Hattendorff’s Theorem

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Hattendorff’s Theorem

Hattendorff’s Theorem (1868) [3] is one of the classical theorems of life insurance mathematics, all the more remarkable for anticipating by more than 100 years one of the main results obtained by formulating life insurance mathematics in a stochastic process setting. It states that the losses in successive years on a life insurance contract have mean zero and are uncorrelated.

We will formulate the theorem in a modern setting, following [4, 5]. Denote the total amount (benefits minus premiums) or net outgo paid in the time interval $(0,t]$ by $B(t)$ (a stochastic process). Assuming that the present value at time 0 of $1$ due at time $t$ is $v(t)$ (a stochastic process); the present value at time 0 of these payments is the random variable

$$ V = \int_0^\infty v(s) \, dB(s). \quad (1) $$

The principle of equivalence (see Life Insurance Mathematics) is satisfied if $E[V] = 0$. Suppose the processes $B(t)$ and $v(t)$ are adapted to a filtration $F = \{F_t\}_{t \geq 0}$. Then $M(t) = E[V | F_t]$ is a $F$-martingale (see Martingales). Moreover, $M(t)$ can be written as

$$ M(t) = \int_0^t v(s) \, dB(s) + v(t)V(t), \quad (2) $$

where the usual prospective reserve (see Life Insurance Mathematics) at time $t$ is

$$ V(t) = \frac{1}{v(t)} \mathbb{E} \left[ \int_t^\infty v(s) \, dB(s) \bigg| F_t \right]. \quad (3) $$

The loss in the time interval $(r, t]$, discounted to time 0, is given by the discounted net outgo between time $r$ and time $t$, plus the present value of the reserve that must be set up at time $t$, offset by the present value of the reserve that was held at time $r$. Denote this quantity $L(r, t)$, then

$$ L(r, t) = \int_r^t v(s) \, dB(s) + v(t)V(t) - v(r)V(r) = M(t) - M(r). \quad (4) $$

The loss $L(r, t)$ is seen to be the increment of the martingale $M(t)$, and Hattendorff’s Theorem follows from the fact that martingales have increments with mean zero, uncorrelated over nonoverlapping periods.

See also [1, 2, 6–8], for the development of this modern form of the theorem.

References


(See also Life Insurance Mathematics)

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