\partial f}{\partial x} = -400$ and $\frac{\partial f}{\partial y} = 200$. Also, $\cos(-30^\circ) = \sqrt{3}/2$ and $\sin(-30^\circ) = -1/2$.  

So $m_\alpha = -400(\sqrt{3}/2) + 200(-1/2) = -200\sqrt{3} - 100$.

### 4.5 Using the chain rule to find derivatives

This looks slightly different for partial derivatives because of the extra variable(s).

For $f = f(x,y)$, $x = x(t)$ and $y = y(t)$, then as $t$ varies, $f$ changes because both $x$ and $y$ change:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$  

(4.4)

If $x$ and $y$ are functions of both $s$ and $t$, $x = x(s,t)$ and $y = y(s,t)$, the ordinary derivatives are replaced by partial derivatives,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t},$$

(4.5)

and there is the obvious formula for the partial derivative of $f$ with respect to $s$,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}.$$  

(4.6)
Question 4.9. Suppose \( f(x) = e^x \) and \( x(t) = t^3 \). Compute \( \frac{df}{dt} \).

Solution. As a function of \( t \) we have \( f(t) = e^{t^3} \). Now

\[
\frac{df}{dx} = e^x \quad \text{and} \quad \frac{dx}{dt} = 3t^2
\]

so the chain rule gives

\[
\frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} = e^x \cdot 3t^2 = 3t^2 e^{t^3}.
\]

Example 4.8. Suppose that \( f(x, y) = ye^x \), where \( x = \sin t \) and \( y = t^2 \). Then

\[
\frac{\partial f}{\partial x} = ye^x, \quad \frac{\partial f}{\partial y} = e^x \quad \text{and} \quad \frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = 2t.
\]

Using the chain rule it follows that

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = ye^x \cdot \cos t + e^x \cdot 2t = t^2 e^{\sin t} \cos t + 2te^{\sin t}.
\]

Note that, as a function of \( t \) we have

\[
f(t) = t^2 e^{\sin t}.
\]

We can also compute \( \frac{df}{dt} \) from this expression (using the product and chain rules for one variable — do this to see that the same result is obtained!)

Question 4.10. Suppose

\[
f(x, y) = x^3 - y^2
\]

where \( x = st \) and \( y = s/t \). Find \( \frac{\partial f}{\partial t} \).

Solution. We have

\[
\frac{\partial f}{\partial x} = 3x^2, \quad \frac{\partial f}{\partial y} = -2y
\]

and

\[
\frac{\partial x}{\partial t} = s, \quad \frac{\partial y}{\partial t} = -\frac{s}{t^2}.
\]

Using the chain rule it follows that

\[
\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = 3x^2 \cdot s + (-2y)\left(-\frac{s}{t^2}\right)
\]

\[
= 3(st)^2 s + 2(s/t)(s/t^2) = 3s^3 t^2 + 2s^2 / t^3.
\]

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4.6 Functions of many variables

Partial differentiation extends easily to functions of many variables.

- To calculate the partial derivative with respect to a certain variable keep all the other variables constant.
- For mixed derivatives the order (generally) doesn’t matter!
- The chain rule contains more terms.

Example 4.9. If \( f \equiv f(x, y, z) \) where \( x, y, z \) are functions of \( s, t \) then

\[
\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}.
\]

Similarly for \( \frac{\partial f}{\partial s} \).

Example 4.10. Consider the function of four variables

\[ f(x, y, z, t) = xy + z^2 t. \]

Here there are four first-order partial derivatives

\[
\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = 2zt, \quad \frac{\partial f}{\partial t} = z^2.
\]

We can also compute higher-order derivatives; for example

\[
\frac{\partial^2 f}{\partial z \partial t} = \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial t} \right) = \frac{\partial}{\partial z} (z^2) = 2z.
\]

Note that we also have \( \frac{\partial^2 f}{\partial t \partial z} = 2z \). (Check this!)

Question 4.11. Let \( f(x, y, z) = 2x^2 + y^2 + 3z \) and let \( x = uv \), \( y = 2v \) and \( z = v + \ln u \). Find \( \frac{\partial f}{\partial u} \) and \( \frac{\partial f}{\partial v} \).

Solution.

\[
\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial u} = 4x \cdot v + 2y \cdot 0 + 3 \cdot 1/u = 4uv^2 + 3/u.
\]

\[
\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial v} = 4x \cdot u + 2y \cdot 2 + 3 \cdot 1 = 4u^2v + 8v + 3.
\]
4.7 Partial differential equations

• A partial differential equation (or PDE) is an equation involving several variables, a function of these variables and its partial derivatives.

• PDEs arise in very many physical situations. They describe how physical quantities change with respect to several different variables.

• In this course we will only show that a given function is a solution of a given PDE — a straightforward calculation!

---

Example 4.11. An example of a PDE is

\[ 3y^2 \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = 2U. \]

where \( U \) is a function of \( x \) and \( y \).

---

Example 4.12. Another example of a PDE is

\[ \frac{\partial^2 w}{\partial t^2} + 2 \frac{\partial^2 w}{\partial x \partial y} = 4w. \]

where \( w \) is a function of \( x, y \) and \( t \).
Question 4.12. Show that $u(x, y) = e^{2x+y^2}$ is a solution of the partial differential equation

$$y^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{y} \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} = 0.$$ 

Solution. Calculating the partial derivatives, we have

$$\frac{\partial u}{\partial y} = 2ye^{2x+y^2}, \quad \frac{\partial^2 u}{\partial y^2} = 4y^2 e^{2x+y^2} + 2e^{2x+y^2}, \quad \frac{\partial u}{\partial x} = 2e^{2x+y^2}, \quad \frac{\partial^2 u}{\partial x^2} = 4e^{2x+y^2}.$$

Thus

$$y^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{y} \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} = y^2 \times 4e^{2x+y^2} + \frac{1}{y} \times 2y e^{2x+y^2} - \left( 4y^2 e^{2x+y^2} + 2e^{2x+y^2} \right) = 0.$$ 

Example 4.13. Let $w(x, t)$ denote the displacement of the point at $x$ on a string at time $t$. A wave moving along the string with speed $c$ can be modelled by the one-dimensional wave equation:

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}. \quad (4.7)$$

A slice of the graph with $t$ fixed shows us the shape of the string at that time $t$. The figure shows three such slices:

A slice of the graph with $x$ fixed would show us how the point at $x$ on the string moves with time.
Question 4.13. Show that the function
\[ w(x, t) = \cos(x - ct) \]
is a solution of the 1D wave equation.

Solution. Direct calculations give
\[
\frac{\partial w}{\partial t} = c \sin(x - ct), \quad \frac{\partial^2 w}{\partial t^2} = -c^2 \cos(x - ct)
\]
and
\[
\frac{\partial w}{\partial x} = -\sin(x - ct), \quad \frac{\partial^2 w}{\partial x^2} = -\cos(x - ct).
\]
It follows that
\[
\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2},
\]
as required.

Note: The alternative notation of subscripts is also used for partial differentiation, so that \( f_x \) means \( \partial f/\partial x \), \( u_{yz} \) is the same as \( \partial^2 u/\partial y \partial z \), etc.
4. Problems: Partial Differentiation

Problem 4.1. Find \( \partial f / \partial x \) and \( \partial f / \partial y \) for each of the following functions:

(a) \( 3x + 4y \)  
(b) \( xy^3 + x^2y^2 \)  
(c) \( x^3y + e^x \)  
(d) \( xe^{2x+3y} \)  
(e) \( (x - y)/(x + y) \)  
(f) \( 2x \sin(x^2y) \).

Problem 4.2. Find \( \partial f / \partial x \), \( \partial f / \partial y \) and \( \partial f / \partial z \) for \( f(x, y, z) = x \cos z + x^2y^3e^z \).

Problem 4.3. Find \( \partial^2 f / \partial x^2 \) and \( \partial^2 f / \partial y^2 \), and check that \( \partial^2 f / \partial x \partial y = \partial^2 f / \partial y \partial x \) for

(i) \( f(x, y) = x^2 \sin y + y^2 \cos x \);  
(ii) \( f(x, y) = \left(\frac{y}{x}\right) \ln x \).

Problem 4.4. Calculate the gradient for the Rosenbrock’s function \( f(x, y) = 100(y - x^2)^2 + (1 - x)^2 \) at

(a) \( (x, y) = (-1, 1) \) at an angle \( \alpha = 45^\circ \) to the \( x \)-axis is an anti-clockwise direction.

(b) \( (x, y) = (-1, 1) \) at an angle \( \alpha = 45^\circ \) to the \( x \)-axis is a clockwise direction.

(c) \( (x, y) = (1, 2) \) at an angle \( \alpha = 80^\circ \) to the \( x \)-axis is an anti-clockwise direction.

Problem 4.5. Suppose that \( f(x, y) = x^2 \sin y + y^2 \cos x \), \( x = r \cos t \) and \( y = r \sin t \). Find \( \partial f / \partial r \) and \( \partial f / \partial t \).

Problem 4.6. Suppose that \( f(x, y) = x^2 + xy - y^2 \), \( x = r \cos \theta \) and \( y = r \sin \theta \). Find \( \partial f / \partial r \) and \( \partial f / \partial \theta \), (i) by direct substitution and (ii) by using the chain rule.

Problem 4.7. Suppose that \( f(x, y, z) = 2y - \sin xz \), \( x = 3t \), \( y = e^t - 1 \) and \( z = \ln t \). Find \( df/dt \).

Problem 4.8. Suppose that \( u(x, y) = x^2 + xy + y^2 \), \( x = uv \) and \( y = u/v \). Show that

\[
\frac{\partial f}{\partial u} + v \frac{\partial f}{\partial v} = 2x \frac{\partial f}{\partial x} \quad \text{and} \quad u \frac{\partial f}{\partial u} - v \frac{\partial f}{\partial v} = 2y \frac{\partial f}{\partial y}.
\]

Is the same true for \( f(x, y) = x \sin y + y \sin x \)?

Problem 4.9. Show that \( u = \ln(1 + xy^2) \) is a solution of the partial differential equation

\[
2 \frac{\partial^2 u}{\partial x^2} + y^3 \frac{\partial^2 u}{\partial x \partial y} = 0.
\]

Problem 4.10. Show that \( u = \ln(1 + xy^2) \) is a solution of the partial differential equation

\[
2x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} - 4u = 0.
\]

Problem 4.11. Show that \( u(x, y) = x^2 \cosh(1 + xy^2) \) is a solution of the partial differential equation

\[
2x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} - 4u = 0.
\]

Problem 4.12. Show that \( u(x, t) = \ln(2x + 2ct) \) is a solution of the one-dimensional wave equation

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.
\]
3. (i) \(2 \sin y - y^2 \cos x, 2 \cos x - x^2 \sin y\).
3. (ii) \(\frac{y}{x^3} (2 \ln x - 3), 0\).

4. (a) \(-2\sqrt{2} = -2.828\), (b) \(-2\sqrt{2} = -2.828\), (c) 127.5

5. \(-2/r^3, 0\).

6. \(2r(\cos^2 \theta + \cos \theta \sin \theta - \sin^2 \theta), r^2(\cos^2 \theta - \sin^2 \theta - 4 \cos \theta \sin \theta)\).

7. \(4w^3v^2 - 4w^3/v^2, 2w^4v + 2w^4/v^3\).

8. \(-3 \cos(3t \ln t)(1 + \ln t) + 2e^{t-1}\).
Chapter 5

Maxima and Minima

As before, we first look at the one-variable case

**Question 5.1.** Find the stationary points of the function

\[ f(x) = \frac{x^3}{3} - x + 2. \]

**Solution.** We have \( f'(x) = x^2 - 1 \) so

\[ f'(x) = 0 \iff x^2 - 1 = 0 \iff x^2 = 1 \iff x = \pm 1. \]

Thus \( f \) has two stationary points, one at \( x = -1 \) and the other at \( x = 1 \).

**Example 5.1.** Returning to the previous example we have \( f''(x) = 2x \). The second derivative test then gives us the following information:

<table>
<thead>
<tr>
<th>Stationary point ( x )</th>
<th>( f''(x) )</th>
<th>Nature of stationary point</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-2 &lt; 0</td>
<td>local maximum</td>
</tr>
<tr>
<td>+1</td>
<td>+2 &gt; 0</td>
<td>local minimum</td>
</tr>
</tbody>
</table>

The graph of \( f \) looks something like the following:

![Graph of f(x) = \( \frac{x^3}{3} - x + 2 \)](image-url)
Example 5.2. Consider the function \( f(x) = x^4 \). Then \( f'(x) = 4x^3 \) so the stationary points of \( f \) occur when

\[
f'(x) = 0 \iff 4x^3 = 0 \iff x = 0.
\]

Now \( f''(x) = 12x^2 \) so at the stationary point \( x = 0 \) we get \( f''(x) = 0 \); it follows that we cannot use the second derivative test to determine the nature of the stationary point! However \( f(x) = x^4 = (x^2)^2 \geq 0 \) for all values of \( x \) while \( f(0) = 0 \). It follows that \( x = 0 \) must be a local minimum.

Question 5.2. You have been asked to design a 1-litre oil can shaped like a cylinder. What dimensions will use the least material?

Solution. Consider the oil can:

The amount of material need to make the oil can is simply \( A \), the surface area of the cylinder; we wish to minimise \( A \) whilst keeping the volume of the can, \( V \) equal to 1 l = 1000 cm\(^3\). Now

\[
A = \text{area of ends} + \text{area of cylinder wall} = 2\pi r^2 + 2\pi rh.
\]

On the other hand

\[
V = \pi r^2 h = 1000 \quad \text{so} \quad h = \frac{1000}{\pi r^2}.
\]

It follows that

\[
A = A(r) = 2\pi r^2 + 2\pi r \times \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}.
\]

Therefore

\[
A'(r) = 4\pi r - \frac{2000}{r^2}
\]

so

\[
A'(r) = 0 \iff 4\pi r = \frac{2000}{r^2} \iff r = \sqrt[3]{\frac{500}{\pi}}.
\]
Thus $A$ has a stationary point when $r = \sqrt[3]{500/\pi}$; we need to check the nature of this stationary point. Now

$$A''(r) = 4\pi + \frac{4000}{r^3}$$

so when $r = \sqrt[3]{500/\pi}$ we get

$$A''(r) = 4\pi + 4000 \times \frac{\pi}{500} = 12\pi > 0.$$

Thus the stationary point is a local minimum. It follows that use of material is minimised if we make a cylindrical oil can with radius

$$r = \sqrt[3]{500/\pi} \approx 5.42 \text{ cm}.$$

Two variables are more complicated! But we shall clearly still need first derivatives to vanish, so, for $f = f(x, y)$, we need both

$$\frac{\partial f}{\partial x}(x, y) = 0 \text{ and, simultaneously, } \frac{\partial f}{\partial y}(x, y) = 0. \quad (5.1)$$

One thing we shall certainly want is an equivalent of the second-derivative test.

**Example 5.3.** What sort stationary points might (i) $f(x, y) = x^2 + y^2$, (ii) $g(x, y) = -x^2 - 2y^2$, (iii) $h(x, y) = x^2 + 4xy + y^2$ have?

The first two are fairly obvious: For $(x, y) \neq (0, 0)$, $f > 0$ and $g < 0$ but $f = g = 0$ if $(x, y) = (0, 0)$. $f$ clearly has a minimum at $x = y = 0$ while $g$ has a maximum there.

The third one is not so straightforward, although it’s apparent that $\partial h/\partial x = 2(x + 2y)$ and $\partial h/\partial y = 2(2x + y)$ are only simultaneously zero at the origin. Taking $x = 0$ gives $h = y^2 \geq 0$, while if $y = 0$, $h = x^2 \geq 0$. However, we can complete the square to get $h = (x + 2y)^2 - 3y^2$. It can now be seen that if $x \neq 0 = y$ then $h > 0$ but if $x = -2y \neq 0$ then $h \leq 0$. Depending upon which direction we move away from the stationary point at $x = y = 0$, the graph of $h = h(x, y)$ can curve either up or down. For (iii), the stationary point, at $x = y = 0$, is a saddle point of $h$.

More generally, suppose that $f(x, y)$ has a stationary point at $(x, y) = (a, b)$, then

$$\frac{\partial f}{\partial x}(a, b) = \frac{\partial^2 f}{\partial x^2}(a, b) = 0.$$ 

Now taking a Taylor expansion of $f(x, y) - f(a, b)$ about $(a, b)$, the first terms will be those in $\frac{\partial^2 f}{\partial x^2},$ $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$. By completing the square for these three terms, like for (iii) in the above example leads to the following version of the second-derivative test:
If \[ \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 > 0 \] and \[ \frac{\partial^2 f}{\partial x^2} < 0 \] then \((a, b)\) is a local maximum. (5.2)

If \[ \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 > 0 \] and \[ \frac{\partial^2 f}{\partial x^2} > 0 \] then \((a, b)\) is a local minimum. (5.3)

If \[ \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 < 0 \] then \((a, b)\) is a saddle point. (5.4)

**Note:** If \[ \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 = 0 \] the test is inconclusive — the stationary point could be either a local maximum, a local minimum or a saddle point!

**Question 5.3.** Find all the stationary points of the function
\[ f(x, y) = x^2 + 3xy - 2y^2 - 8x + 5y. \]

**Solution.** We have
\[ \frac{\partial f}{\partial x}(x, y) = 2x + 3y - 8 \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = 3x - 4y + 5, \]
so
\[ \begin{align*}
\frac{\partial f}{\partial x} &= 0 \quad \iff \quad 2x + 3y - 8 = 0 \\
\frac{\partial f}{\partial y} &= 0 \quad \iff \quad 3x - 4y + 5 = 0
\end{align*} \] \((A)\)
\[ \begin{align*}
\frac{\partial f}{\partial x} &= 0 \quad \iff \quad 2x + 3y = 8 \\
\frac{\partial f}{\partial y} &= 0 \quad \iff \quad 3x - 4y = -5
\end{align*} \] \((B)\)

We need to solve these simultaneous equations; taking 3\((A)\) − 2\((B)\) gives
\[ (6x + 9y) - (6x - 8y) = 24 - (-10) \quad \iff \quad 17y = 34 \quad \iff \quad y = 2. \]
Substituting this back into \((A)\) then gives
\[ 2x + 3(2) = 8 \quad \iff \quad x = 1. \]
It follows that \(f\) has exactly one stationary point, namely \((1, 2)\).

**Example 5.4.** In the above previous question we found that \((1, 2)\) was the unique stationary point of the function
\[ f(x, y) = x^2 + 3xy - 2y^2 - 8x + 5y. \]
Now
\[ \frac{\partial^2 f}{\partial x^2}(x, y) = 2, \quad \frac{\partial^2 f}{\partial y^2}(x, y) = -4, \quad \frac{\partial^2 f}{\partial x \partial y}(x, y) = 3, \]
so the second-derivative test gives us the following information:
<table>
<thead>
<tr>
<th>Stationary point</th>
<th>$\frac{\partial^2 f}{\partial x^2}$</th>
<th>$\frac{\partial^2 f}{\partial x\partial y}$</th>
<th>$\frac{\partial^2 f}{\partial y^2}$</th>
<th>$\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x\partial y} \right)^2$</th>
<th>Nature of stationary point</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1,2)$</td>
<td>2</td>
<td>3</td>
<td>$-4$</td>
<td>$-17 &lt; 0$</td>
<td>saddle point</td>
</tr>
</tbody>
</table>

**Question 5.4.** Find and classify all the stationary points of the function

$$f(x, y) = xy(1 - x - y).$$

**Solution.** We have

$$\frac{\partial f}{\partial x}(x, y) = -xy + y(1 - x - y) = y(1 - 2x - y)$$

and

$$\frac{\partial f}{\partial y}(x, y) = -xy + x(1 - x - y) = x(1 - x - 2y).$$

Therefore

$$\begin{align*}
\frac{\partial f}{\partial x}(x, y) &= 0 \\
\frac{\partial f}{\partial y}(x, y) &= 0
\end{align*} \quad \iff \quad \begin{align*}
y(1 - 2x - y) &= 0 \\
x(1 - x - 2y) &= 0
\end{align*}$$

We must solve these simultaneous equations; this can be done by considering four separate cases:

**Case (i):**

$$\begin{align*}
y &= 0 \\
x &= 0
\end{align*}$$

This immediately gives the stationary point $(0,0)$.

**Case (ii):**

$$\begin{align*}
y &= 0 \\
1 - x - 2y &= 0
\end{align*}$$

Putting $y = 0$ in the second equation gives $1 - x - 0 = 0$ or $x = 1$. Thus we get the stationary point $(1,0)$.

**Case (iii):**

$$\begin{align*}
1 - 2x - y &= 0 \\
x &= 0
\end{align*}$$

Putting $x = 0$ in the first equation gives $1 - 0 - y = 0$ or $y = 1$. Thus we get the stationary point $(0,1)$.

**Case (iv):**

$$\begin{align*}
1 - 2x - y &= 0 \\
1 - x - 2y &= 0
\end{align*} \quad \iff \quad \begin{align*}
2x + y &= 1 \\
x + 2y &= 1
\end{align*} \quad \text{(A)} \quad \text{(B)}$$

Taking $(A) - 2(B)$ gives

$$2x + y - (2x + 4y) = 1 - 2 \quad \iff \quad -3y = -1 \quad \iff \quad y = \frac{1}{3}.$$
Substituting this back into \((B)\) then gives
\[
x + 2\left(\frac{1}{3}\right) = 1 \iff x = \frac{1}{3}.
\]
Thus we get the stationary point \((\frac{1}{3}, \frac{1}{3})\).

We have thus found four stationary points for \(f\). To determine their natures we first calculate
\[
\frac{\partial^2 f}{\partial x^2}(x, y) = -2y, \quad \frac{\partial^2 f}{\partial y^2}(x, y) = -2x, \quad \frac{\partial^2 f}{\partial x \partial y}(x, y) = 1 - 2x - 2y.
\]

Applying the second-derivative test we now get the following information:

<table>
<thead>
<tr>
<th>Stationary point</th>
<th>(\frac{\partial^2 f}{\partial x^2})</th>
<th>(\frac{\partial^2 f}{\partial x \partial y})</th>
<th>(\frac{\partial^2 f}{\partial y^2})</th>
<th>(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2})</th>
<th>Nature of stationary point</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1 &lt; 0</td>
<td>saddle point</td>
</tr>
<tr>
<td>((1, 0))</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td>-1 &lt; 0</td>
<td>saddle point</td>
</tr>
<tr>
<td>((0, 1))</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>-1 &lt; 0</td>
<td>saddle point</td>
</tr>
<tr>
<td>((\frac{1}{3}, \frac{1}{3}))</td>
<td>-\frac{2}{3}</td>
<td>-\frac{1}{3}</td>
<td>-\frac{2}{3}</td>
<td>\frac{1}{3} &gt; 0</td>
<td>local maximum</td>
</tr>
</tbody>
</table>

**Question 5.5.** A company makes open rectangular crates of volume \(8\text{m}^3\). If the material for the bottom costs twice as much as that for the sides, find the dimensions that minimise the cost.

**Solution.** Suppose the crate has a width, depth and height of \(x, y\) and \(z\) respectively:

Suppose the cost per \(\text{m}^2\) of the side material is \(\£K\) (so the cost of the bottom material is \(\£2K\)). Then the total cost \(C\) is

\[
C = \text{cost of sides} + \text{cost of bottom} = (2xz + 2yz)K + (xy)2K.
\]

On the other hand we know that

\[
\text{volume of crate} = xyz = 8 \quad \text{so} \quad z = \frac{8}{xy}.
\]

It follows that

\[
C = C(x, y) = 16K\left(\frac{1}{y} + \frac{1}{x}\right) + 2Kxy.
\]
We now find the stationary points of $C$; firstly we calculate
\[
\frac{\partial C}{\partial x}(x, y) = -\frac{16K}{x^2} + 2Ky \quad \text{and} \quad \frac{\partial C}{\partial y}(x, y) = -\frac{16K}{y^2} + 2Kx.
\]

The stationary points are then given by solving
\[
\begin{align*}
\frac{\partial C}{\partial x}(x, y) &= 0 \iff -\frac{16K}{x^2} + 2Ky = 0 \iff y = \frac{8}{x^2} \quad \text{(A)} \\
\frac{\partial C}{\partial y}(x, y) &= 0 \iff -\frac{16K}{y^2} + 2Kx = 0 \iff x = \frac{8}{y^2} \quad \text{(B)}
\end{align*}
\]
Substituting (A) into (B) gives
\[
x = \frac{8}{\left(\frac{8}{x^2}\right)^2} \iff x = \frac{x^4}{8} \iff x^3 = 8 \iff x = 2.
\]
Substituting this back into (A) then gives
\[
y = \frac{8}{2^2} = 2.
\]
Thus $C(x, y)$ has the stationary point $(2, 2)$. We can use the second derivative test to check that this stationary point is a minimum for the cost; alternatively this point is quite clear if one thinks about the set up of the problem.

Finally we note that when $x = y = 2$ we have $z = 8/(xy) = 2$ so the cheapest shape to make the crate is as a cube measuring $2m \times 2m \times 2m$. 

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5. Problems: Maxima and Minima

Problem 5.1. Find the stationary points of the following functions and determine their nature:
(a) \( f(x, y) = 7 - 2x + 6y + 2x^2 + 3y^2; \)
(b) \( f(x, y) = x^2 - xy + y^2; \)
(c) \( f(x, y) = x^2 - 3xy - 2y^2 + 2x; \)
(d) \( f(x, y) = xy(6 - x - 2y); \)
(e) \( f(x, y) = x \sin y. \)

Problem 5.2. Find the shortest distance from the point \((-1, 3, -2)\) to the plane \( x - y + 2z = 4. \)

Hint: A point on the plane satisfies \( z = \frac{4 - x + y}{2} \) and so the distance of \((-1, 3, -2)\) from \((x, y, z)\) is
\[
\sqrt{(x + 1)^2 + (y - 3)^2 + (z + 2)^2} = \sqrt{(x + 1)^2 + (y - 3)^2 + \left[\frac{1}{2}(4 - x + y) + 2\right]^2}
\]
\[
= \sqrt{(x + 1)^2 + (y - 3)^2 + (4 - \frac{1}{2}x + \frac{1}{2}y)^2}.
\]
Hence we must minimise \( f(x, y) = (x + 1)^2 + (y - 3)^2 + (4 - \frac{1}{2}x + \frac{1}{2}y)^2. \)

Problem 5.3. A company plans to manufacture closed rectangular crates which have a volume of \(8 \text{ m}^3. \) Given that the material for the top and bottom costs twice as much as the material for the sides, find the dimensions of the crate that will minimise cost.

Problem 5.4. If an open rectangular box is to have a fixed surface area \(A, \) what dimensions will make the volume a maximum?

Answers
1. (a) \( \left(\frac{1}{2}, -1\right), \) minimum. (b) \( (0, 0), \) minimum. (c) \( \left(-\frac{8}{17}, \frac{6}{17}\right), \) saddle.
1. (d) \( (0, 0), \) saddle; \( (6, 0), \) saddle; \( (0, 3), \) saddle; \( (2, 1), \) maximum. (e) \( (0, n\pi), \) saddles.

2. \( \sqrt{24}. \)

3. \( 4^{1/3}, 4^{1/3}, 2 \times 4^{1/3}. \)

4. \( \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \frac{1}{2} \sqrt{\frac{2}{3}}. \)
Chapter 6

Taylor Series and Linear Approximation

Recall, first, what happens for a function of only one variable.

Example 6.1. If \( h \) is small then \( h^2 \) and \( h^3 \) etc., are very small compared to \( h \). For example, if

\[
  h = 0.02 \quad \text{then} \quad h^2 = 0.0004.
\]

A function \( f(x) \) with \( x \) “near” some value \( a \), say \( x = a + h \), can then be given by its Taylor series:

\[
  f(x) = f(a + h) = f(a) + h \frac{df}{dx}(a) + \frac{h^2}{2} \frac{d^2f}{dx^2}(a) + \ldots.
\]

With \( h \) small enough we might stop with the second term on the right-hand side to get the linear approximation

\[
  f(x) = f(a + h) \approx f(a) + h \frac{df}{dx}(a).
\]

Question 6.1. Use linear approximation to estimate \((1.01)^{1/3}\).

Solution. Take \( f(x) = x^{1/3} \) so \( f'(x) = \frac{1}{3}x^{-2/3} \). Then

\[
  (1.01)^{1/3} = f(1 + 0.01)
  \approx f(1) + f'(1) \times 0.01
  = 1 + \frac{1}{3} \times 0.01
  \approx 1.0033.
\]

Linear approximation replaces the graph of \( f \) with a section of straight line.
Changes in $x$ produce changes in $f(x)$. We often use the notation
\[ \delta x = \text{change in } x, \quad \delta f = \text{change in } f. \]
Then
\[ \delta f \approx \frac{df}{dx} \delta x. \]

### 6.1 Taylor series

With more variables we have to allow for changes in each of them.

---

**Example 6.2.** Taking $f = f(x, y)$, the first few terms in a Taylor series about a point $(a, b)$, taking $x = a + h$, $y = b + k$, gives us

\[ f(x, y) = f(a + h, b + k) = f(a, b) + \frac{\partial f}{\partial x}(a, b)h + \frac{\partial f}{\partial y}(a, b)k + \ldots. \quad (6.1) \]

With more variables we naturally get more terms!

Using the delta notation,
\[ \delta f \approx \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y. \quad (6.2) \]

---

**Question 6.2.** Using linear approximation, estimate the change in
\[ f(x, y) = xy^2 - 2y \]
if $(x, y)$ changes from $(1, -2)$ to $(1.01, -1.98)$.

**Solution.** We have
\[ \frac{\partial f}{\partial x} = y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy - 2, \]

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while \[ \delta x = 1.01 - 1 = 0.01 \quad \text{and} \quad \delta y = -1.98 - (-2) = 0.02. \]

Therefore

\[
\delta f \approx \frac{\partial f}{\partial x}(1, -2) \times \delta x + \frac{\partial f}{\partial y}(1, -2) \times \delta y \\
= 4 \times 0.01 + (-6) \times 0.02 \\
= -0.08.
\]

**Question 6.3.** Using linear approximation, estimate the change in

\[ f(x, y) = xy^3 \]

if \((x, y)\) changes from \((1, 2)\) to \((0.99, 2.02)\).

**Solution.** We have

\[
\frac{\partial f}{\partial x} = y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3xy^2,
\]

while

\[
\delta x = 0.99 - 1 = -0.01 \quad \text{and} \quad \delta y = 2.02 - 2 = 0.02.
\]

Therefore

\[
\delta f \approx \frac{\partial f}{\partial x}(1, 2) \times \delta x + \frac{\partial f}{\partial y}(1, 2) \times \delta y = 8 \times (-0.01) + 12 \times 0.02 = 0.16.
\]

### 6.2 Estimation of errors

We can use linear approximation to estimate the effect of errors in measurements:

- Suppose that we are interested in a quantity \(Q(x, y)\).

- We make measurements of \(x\) and \(y\) with errors \(\delta x\) and \(\delta y\).

- The corresponding error in \(Q\) is

\[
\delta Q \approx \frac{\partial Q}{\partial x} \delta x + \frac{\partial Q}{\partial y} \delta y.
\]

**Example 6.3.** A company makes tanks 6m high with a of radius 2m. How do errors in construction affect the volume? Consider a tank of height \(h\) and radius \(r\):
Then the volume of the tank is $V(h, r) = \pi r^2 h$. In particular
\[
\frac{\partial V}{\partial h} = \pi r^2 \quad \text{and} \quad \frac{\partial V}{\partial r} = 2\pi rh.
\]
Now suppose: actual height of tank = $6 + \delta h$ and actual radius of tank = $2 + \delta r$.
Then the error in the volume of the tank is given by
\[
\delta V \approx \frac{\partial V}{\partial h}(6, 2) \times \delta h + \frac{\partial V}{\partial r}(6, 2) \times \delta r
= (\pi \times 2^2)\delta h + (2\pi \times 2 \times 6)\delta r
= 4\pi \delta h + 24\pi \delta r.
\]
Conclusion: pay attention to the quality of the radius!

Often the percentage error is of at least as much concern. For a quantity $A$ this is defined by
\[
\% \text{ error in } A = \frac{\text{error in } A}{A} \times 100. \tag{6.3}
\]

**Question 6.4.** Consider again the tank from the previous example. If the percentage error in the height is 2% and the percentage error in the radius is 1%, estimate the percentage error in the volume.

**Solution.** Using the previous calculations, we have
\[
(\% \text{ error in } V) = \frac{\delta V}{V} \times 100
= \frac{\pi r^2 \delta h + 2\pi rh \delta r}{\pi r^2 h} \times 100
= \frac{\delta h}{h} \times 100 + 2 \frac{\delta r}{r} \times 100
= (\% \text{ error in } h) + 2 \times (\% \text{ error in } r).
\]
Using the given values it follows that $(\% \text{ error in } V) = 2 + 2 \times 1 = 4\%$. 
Question 6.5. A company makes boxes with dimensions $x$, $y$ and $z$.

Estimate the percentage error in the volume in terms of the percentage errors in $x$, $y$ and $z$.

Solution. The volume of the box is given as $V(x, y, z) = xyz$ so

$$\delta V \approx \frac{\partial V}{\partial x} \delta x + \frac{\partial V}{\partial y} \delta y + \frac{\partial V}{\partial z} \delta z$$

$$= yz \delta x + xz \delta y + xy \delta z.$$ 

Hence

$$\frac{\delta V}{V} \approx \frac{yz \delta x}{xyz} + \frac{xz \delta y}{xyz} + \frac{xy \delta z}{xyz} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}.$$ 

It follows that

$$(\% \text{ error in } V) = \frac{\delta V}{V} \times 100$$

$$\approx \frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100 + \frac{\delta z}{z} \times 100$$

$$= (\% \text{ error in } x) + (\% \text{ error in } y) + (\% \text{ error in } z).$$
6. Problems: Linear Approximation and Error Analysis

Problem 6.1. Consider the function $f(x) = x^{1/3}$.

Derive a linear approximation (using Taylor series) centred at $x_0 = 8$. Use the linear approximation to approximate $9^{1/3}$.

What is the percentage error in the approximation compared to the true value of $9^{1/3}$.

Problem 6.2. Use linear approximation to estimate $\sqrt{24}$.

Problem 6.3. Use linear approximation to estimate the change in $f(x, y) = x^2 - 3x^3y + 4x - 2y^3 + 6$ if $(x, y)$ changes from $(-2, 3)$ to $(-2.02, 3.01)$ and hence find an approximate value for $f(-2.02, 3.01)$.

Problem 6.4. The dimensions of a closed rectangular box are measured as 3 m, 4 m, and 5m with maximum possible error of 0.1 cm. Using linear approximation find the maximum error in the calculated values of (i) the volume of the box, and (ii) the surface area of the box.

Problem 6.5. The height of a tower is calculated by measuring the horizontal distance $x$ from the foot of the tower to a point $A$ and the angle of elevation $\theta$ of the top of the tower from $A$. Using linear approximation, estimate the percentage error in the height in terms of the percentage errors in $x$ and $\theta$.

Problem 6.6. The rate of flow of gas in a pipe is given by $v = C d^{\frac{1}{2}} T^{-\frac{5}{6}}$ where $C$ is a constant, $d$ is the diameter of the pipe and $T$ is the temperature of the gas. By using linear approximation, find the % change in $v$ when the diameter of the pipe is increased by 1% and the gas temperature is increased by 2%.

Answers

1. $f(x) \approx 2 + \frac{1}{12}(x - 8)$, $9^{1/3} \approx 2.0833$, 0.16%.

2. 4.9.

3. 1.86, 21.86.

4. 0.047 m$^2$, 0.048 m$^2$.

5. % error in $x + \frac{\theta}{\sin \theta \cos \theta} \times$ % error in $\theta$.

6. $-\frac{7}{6}\%$. 

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Chapter 7

Multiple Integrals

7.1 Integration of functions of one variable

We start by recalling the basics of integration with respect to a single variable. We have results such as:

\[ \int_a^b f(x) \, dx = \left[ F(x) \right]_a^b := F(b) - F(a) \]

where \( F \) is an “antiderivative” or indefinite integral of \( f \). \( (f = dF/dx.) \)

\[ \int \lambda f(x) \, dx = \lambda \int f(x) \, dx, \]

\[ \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx. \]

Useful techniques include:

- Integration by parts.
- Integration by substitution.
- Use of partial fractions.

Some specific integrals can be found on the formula sheet.

**Question 7.1.** Calculate \( \int_0^1 (3x^4 + x^2e^{x^3}) \, dx \).

**Solution.** This integral is designed to use a few different techniques! Firstly we can split the integral as

\[ \int_0^1 (3x^4 + x^2e^{x^3}) \, dx = 3\int_0^1 x^4 \, dx + \int_0^1 x^2e^{x^3} \, dx. \]

The first integral on the right-hand side is straightforward:

\[ \int_0^1 x^4 \, dx = \left[ \frac{x^5}{5} \right]_0^1 = \frac{1}{5}. \]
To calculate the second integral on the right hand side we use integration by substitution; taking \( u = x^3 \) we have \( \frac{du}{dx} = 3x^2 \) so \( x^2 \, dx = \frac{1}{3} \, du \). Therefore

\[
\int x^2 e^{x^3} \, dx = \int \frac{1}{3} e^u \, du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C.
\]

The definite integral can then be computed as

\[
\int_0^1 x^2 e^{x^3} \, dx = \left[ \frac{1}{3} e^{x^3} \right]_0^1 = \frac{1}{3} e^1 - \frac{1}{3} e^0 = \frac{e - 1}{3}.
\]

Combining the above calculations we finally get

\[
\int_0^1 (3x^4 + x^2 e^{x^3}) \, dx = 3 \times \frac{1}{5} + \frac{e - 1}{3} = \frac{4 + 5e}{15}.
\]

One interpretation of an integral like this is the area under a graph: assuming that \( f \) is positive, the area of a strip of width \( dx \) and length \( f \) is \( f \, dx \) and then \( A = \int_a^b f(x) \, dx \) is the total area between the \( x \) axis and the curve \( y = f(x) \) for \( a < x < b \).

![Graph of y = f(x) with shaded area A]

More generally, if \( f \) represents the density of some quantity, \( i.e. \) it is the amount per unit length, \( f \, dx \) is the amount in a short length \( dx \) and \( A = \int_a^b f(x) \, dx \) is the total amount between \( x = a \) and \( x = b \).

### 7.2 Integration of functions of two variables

Thinking of a single integral as giving an area, we might ask: what is the volume under a surface \( z = f(x, y) \) lying above a rectangle in the \( x - y \) plane, \( a < x < b, c < y < d \)? (We again take \( f > 0 \) for simplicity.)

First look at a small rectangle of length \( dx \) and width \( dy \). Its area is \( dA = dx \, dy \). Then the volume between that small rectangle and the surface is the height of the enclosed (approximately) cuboidal region times the area: \( f \, dA = f \, dx \, dy \).
The volume between the $x$-$y$ plane and the surface in the slab of width $dx$ between $y = c$ and $y = d$ is then given by integrating with respect to $y$: $(\int_c^d f(x, y) \, dy) \, dx$. (In doing this integral, $x$ is held fixed.) Finally, to get the total volume, we must integrate with respect to $x$ from $a$ to $b$:

$$
\text{volume} = V = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx .
$$

We usually abbreviate this double integral as

$$
V = \int_a^b \int_c^d f(x, y) \, dy \, dx .
$$

Noting that, for cases we’ll be considering at least, we could have integrated with respect to $x$ first and then $y$, we also have

$$
V = \int_c^d \int_a^b f(x, y) \, dx \, dy = \int \int_D f(x, y) \, dx \, dy ,
$$

where $D$ is our region of integration, here $D$ is the rectangle $a < x < b$, $c < y < d$. More generally if $f$ once again represents some sort of density, e.g. mass per unit area in a sheet of metal, the double integral $\int \int_D f(x, y) \, dx \, dy$ is the total amount in the region $D$.

---

**Example 7.1.** An example of a double integral is $\int_1^4 \int_{-3}^{-1} (x^2 y + \sin(xy)) \, dy \, dx$.

---

**Question 7.2.** Calculate $\int_2^4 \int_1^3 (xy^2 + y) \, dy \, dx$.

**Solution.** Calculating the inner integral first gives

$$
\int_1^3 (xy^2 + y) \, dy = \left[ \frac{xy^3}{3} + \frac{y^2}{2} \right]_1^3 = \left( 9x + \frac{9}{2} \right) - \left( \frac{x}{3} + \frac{1}{2} \right) = \frac{26x}{3} + 4 .
$$
Therefore
\[
\int_2^4 \int_1^3 (xy^2 + y) \, dy \, dx = \int_2^4 \left( \frac{26x}{3} + 4 \right) \, dx
\]
\[
= \left[ \frac{13x^2}{3} + 4x \right]^4_2 = \left( \frac{13 \times 16}{3} + 16 \right) - \left( \frac{13 \times 4}{3} + 8 \right) = 13 \times 4 + 8 = 60.
\]

**Question 7.3.** Calculate \( \int_0^1 \int_2^3 2xy \, dx \, dy \).

**Solution.** Calculating the inner integral first gives
\[
\int_2^3 2xy \, dx = [x^2y]^3_2 = 9y - 4y = 5y.
\]
Therefore
\[
\int_0^1 \int_2^3 2xy \, dx \, dy = \int_0^1 5y \, dy = \left[ \frac{5y^2}{2} \right]^1_0 = \frac{5}{2}.
\]

**Example 7.2.** For the double integral \( \int_2^3 \int_0^1 2xye^{xy^2} \, dy \, dx \) the region of integration \( D \) is the rectangle given by
\[
0 \leq y \leq 1, \quad 2 \leq x \leq 3.
\]

**Question 7.4.** Calculate \( \int_2^3 \int_0^1 2xye^{xy^2} \, dy \, dx \).

**Solution.** Firstly consider the inner integral
\[
\int_0^1 2xye^{xy^2} \, dy.
\]
We will use integration by substitution for this integral; taking \( u = xy^2 \) we have \( \frac{\partial u}{\partial y} = 2xy \) so \( du = 2xy \, dy \) and hence
\[
\int 2xye^{xy^2} \, dy = \int e^u \, du = e^u + C = e^{xy^2} + C.
\]
Thus
\[ \int_0^1 2xye^{xy^2} \, dy = [e^{xy^2}]_0^1 = e^x - 1. \]

It follows that
\[ \int_2^3 \int_0^1 2xye^{xy^2} \, dy \, dx = \int_2^3 (e^x - 1) \, dx = [e^x - x]_2^3 = (e^3 - 3) - (e^2 - 2) = e^3 - e^2 - 1. \]

**Question 7.5.** Let \( D \) be the rectangle given by
\[ -1 \leq y \leq 0, \quad 1 \leq x \leq 3. \]

Calculate \( \iint_D (x^2 + y) \, dy \, dx \).

**Solution.** The region of integration is the following:

![Diagram of the region D]

Now
\[
\iint_D (x^2 + y) \, dy \, dx = \int_1^3 \int_{-1}^0 (x^2 + y) \, dy \, dx = \int_1^3 \left[ x^2 y + \frac{y^2}{2} \right]_{-1}^0 \, dx = \int_1^3 \left( x^2 - \frac{1}{2} \right) \, dx
\]
\[= \left[ \frac{x^3}{3} - \frac{x}{2} \right]_1 = \left( \frac{9}{3} - \frac{3}{2} \right) - \left( \frac{1}{3} - \frac{1}{2} \right) = 23 \frac{3}{3}.
\]

Of course a region of integration need not be rectangular. If \( D \) can be described by \( g(x) < y < h(x) \) for \( a < x < b \), the (double) integral of \( f(x, y) \) over \( D \) will be
\[
\int \int_D f(x, y) \, dy \, dx = \int_a^b \left( \int_{g(x)}^{h(x)} f(x, y) \, dy \right) \, dx.
\]

We can drop the brackets and simply write this as \( \int_a^b f(x, y) \, dy \, dx \).

**Example 7.3.** Consider the region given by
\[ \frac{-1}{2} \leq x \leq \frac{1}{2}, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}. \]

This region is part of a disc.
Example 7.4. Consider the region $D$:

In this region we have $0 \leq x \leq 1$ whilst, for a given $x$, $0 \leq y \leq 1 - x$. Thus the region $D$ is described by

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x.$$ 

Question 7.6. Calculate $\int \int_D y \, dA$ over the region $D$ below:

Solution. In this region we have $0 \leq x \leq 1$, whilst, for a given $x$, we have

$$\sqrt{x} \leq y \leq 1.$$ 

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Therefore
\[ \int \int_{D} y \, dA = \int_{0}^{1} \int_{\sqrt{x}}^{1} y \, dy \, dx. \]
Calculating the inner integral first gives
\[ \int_{\sqrt{x}}^{1} y \, dy = \left[ \frac{y^2}{2} \right]_{\sqrt{x}}^{1} = \frac{1}{2} - \frac{x}{2}. \]
Hence
\[ \int_{D} y \, dA = \int_{0}^{1} \left( \frac{1}{2} - \frac{x}{2} \right) \, dx = \left[ \frac{3}{4} - \frac{x^2}{4} \right]_{0}^{1} = 0 - \left( \frac{1}{4} - \frac{1}{4} \right) = \frac{1}{4}. \]

**Example 7.5.** Consider the region $D$:

![Diagram of region D](image)

In this region we have $0 \leq y \leq 1$, whilst, for a given $y$, $y \leq x \leq 1$.

**Question 7.7.** Calculate $\int \int_{D} (3 - x - y) \, dA$ where $D$ is the region described in the previous question.

**Solution.** Recall that $D$ is described by
\[ 0 \leq y \leq 1, \quad y \leq x \leq 1. \]
Therefore
\[ \int \int_{D} (3 - x - y) \, dA = \int_{0}^{1} \int_{y}^{1} (3 - x - y) \, dx \, dy. \]
Now
\[ \int_{y}^{1} (3 - x - y) \, dx = \left[ 3x - \frac{x^2}{2} - xy \right]_{y}^{1} = \left( 3 - \frac{1}{2} - y \right) - \left( 3y - \frac{y^2}{2} - y^2 \right) = \frac{5}{2} - 4y + \frac{3y^2}{2}. \]
It follows that
\[ \int \int_{D} (3 - x - y) \, dA = \int_{0}^{1} \left( \frac{5}{2} - 4y + \frac{3y^2}{2} \right) \, dy = \left[ \frac{5y}{2} - 2y^2 + \frac{y^3}{2} \right]_{0}^{1} = \left( \frac{5}{2} - 2 + \frac{1}{2} \right) - 0 = 1. \]
7.3 Interchanging the order of integration

As noted before, we can swap the order in which the order in which the integrals are carried out: \( \int \int_D f \, dy \, dx = \int \int_D f \, dA = \int \int_D f \, dx \, dy \). It is sometimes easier to calculate the value of a double integral doing the integrations in one order than the other.

Example 7.6. Consider the region \( D \) which lies between the line \( y = x \) and the parabola \( y = x^2 \):

This region can be described by

\[
0 \leq x \leq 1, \quad x^2 \leq y \leq x.
\]

On the other hand, the parabola is also given by the equation \( x = \sqrt{y} \) so the region \( D \) can also be described by

\[
0 \leq y \leq 1, \quad y \leq x \leq \sqrt{y}.
\]

It follows that we have

\[
\int_0^1 \int_{x^2}^x f(x, y) \, dy \, dx = \int \int_D f(x, y) \, dA = \int_0^1 \int_y^{\sqrt{y}} f(x, y) \, dx \, dy.
\]

Question 7.8. Calculate \( \int \int_T e^{y^2} \, dA \) where \( T \) is the triangular region with vertices \((0, 0), (0, 1)\) and \((1, 1)\):
Solution. As a first attempt we may describe $T$ by

$$0 \leq x \leq 1, \quad x \leq y \leq 1.$$  

It follows that

$$\int \int_T e^{y^2} \, dA = \int_0^1 \int_x^1 e^{y^2} \, dy \, dx.$$  

The inner integral is then

$$\int_x^1 e^{y^2} \, dy$$

which can’t be evaluated very easily! We’ve got stuck!

As a second attempt let us describe $T$ the other way; that is

$$0 \leq y \leq 1, \quad 0 \leq x \leq y.$$  

Then

$$\int \int_T e^{y^2} \, dA = \int_0^y \int_0^x e^{y^2} \, dx \, dy.$$  

Evaluating the inner integral first gives

$$\int_0^y e^{y^2} \, dx = [xe^{y^2}]_0^y = ye^{y^2}.$$  

It follows that

$$\int \int_T e^{y^2} \, dA = \int ye^{y^2} \, dy = \left[ \frac{e^{y^2}}{2} \right]_0^1 = e - 1 = \frac{1}{2}(e - 1),$$

where we used the substitution $u = y^2$ to evaluate the integral.

Note. On evaluating a double integral $I = \int \int_D f(x, y) \, dA$ by doing the $y$ integral first, i.e. taking $I = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx$, we integrate along vertical strips between the lower boundary, say $y = g_1(x)$, and the upper boundary, say $y = g_2(x)$ (this gives a result which generally depends upon $x$, but definitely does not depend on $y$): $I_y(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy$.

We then total up the contributions of strips by integrating $I_y$ from the lowest value of $x$ taken in $D$, say $a$, to the largest, say $b$ (see Fig. 7.1(i)): $I = \int_a^b I_y(x) \, dx$. On the other hand, doing the $y$ integral first, we take $I = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy$, we integrate along horizontal strips between the left-hand boundary, say $x = h_1(y)$, and the right-hand boundary, say $x = h_2(y)$ (this gives a result which generally depends upon $y$, but definitely does not depend on $x$): $I_x(y) = \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx$. We then total up the contributions of strips by integrating $I_x$ from the lowest value of $y$ taken in $D$, say $c$, to the largest, say $d$ (see Fig. 7.1(ii)). (Either way, the final result does not depend on either $x$ or $y$!)
Figure 7.1: Different orders of integration over $D$. 
7. Problems: Multiple Integrals

Problem 7.1. Evaluate the integrals:

(i) \( \int_{0}^{4} \int_{0}^{6} x^2 + y^2 \, dx \, dy \);

(ii) \( \int_{1}^{2} \int_{0}^{1} x^2y^2 + 2xy^3 \, dx \, dy \).

Problem 7.2. Evaluate the following integrals:

(i) \( \int \int_{D} x^2 y \, dx \, dy \) where \( D \) is the rectangle \( 0 < x < 2, 1 < y < 3 \);

(ii) \( \int \int_{D} e^{2x+y} \, dx \, dy \) where \( D \) is the rectangle \( 0 < x < 1, 0 < y < 3 \);

(iii) \( \int \int_{T} (x - y) \, dx \, dy \) where \( T \) is the triangle with vertices \( (0,0), (1,0), (2,1) \);

(iv) \( \int \int_{P} xy \, dx \, dy \) where \( P \) is the parallelogram with vertices \( (0,0), (1,1), (1,0), (2,1) \).

Problem 7.3. Sketch the regions of integration of the following:

(a) \( \int_{1}^{2} \left\{ \int_{y}^{2y} f(x,y) \, dx \right\} \, dy \);  
(b) \( \int_{0}^{1} \left\{ \int_{x}^{\sqrt{y}} f(x,y) \, dy \right\} \, dx \).

Problem 7.4. Sketch the region of integration and evaluate the following iterated integrals in each case:

(i) \( \int_{1}^{2} \int_{\sqrt{x}}^{y} x^2y \, dy \, dx \);  
(ii) \( \int_{-1}^{1} \int_{x^3}^{x+1} (3x + 2y) \, dy \, dx \);  
(iii) \( \int_{0}^{1} \int_{y}^{1} \frac{1}{1+y^2} \, dx \, dy \).

Problem 7.5. Find the following integrals by first interchanging the order of integration:

(i) \( \int_{0}^{2} \int_{y}^{2} e^{x^2} \, dx \, dy \);  
(ii) \( \int_{0}^{9} \int_{\sqrt{y}}^{3} \sin x^3 \, dx \, dy \);

(iii) \( \int_{0}^{2} \int_{y/2}^{1} \ln(1 + y^2) \, dy \, dx \);  
(iv) \( \int_{0}^{\pi/4} \int_{2x}^{\pi/2} \frac{\cos y}{y} \, dy \, dx \).

Answers

1. (i) 416, (ii) 163/36.

2. (i) \( \frac{32}{3} \). (ii) \( \frac{1}{2}(e^2 - 1)(e^3 - 1) \). (iii) \( \frac{1}{3} \). (iv) \( \frac{7}{12} \).

4. (i) \( \frac{163}{120} \). (ii) \( \frac{334}{105} \). (iii) \( \frac{7}{4} - \frac{1}{2} \ln 2 \).

5. (i) \( \frac{1}{2}(e^4 - 1) \). (ii) \( \frac{1}{3}(1 - \cos 27) \). (iii) \( 2 \ln 2 - 1 \). (iv) \( \frac{1}{2} \).
Chapter 8
Double Integrals: Applications and Polar Coordinates

8.1 Change of variables to polar coordinates

We must first be familiar with polar coordinates.

Question 8.1. Convert the function

\[ f(x, y) = \frac{2xy}{x^2 + y^2}. \]

into a function of polar coordinates.

Solution. We need to make the substitutions

\[ x = r \cos \theta, \quad y = r \sin \theta. \]

This gives us

\[ f(r, \theta) = \frac{2r \cos \theta \times r \sin \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = \frac{r^2 2 \cos \theta \sin \theta}{r^2 (\cos^2 \theta + \sin^2 \theta)} = \sin 2\theta. \]

Now we can look at how they are used for double integrals.
A key step is to see what replaces the \(dx \, dy\) we were using for Cartesian co-ordinates we had our small area element a rectangle of area \(dA = dx \, dy\). For polars, with change in angle, \(d\theta\), and change in distance from the origin, \(dr\), both small, we get something which is approximately rectangular, with sides of length \(rd\theta\) and \(dr\).
We can then write a double integral $\int \int f \, dA = \int \int f \, dx \, dy$ as $\int \int f \, dA = \int \int fr \, dr \, d\theta$.

**Example 8.1.** The quarter circle

![Quarter Circle Diagram]

is given by

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

whilst the annulus

![Annulus Diagram]

is given by

$$1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi.$$

**Question 8.2.** Calculate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} 5\sqrt{x^2+y^2} \, dy \, dx.$$

**Solution.** Clearly this is going to be very messy if we use cartesian coordinates. On the other hand, converting the integrand to polar coordinates gives

$$5\sqrt{x^2+y^2} = 5\sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = 5r,$$

whilst the region of integration

$$0 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{1-x^2}$$

is a quarter circle (n.b. if $y = \sqrt{1-x^2}$ then $x^2 + y^2 = 1$ which is the equation for the circle of radius 1, centred at the origin):
Thus, in polar coordinates, the region of integration is given by

\[ 0 \leq r \leq 1, \quad 0 \leq \theta \leq \frac{\pi}{2}. \]

It follows that

\[ \int_0^1 \int_0^{\sqrt{1-x^2}} 5\sqrt{x^2+y^2} \, dy \, dx = \int_{\pi/2}^0 \int_0^1 5r \, r \, dr \, d\theta. \]

Calculating the inner integral gives

\[ \int_0^1 5r^2 \, dr = \left[ \frac{5r^3}{3} \right]_0^1 = \frac{5}{3}. \]

Therefore

\[ \int_{\pi/2}^0 \int_0^1 5r \, r \, dr \, d\theta = \int_{\pi/2}^0 \frac{5}{3} \, d\theta = \left[ \frac{5\theta}{3} \right]_0^{\pi/2} = \frac{5\pi}{6}. \]

### 8.2 Volume

**Question 8.3.** Find the volume of the solid which lies under the graph of

\[ f(x, y) = x^2 + 3y^2 \]

over the triangle with corners \((0, 0), (0, 1)\) and \((1, 0)\).

**Solution.** The region of integration is the following

which can be defined by

\[ 0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x. \]
Therefore
\[
\text{Volume} = \int_0^1 \int_0^{1-x} (x^2 + 3y^2) \, dy \, dx.
\]
Calculating the inner integral first gives
\[
\int_0^{1-x} (x^2 + 3y^2) \, dy = \left[ x^2 y + y^3 \right]_0^{1-x} = x^2 (1-x) + (1-x)^3.
\]
Hence
\[
\text{Volume} = \int_0^1 (x^2 - x^3 + (1-x)^3) \, dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{4} \right]_0^1 = \left( \frac{1}{3} - \frac{1}{4} - 0 \right) - \left( 0 - 0 - \frac{1}{4} \right) = \frac{1}{3}.
\]

### 8.3 Average values

**Example 8.2.** Consider a 3m\times3m metal plate \(R\). For convenience take the origin to be the bottom left corner.

Suppose the temperature at the point \((x, y)\) is given by
\[
T(x, y) = 40 + 2x + 2y.
\]
Then we can calculate
\[
\text{Average temperature of } R = \frac{1}{\text{Area of } R} \iint_R (40+2x+2y) \, dA = \frac{1}{9} \int_0^3 \int_0^3 (40+2x+2y) \, dy \, dx = \frac{1}{9} \int_0^3 [40y + 2xy + y^2]_0^3 \, dx = \frac{1}{9} \int_0^3 (129 + 6x) \, dx = \frac{1}{9} \left[ 129x + 3x^2 \right]_0^3 = \frac{1}{9} (378 + 27) = 46.
\]

### 8.4 Mass and centre of mass

**Question 8.4.** Find the centre of mass of a triangular metal plate \(P\) with vertices at \((0, 0)\), \((1, 0)\) and \((0, 1)\), and which has uniform density given by \(\rho(x, y) = 1 \text{ g/cm}^2\).
Since $P$ has a uniform density its mass is

$$M = (\text{Area of } P) \times (\text{Density of } P) = \frac{1}{2} \times 1 = \frac{1}{2}.$$ 

Now $P$ can be described by

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x.$$

Thus the $x$-coordinate of the centre of mass is given by

$$\bar{x} = \frac{1}{M} \int \int x\, \rho(x, y)\, dA = 2 \int_0^1 \int_0^{1-x} x\, dy\, dx = 2 \int_0^1 [xy]_0^{1-x} dx$$

$$= 2 \int_0^1 (x - x^2)\, dx = 2 \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2 \left( \frac{1}{2} - \frac{1}{3} - 0 \right) = \frac{1}{3},$$

whilst the $y$-coordinate of the centre of mass is given by

$$\bar{y} = \frac{1}{M} \int \int y\, \rho(x, y)\, dA = 2 \int_0^1 \int_0^{1-x} y\, dy\, dx = \int_0^1 \left[ \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= 2 \int_0^1 \frac{1}{2}(1 - x)^2\, dx = \left[ -\frac{1}{3}(1 - x)^3 \right]_0^1 = \left( -0 + \frac{1}{3}(1 - 0)^3 \right) = \frac{1}{3}.$$

Hence the centre of mass of $P$ is at \( \left( \frac{1}{3}, \frac{1}{3} \right) \).
8. Problems: Applications of Multiple Integrals

Problem 8.1. Find \( \int \int_C \sqrt{x^2 + y^2} \, dA \) where \( C \) is the circle \( x^2 + y^2 < a^2 \).

Problem 8.2. For each of the following sketch the region of integration evaluate the integral by using polar coordinates.

\[
(a) \int_0^3 \int_0^{\sqrt{9-y^2}} x^2 y \, dx \, dy \quad (b) \int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} \sqrt{1 + x^2 + y^2} \, dy \, dx.
\]

Problem 8.3. Find \( \int \int_D \sin(\sqrt{x^2 + y^2}) \, dA \) where \( D \) is

(i) the semi-circle \( x^2 + y^2 < 1, \ y > 0 \);

(ii) the ring-shaped region \( 1 < x^2 + y^2 < 4 \).

Problem 8.4. Find the volume \( V \) of the solid bounded by the paraboloid \( z = 4 - x^2 - y^2 \) and the \( x - y \) plane.

Problem 8.5. Find the volume of the solid which lies under the surface \( z = x^2 + 2y^2 \) and over the triangle in the \( x - y \) plane with vertices \( (0, 0), (1, 0) \) and \( (1, 2) \).

Problem 8.6. Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane of equation \( 2x + y + z = 2 \).

Problem 8.7. A landscape is described by a function \( H(x, y) = x^2 + y^2 \) where \( x \) and \( y \) are co-ordinates on some map. This function gives the height above sea level at the point \( (x, y) \). Find the average height of the landscape above the rectangle \( 0 < x < 3, 1 < y < 2 \).

Problem 8.8. A flat plate in the shape of a triangle with vertices \( (0, 0), (1, 1), (2, 0) \) (using cm) lies in the \( x - y \) plane. The density of the plate, \( \rho(x, y) \), at the point \( (x, y) \) is equal to \( 1 + 6x + 12y \, \text{g/cm}^2 \). Find the mass of the plate (given by \( \int \int_T \rho(x, y) \, dx \, dy \)). Write down the co-ordinates of the centre of mass of the triangle in terms of double integrals.
Answers

1. \(2\pi a^3/3\).

2. (a) \(\frac{81}{5}\).  \hspace{1em} (b) \(\frac{\pi}{17}(2\sqrt{2} - 1)\).

3. (i) \(\pi(\sin 1 - \cos 1)\).  \hspace{1em} (ii) \(2\pi(\sin 2 - 2\cos 2 - \sin 1 + \sin 2)\).

4. \(8\pi\).

5. \(\frac{11}{6}\).

6. \(\frac{2}{3}\).

7. \(\frac{16}{3}\).

8. 11; \(\frac{1}{11} \int_0^1 \int_{2-y}^{1-y} x(1 + 6x + 12y) \, dx \, dy\); \(\frac{1}{11} \int_0^1 \int_{2-y}^{1-y} y(1 + 6x + 12y) \, dx \, dy\).