Present Values and Accumulations

Angus S. Macdonald

Volume 3, pp. 1331–1336

In

Encyclopedia Of Actuarial Science
(ISBN 0-470-84676-3)

Edited by

Jozef L. Teugels and Bjørn Sundt

Present Values and Accumulations

Effective Interest

Money has a *time value*; if we invest \$1 today, we expect to get back more than \$1 at some future time as a reward for lending our money to someone else who will use it productively. Suppose that we invest \$1, and a year later we get back $\$(1+i)$. The amount invested is called the *principal*, and we say that i is the *effective rate of interest per year*. Evidently, this definition depends on the time unit we choose to use. In a riskless world, which may be well approximated by the market for good quality government bonds, i will be certain, but if the investment is risky, i is uncertain, and our expectation at the outset to receive $\$(1+i)$ can only be in the probabilistic sense.

We can regard the accumulation of invested money in either a retrospective or prospective way. We may take a given amount, $\$X$ say, to be invested now and ask, as above, to what amount will it accumulate after T years? Or, we may take a given amount, $\$Y$ say, required in T years' time (to meet some liability perhaps) and ask, how much we should invest now, so that the accumulation in T years' time will equal $\$Y$? The latter quantity is called the *present value* of $\$Y$ in T years' time. For example, if the effective annual rate of interest is i per year, then we need to invest $\$1/(1+i)$ now, in order to receive \$1 at the end of one year. In standard actuarial notation, $1/(1+i)$ is denoted v, and is called the discount factor. It is immediately clear that in a deterministic setting, accumulating and taking present values are inverse operations.

Although a time unit must be introduced in the definition of i and v, money may be invested over longer or shorter periods. First, consider an amount of \$1 to be invested for n complete years, at a rate i per year effective.

- Under *simple interest*, only the amount originally invested attracts interest payments each year, and after n years the accumulation is $\$(1+ni)$.
- Under *compound interest*, interest is earned each year on the amount originally invested *and* interest already earned, and after n years the accumulation is $\$(1+i)^n$.

Since $(1+i)^n \geq (1+ni)$ $(i > 0)$, an astute investor will turn simple interest into compound interest just by withdrawing his money each year and investing it afresh, if he is able to do so; therefore the use of simple interest is unusual, and unless otherwise stated, interest is always compound.

Given effective interest of i per year, it is easily seen that \$1 invested for any length of time $T \geq 0$ will accumulate to $\$(1+i)^T$. This gives us the rule for changing the time unit; for example, if it was more convenient to use the month as time unit, interest of i per year effective would be equivalent to interest of $j = (1+i)^{1/12} - 1$ per month effective, because $(1+j)^{12} = 1+i$.

Changing Interest Rates and the Force of Interest

The rate of interest need not be constant. To deal with variable interest rates in the greatest generality, we define the accumulation factor $A(t,s)$ to be the amount to which \$1 invested at time t will accumulate by time $s > t$. The corresponding discount factor is $V(t,s)$, the amount that must be invested at time t to produce \$1 at time s, and clearly $V(t,s) = 1/A(t,s)$. The fact that interest is compound is expressed by the relation

$$A(t,s) = A(t,r)A(r,s) \text{ for } t < r < s. \qquad (1)$$

The *force of interest* at time t, denoted $\delta(t)$, is defined as

$$\delta(t) = \frac{1}{A(0,t)} \frac{\mathrm{d}A(0,t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \log A(0,t). \qquad (2)$$

The first equality gives an ordinary differential equation for $A(0,t)$, which with boundary condition $A(0,0) = 1$ has the following solution:

$$A(0,t) = \exp\left(\int_0^t \delta(s)\,\mathrm{d}s\right)$$

$$\left(\text{so } V(0,t) = \exp\left(-\int_0^t \delta(s)\,\mathrm{d}s\right)\right). \qquad (3)$$

The special case of constant interest rates is now given by setting $\delta(t) = \delta$, a constant, from which we obtain the following basic relationships:

$$(1+i) = \mathrm{e}^{\delta} \text{ and } \delta = \log(1+i). \qquad (4)$$

The theory of cash flows and their accumulations and present values has been put in a very general framework by Norberg [10].

Nominal Interest

In some cases, interest may be expressed as an annual amount payable in equal instalments during the year; then the annual rate of interest is called *nominal*. For example, under a nominal rate of interest of 8% per year, payable quarterly, interest payments of 2% of the principal would be made at the end of each quarter-year. A nominal rate of i per year payable m times during the year is denoted $i^{(m)}$. This is equivalent to an effective rate of interest of $i^{(m)}/m$ per $1/m$ year, and by the rule for changing time unit, this is equivalent to effective interest of $(1+i^{(m)}/m)^m - 1$ per year.

Rates of Discount

Instead of supposing that interest is always paid at the end of the year (or other time unit), we can suppose that it is paid in advance, at the start of the year. Although this is rarely encountered in practice, for obvious reasons, it is important in actuarial mathematics. The *effective rate of discount* per year, denoted d, is defined by $d = i/(1+i)$, and receiving this in advance is clearly equivalent to receiving i in arrears. We have the simple relation $d = 1 - v$. Nominal rates of discount $d^{(m)}$ may also be defined, exactly as for interest.

Annuities Certain

We often have to deal with more than one payment, for example, we may be interested in the accumulation of regular payments made into a bank account. This is simply done; both present values and accumulations of multiple payments can be found by summing the present values or accumulations of each individual payment.

An **annuity** is a series of payments to be made at defined times in the future. The simplest are level annuities, for example, of amount \$1 per annum. The payments may be contingent on the occurrence or nonoccurrence of a future event – for example, a **pension** is an annuity that is paid as long as the recipient survives – but if they are guaranteed regardless of events, the annuity is called an *annuity certain*. Actuarial notation extends to annuities certain as follows:

- A temporary annuity certain is one payable for a limited term. The simplest example is a level annuity of \$1 per year, payable at the end of each of the next n years. Its accumulation at the end of n years is denoted $s_{\overline{n}|}$, and its present value at the outset is denoted $a_{\overline{n}|}$. We have

$$s_{\overline{n}|} = \sum_{r=0}^{n-1}(1+i)^r = \frac{(1+i)^n - 1}{i}, \qquad (5)$$

$$a_{\overline{n}|} = \sum_{r=1}^{n} v^r = \frac{1-v^n}{i}. \qquad (6)$$

  There are simple recursive relationships between accumulations and present values of annuities certain of successive terms, such as $s_{\overline{n+1}|} = 1 + (1+i)s_{\overline{n}|}$ and $a_{\overline{n+1}|} = v + va_{\overline{n}|}$, which have very intuitive interpretations and can easily be verified directly.
- A perpetuity is an annuity without a limited term. The present value of a perpetuity of \$1 per year, payable in arrear, is denoted $a_{\overline{\infty}|}$, and by taking the limit in equation (5) we have $a_{\overline{\infty}|} = 1/i$. The accumulation of a perpetuity is undefined.
- An annuity may be payable in advance instead of in arrears, in which case it is called an *annuity-due*. The actuarial symbols for accumulations and present values are modified by placing a pair of dots over the s or a. For example, a temporary annuity-due of \$1 per year, payable yearly for n years would have accumulation $\ddot{s}_{\overline{n}|}$ after n years or present value $\ddot{a}_{\overline{n}|}$ at outset; a perpetuity of \$1 per year payable in advance would have present value $\ddot{a}_{\overline{\infty}|}$; and so on.
  We have

$$\ddot{s}_{\overline{n}|} = \sum_{r=1}^{n}(1+i)^r = \frac{(1+i)^n - 1}{d}, \qquad (7)$$

$$\ddot{a}_{\overline{n}|} = \sum_{r=0}^{n-1} v^r = \frac{1-v^n}{d}, \qquad (8)$$

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{d}. \qquad (9)$$

- Annuities are commonly payable more frequently than annually, say m times per year. A level annuity of \$1 per year, payable in arrears m times a year for n years has accumulation denoted $s_{\overline{n}|}^{(m)}$ after n years and present value denoted $a_{\overline{n}|}^{(m)}$ at outset; the symbols for annuities-due, perpetuities, and so on are modified similarly. We have

$$s_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}}, \tag{10}$$

$$a_{\overline{n}|}^{(m)} = \frac{1 - v^n}{i^{(m)}}, \tag{11}$$

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}}, \tag{12}$$

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{d^{(m)}}. \tag{13}$$

Comparing, for example, equations (5) and (10), we find convenient relationships such as

$$s_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} s_{\overline{n}|}. \tag{14}$$

In precomputer days, when all calculations involving accumulations and present values of annuities had to be performed using tables and logarithms, these relationships were useful. It was only necessary to tabulate $s_{\overline{n}|}$ or $a_{\overline{n}|}$, and the ratios $i/i^{(m)}$ and $i/d^{(m)}$, at each annual rate of interest needed, and all values of $s_{\overline{n}|}^{(m)}$ and $a_{\overline{n}|}^{(m)}$ could be found. In modern times this trick is superfluous, since, for example, $s_{\overline{n}|}^{(m)}$ can be found from first principles as the accumulation of an annuity of $\$1/m$, payable in arrears for nm time units at an effective rate of interest of $(1+i)^{1/m} - 1$ per time unit. Accordingly, the $i^{(m)}$ and $s_{\overline{n}|}^{(m)}$ notation is increasingly of historical interest only.

- A few special cases of nonlevel annuities arise often enough so that their accumulations and present values are included in the **international actuarial notation**, namely, arithmetically increasing annuities. An annuity payable annually for n years, of amount $\$t$ in the tth year, has accumulation denoted $(Is)_{\overline{n}|}$ and present value denoted $(Ia)_{\overline{n}|}$ if payable in arrears, or $(I\ddot{s})_{\overline{n}|}$ and $(I\ddot{a})_{\overline{n}|}$ if payable in advance.

$$(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}, \tag{15}$$

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}, \tag{16}$$

$$(I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d}, \tag{17}$$

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}. \tag{18}$$

$(Is)_{\overline{n}|}^{(m)}$ (and so on) is a valid notation for increasing annuities payable m times a year, but note that the payments are of amount $\$1/m$ during the first year, $\$2/m$ during the second year and so on, *not* the arithmetically increasing sequence $\$1/m$, $\$2/m$, $\$3/m, \ldots$ at intervals of $1/m$ year. The notation for the latter is $(I^{(m)}s)_{\overline{n}|}^{(m)}$ (and so on).

- In theory, annuities or other cash flows may be payable continuously rather than discretely. In practice, this is rarely encountered but it may be an adequate approximation to payments made daily or weekly. In the international actuarial notation, continuous payment is indicated by a bar over the annuity symbol. For example, an annuity of \$1 per year payable continuously for n years has accumulation $\overline{s}_{\overline{n}|}$ and present value $\overline{a}_{\overline{n}|}$. We have

$$\begin{aligned}\overline{s}_{\overline{n}|} &= \int_0^n (1+i)^{n-t}\,\mathrm{d}t = \int_0^n \mathrm{e}^{\delta(n-t)}\,\mathrm{d}t \\ &= \frac{(1+i)^n - 1}{\delta},\end{aligned} \tag{19}$$

$$\begin{aligned}\overline{a}_{\overline{n}|} &= \int_0^n (1+i)^{-t}\,\mathrm{d}t = \int_0^n \mathrm{e}^{-\delta t}\,\mathrm{d}t \\ &= \frac{1 - v^n}{\delta},\end{aligned} \tag{20}$$

$$\overline{a}_{\overline{\infty}|} = \int_0^\infty (1+i)^{-t}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\delta t}\,\mathrm{d}t = \frac{1}{\delta}. \tag{21}$$

Increasing continuous annuities may have a rate of payment that increases continuously, so that at time t the rate of payment is $\$t$ per year, or that increases at discrete time points, for example, a rate of payment that is level at $\$t$ per year during the tth year. The former is indicated by a bar that extends over the I, the latter by a bar that does not. We have

$$\begin{aligned}(I\overline{s})_{\overline{n}|} &= \sum_{r=0}^{n-1} (r+1) \int_r^{r+1} (1+i)^{n-t}\,\mathrm{d}t \\ &= \frac{\ddot{s}_{\overline{n}|} - n}{\delta},\end{aligned} \tag{22}$$

$$(I\overline{a})_{\overline{n}|} = \sum_{r=0}^{n-1}(r+1)\int_{r}^{r+1}(1+i)^{-t}\,\mathrm{d}t$$
$$= \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}, \quad (23)$$

$$(\overline{I}\overline{s})_{\overline{n}|} = \int_0^n t(1+i)^{n-t}\,\mathrm{d}t = \frac{\overline{s}_{\overline{n}|} - n}{\delta}, \quad (24)$$

$$(\overline{I}\overline{a})_{\overline{n}|} = \int_0^n t(1+i)^{-t}\,\mathrm{d}t = \frac{\overline{a}_{\overline{n}|} - nv^n}{\delta}. \quad (25)$$

Much of the above actuarial notation served to simplify calculations before widespread computing power became available, and it is clear that it is now a trivial task to calculate any of these present values and accumulations (except possibly continuous cash flows) with a simple spreadsheet; indeed restrictions such as constant interest rates and regular payments are no longer important. Only under very particular assumptions can any of the above actuarial formulae be adapted to nonconstant interest rates [16].

For full treatments of the mathematics of interest rates, see [8, 9].

Accumulations and Present Values Under Uncertainty

There may be uncertainty about the timing and amount of future cash flows, and/or the rate of interest at which they may be accumulated or discounted. Probabilistic models have been developed that attempt to model each of these separately or in combination. Many of these models are described in detail in other articles; here we just indicate some of the major lines of development.

Note that when we admit uncertainty, present values and accumulations are no longer equivalent, as they were in the deterministic model. For example, if a payment of \$1 now will accumulate to a random amount \$$X$ in a year, Jensen's inequality (*see* **Convexity**) shows that $\mathrm{E}[1/X] \neq 1/\mathrm{E}[X]$. In fact, the only way to restore equality is to condition on knowing X, in other words, to remove all the uncertainty. Financial institutions are usually concerned with managing future uncertainty, so both actuarial and financial mathematics tend to stress present values much more than accumulations.

- **Life insurance** contracts define payments that are contingent upon the death or survival of one or more individuals. The simplest insurance contracts such as whole life insurance guarantee to pay a fixed amount on death, while the simplest annuities guarantee a level amount throughout life. For simplicity, we will suppose that cash flows are continuous, and death benefits are payable at the moment of death. We can (a) represent the future lifetime of a person now age x by the random variable T_x; and (b) assume a fixed rate of interest of i per year effective; and then the present value of \$1 paid upon death is the random variable v^{T_x}, and the present value of an annuity of \$1 per annum, payable continuously while they live, is the random variable $\overline{a}_{\overline{T_x}|}$. The *principle of equivalence* states that two series of contingent payments that have equal expected present values can be equated in value; this is just the law of large numbers (*see* **Probability Theory**) applied to random present values. For example, in order to find the rate of premium $\overline{P}_x$ that should be paid throughout life by the person now age x, we should solve
$$\mathrm{E}[v^{T_x}] = \overline{P}_x \mathrm{E}[\overline{a}_{\overline{T_x}|}]. \quad (26)$$
In fact, these expected values are identical to the present values of contingent payments obtained by regarding the life table as a deterministic model of mortality, and many of them are represented in the international actuarial notation. For example, $\mathrm{E}[v^{T_x}] = \overline{A}_x$ and $\mathrm{E}[\overline{a}_{\overline{T_x}|}] = \overline{a}_x$. Calculation of these expected present values requires a suitable life table (*see* **Life Table**; **Life Insurance Mathematics**). In this model, expected present values may be the basis of pricing and reserving in life insurance and pensions, but the higher moments and distributions of the present values are of interest for **risk management** (see [15] for an early example, which is an interesting reminder of just how radically the scope of actuarial science has expanded since the advent of computers). For more on this approach to life insurance mathematics, see [1, 2].
- For more complicated contracts than life insurance, such as **disability insurance** or income protection insurance, multiple state models were developed and expected present values of extremely general contingent payments were obtained as solutions of Thiele's differential equations (*see* **Life Insurance Mathematics**) [4, 5]. This development reached its logical conclusion when

life histories were formulated as **counting processes**, in which setting the familiar expected present values could again be derived [6] as well as computationally tractable equations for the higher moments [13], and distributions [3] of present values. All of classical life insurance mathematics is generalized very elegantly using counting processes [11, 12], an interesting example of Jewell's advocacy that actuarial science would progress when models were formulated in terms of the basic random events instead of focusing on expected values [7].

- Alternatively, or in addition, we may regard the interest rates as random (*see* **Interest-rate Modeling**), and develop accumulations and present values from that point of view. Under suitable distributional assumptions, it may be possible to calculate or approximate moments and distributions of present values of simple contingent payments; for example, [14] assumed that the force of interest followed a second-order autoregressive process, while [17] assumed that the rate of interest was log-normal. The application of such stochastic asset models (*see* **Asset–Liability Modeling**) to actuarial problems has since become extremely important, but the derivation of explicit expressions for moments or distributions of expected values and accumulations is not common. Complex asset models may be applied to complex models of the entire insurance company, and it would be surprising if analytical results could be found; as a rule it is hardly worthwhile to look for them, instead, numerical methods such as Monte Carlo simulation are used (*see* **Stochastic Simulation**).

References

[1] Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. & Nesbitt, C.J. (1986). *Actuarial Mathematics*, The Society of Actuaries, Itasca, IL.

[2] Gerber, H.U. (1990). *Life Insurance Mathematics*, Springer-Verlag, Berlin.

[3] Hesselager, O. & Norberg, R. (1996). On probability distributions of present values in life insurance, *Insurance: Mathematics & Economics* **18**, 35–42.

[4] Hoem, J.M. (1969). Markov chain models in life insurance, *Blätter der Deutschen Gesellschaft für Versicherungsmathematik* **9**, 91–107.

[5] Hoem, J.M. (1988). The versatility of the Markov chain as a tool in the mathematics of life insurance, in Transactions of the 23rd International Congress of Actuaries, Helsinki, **S**, pp. 171–202.

[6] Hoem, J.M. & Aalen, O.O. (1978). Actuarial values of payment streams, *Scandinavian Actuarial Journal* 38–47.

[7] Jewell, W.S. (1980). Generalized models of the insurance business (life and/or non-life insurance), in Transactions of the 21st International Congress of Actuaries, Zurich and Lausanne, **S**, pp. 87–141.

[8] Kellison, S.G. (1991). *The Theory of Interest*, 2nd Edition, Irwin, Burr Ridge, IL.

[9] McCutcheon, J.J. & Scott, W.F. (1986). *An Introduction to the Mathematics of Finance*, Heinemann, London.

[10] Norberg, R. (1990). Payment measures, interest, and discounting. An axiomatic approach with applications to insurance, *Scandinavian Actuarial Journal* 14–33.

[11] Norberg, R. (1991). Reserves in life and pension insurance, *Scandinavian Actuarial Journal* 3–24.

[12] Norberg, R. (1992). Hattendorff's theorem and Thiele's differential equation generalized, *Scandinavian Actuarial Journal* 2–14.

[13] Norberg, R. (1995). Differential equations for moments of present values in life insurance, *Insurance: Mathematics & Economics* **17**, 171–180.

[14] Pollard, J.H. (1971). On fluctuating interest rates, *Bulletin de L'Association Royale des Actuaires Belges* **66**, 68–97.

[15] Pollard, A.H. & Pollard, J.H. (1969). A stochastic approach to actuarial functions, *Journal of the Institute of Actuaries* **95**, 79–113.

[16] Stoodley, C.L. (1934). The effect of a falling interest rate on the values of certain actuarial functions, *Transactions of the Faculty of Actuaries* **14**, 137–175.

[17] Waters, H.R. (1978). The moments and distributions of actuarial functions, *Journal of the Institute of Actuaries* **105**, 61–75.

(*See also* **Annuities**; **Interest-rate Modeling**; **Life Insurance Mathematics**)

ANGUS S. MACDONALD