

# **Lidstone's Theorem**

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# Lidstone's Theorem

Lidstone's theorem, first presented in 1905 [5], gives a sufficient condition for being able to determine whether the prospective reserve held (*see* **Life Insurance Mathematics**) under a **life insurance** policy will be increased or decreased if one or more elements of the **technical basis** are changed. **Lidstone** obtained his results in a discrete-time framework, using the usual one-year probabilities of the **life table**. In this setting, the theorem was extended (*see* [1–3]) and in 1985, it was generalized to continuous time by Norberg [6], which we follow closely here.

Let the technical basis be given by a constant force of interest  $\delta$  and a force of mortality  $\mu_y$  at age  $y$ . Suppose a life insurance contract is taken out at age  $x$ , for a term of  $n$  years, with death benefit  $b_t$  payable on death at age  $x + t$ , and maturity benefit  $b_n$ . Let the rate of premium be  $\pi_t$  at age  $x + t$ , possibly with a single premium of  $\pi_0$  at outset. The prospective **policy** reserve at age  $x + t$  is denoted  $V_t$ , and we suppose that the same technical basis is used for calculating both premiums and reserves, so **Thiele's** differential equation (*see* **Life Insurance Mathematics**) is satisfied

$$\frac{d}{dt}V_t = \delta V_t + \pi_t - \mu_{x+t}(b_t - V_t). \quad (1)$$

A new technical basis is specified by new interest and mortality parameters,  $\delta'$  and  $\mu'_y$ , which lead to a new net premium  $\pi'_t$ , and a new prospective reserve  $V'_t$ , also satisfying Thiele's equation:

$$\frac{d}{dt}V'_t = \delta' V'_t + \pi'_t - \mu'_{x+t}(b_t - V'_t). \quad (2)$$

Lidstone introduced a *critical function*, which we denote  $c_t$ , and in the notation used here it is

$$c_t = (\pi'_t - \pi_t) - (\mu'_{x+t} - \mu_{x+t})b_t + (\delta' - \delta + \mu'_{x+t} - \mu_{x+t})V_t. \quad (3)$$

Informally, Lidstone's theorem then states that if the reserves on the old and new bases are equal at outset and at expiry, and if the critical function changes sign once during the policy term, then  $V'_t$  is

either always not less than  $V_t$ , or always not greater than  $V_t$ . More precisely (*see* [6]), if  $V_{0+} = V'_{0+}$  and  $V_{n-} = V'_{n-}$  (allowing for the possible initial and final payments  $\pi_0$  and  $b_n$ ) and there exists a single time  $t_0$ , ( $0 < t_0 < n$ ) at which  $c_t$  may change sign, then

$$c_t \leq 0 \text{ for } t \in (0, t_0) \text{ and } c_t \geq 0 \text{ for } t \in (t_0, n) \\ \Rightarrow V'_t \leq V_t \text{ for } t \in (0, n) \quad (4)$$

$$c_t \geq 0 \text{ for } t \in (0, t_0) \text{ and } c_t \leq 0 \text{ for } t \in (t_0, n) \\ \Rightarrow V'_t \geq V_t \text{ for } t \in (0, n) \quad (5)$$

and the result is also valid if all the inequalities are strict.

The theorem has some corollaries for contracts with level benefits, level premiums, and increasing reserves, such as endowment assurances [6].

- An increase/decrease in the force of interest produces a decrease/increase in the reserve.
- An increase/decrease in the force of mortality at all ages, such that  $|\mu'_y - \mu_y|$  is nonzero and nonincreasing, produces a decrease/increase in the reserve.

The theorem is, in essence, about testing the sensitivity of reserves to the elements of the valuation basis. It applies only to the special case of the single-decrement mortality model, and it seems not to have any obvious extension to generalizations of this, such as the Markov illness–death model (*see* **Disability Insurance**). Kalashnikov & Norberg [4] took this step by direct differentiation of Thiele's equations, and numerical solution of the resulting partial differential equations.

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(See also **Lidstone, George James (1870–1952); Life Insurance Mathematics; Technical Bases in Life Insurance**)

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