

VIRTUAL AMALGAMATION OF RELATIVELY QUASICONVEX SUBGROUPS

EDUARDO MARTÍNEZ-PEDROZA AND ALESSANDRO SISTO

ABSTRACT. For relatively hyperbolic groups, we investigate conditions guaranteeing that the subgroup generated by two relatively quasiconvex subgroups Q_1 and Q_2 is relatively quasiconvex and isomorphic to $Q_1 *_{Q_1 \cap Q_2} Q_2$. The main theorem extends results for quasiconvex subgroups of word-hyperbolic groups, and results for discrete subgroups of isometries of hyperbolic spaces. An application on separability of double cosets of quasiconvex subgroups is included.

1. INTRODUCTION

This paper continues the work that started in [10] motivated by the following question:

Problem 1. Suppose G is a relatively hyperbolic group, Q_1 and Q_2 are relatively quasiconvex subgroups of G . Investigate conditions guaranteeing that the natural homomorphism

$$Q_1 *_{Q_1 \cap Q_2} Q_2 \longrightarrow G$$

is injective and that its image $\langle Q_1 \cup Q_2 \rangle$ is relatively quasiconvex.

Let G be a group hyperbolic relative to a finite collection of subgroups $\mathbb{P}$, and let dist be a proper left invariant metric on G .

Definition 1. Two subgroups Q and R of G have *compatible parabolic subgroups* if for any maximal parabolic subgroup P of G either $Q \cap P < R \cap P$ or $R \cap P < Q \cap P$.

Theorem 2. *For any pair of relatively quasiconvex subgroups Q and R of G with compatible parabolic subgroups, and any finite index subgroup H of $Q \cap R$, there is a constant $M = M(Q, R, H, \text{dist}) \geq 0$ with the following property. Suppose that $Q' < Q$ and $R' < R$ are subgroups such that*

- (1) $H = Q' \cap R'$, and
- (2) $\text{dist}(1, g) \geq M$ for any g in $Q' \setminus Q' \cap R'$ or $R' \setminus Q' \cap R'$.

Then the subgroup $\langle Q' \cup R' \rangle$ of G satisfies:

- (1) *The natural homomorphism*

$$Q' *_{Q' \cap R'} R' \longrightarrow \langle Q' \cup R' \rangle$$

is an isomorphism.

- (2) *If Q' and R' are relatively quasiconvex, then so is $\langle Q' \cup R' \rangle$.*

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Theorem 2 extends results by Gitik [6, Theorem 1] for word-hyperbolic groups and by the first author [10, Theorem 1.1] for relatively hyperbolic groups. Combination results with stronger hypothesis along the lines of Theorem 2 have recently obtained by Yang including a version for HNN-extensions and applications to subgroup separability [14].

Definition 3. Two subgroups Q and R of a group G can be *virtually amalgamated* if there are finite index subgroups $Q' < Q$ and $R' < R$ such that the natural map $Q' *_{Q' \cap R'} R' \rightarrow G$ is injective.

Let Q and R be relatively quasiconvex subgroups of G with compatible parabolic subgroups, and let $M = M(Q, R, Q \cap R)$ be the constant provided by Theorem 2. If $Q \cap R$ is a separable subgroup of G , then there is a finite index subgroup G' of G containing $Q \cap R$ such that $\text{dist}(1, g) > M$ for every $g \in G$ with $g \notin Q \cap R$. In this case, the subgroups $Q' = G' \cap Q$ and $R' = G' \cap R$ satisfy the hypothesis of Theorem 2; hence they have a quasiconvex virtual amalgam.

Corollary 4 (Virtual Quasiconvex Amalgam Theorem). *Let Q and R quasiconvex subgroups of G with compatible parabolic subgroups, and suppose that $Q \cap R$ is separable. Then Q and R can be virtually amalgamated in G .*

It is known that many (relatively) hyperbolic groups have that property that all quasiconvex or all finitely generated subgroups are separable [2, 9, 8, 13, 12, 1]. Still, it is a natural question to ask whether the corollary above holds under the hypothesis that G is residually finite.

A special case of the Virtual Quasiconvex Amalgam Theorem is the following (cfr. [3, Theorem 5.3]).

Corollary 5. *Let G be a geometrically finite subgroup of $\text{Isom}(\mathbb{H}^n)$, and let Q and R be geometrically finite subgroups of G with compatible parabolic subgroups. Suppose that $Q \cap R$ is separable in G . Then Q and R have a geometrically finite virtual amalgam.*

Separability of quasiconvex subgroups and double cosets of quasiconvex subgroups is of interest in the construction of actions on special cube complexes [12]. The machinery we use to prove the main result also gives the following.

Corollary 6 (Double cosets are separable). *Let G be a relatively hyperbolic group such that all its quasiconvex subgroups are separable. If Q and R are quasiconvex subgroups with compatible parabolic subgroups then the double coset QR is separable.*

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2. PRELIMINARIES

2.1. Gromov-hyperbolic Spaces. Let (X, dist) be a proper and geodesic δ -hyperbolic space. Recall that a (λ, μ) -quasi-geodesic is a curve $\gamma: [a, b] \rightarrow X$ parametrize by arc-length such that

$$|x - y|/\lambda - \mu \leq \text{dist}(\gamma(x), \gamma(y)) \leq \lambda|x - y| + \mu$$

for all $x, y \in [a, b]$. The curve γ is a k -local (λ, μ) -quasi-geodesic if the above condition is required only for $x, y \in [a, b]$ such that $|x - y| \leq k$.

Lemma 7. [5, Chapter 3, Theorem 1.2] (*Morse Lemma*) *For each λ, μ, δ there exists $k > 0$ with the following property. In an δ -hyperbolic geodesic space, any (λ, μ) -quasi-geodesic at k -Hausdorff-distance from the geodesic between its endpoints.*

Lemma 8. [5, Chapter 3, Theorem 1.4] *For each λ, μ, δ there exist k, λ', μ' so that any k -local (λ, μ) -quasi-geodesic in a δ -hyperbolic geodesic space is a (λ', μ') -quasi-geodesic.*

Fix a basepoint $x_0 \in X$. If G is a subgroup of $\text{Isom}(X)$, we identify each element g of G with the point gx_0 of X . For $g_1, g_2 \in G$ denote by $\text{dist}(g_1, g_2)$ the distance $\text{dist}(g_1x_0, g_2x_0)$. Since X is a proper space, if G is a discrete subgroup of $\text{Isom}(X)$, this is a proper and left invariant pseudo-metric on G .

Lemma 9 (Bounded Intersection). [10, Lemma 4.2] *Let G be a discrete subgroup of $\text{Isom}(X)$, let Q and R be subgroups of G , and let $\mu > 0$ be a real number. Then there is a constant $M = M(Q, R, \mu) \geq 0$ so that*

$$Q \cap \mathcal{N}_\mu(R) \subset \mathcal{N}_M(Q \cap R).$$

2.2. Relatively Quasiconvex Subgroups. We follow the approach to relatively hyperbolic groups as developed by Hruska [7].

Definition 10 (Relative Hyperbolicity). A group G is *relatively hyperbolic with respect to a finite collection of subgroups* $\mathbb{P}$ if G acts properly discontinuously and by isometries on a proper and geodesic δ -hyperbolic space X with the following property: X has a G -equivariant collection of pairwise disjoint horoballs whose union is an open set U , G acts cocompactly on $X - U$, and $\mathbb{P}$ is a set of representatives of the conjugacy classes of parabolic subgroups of G .

Throughout the rest of the paper, G is a relatively hyperbolic group acting on a proper and geodesic δ -hyperbolic space X with a G -equivariant collection of horoballs satisfying all conditions of Definition 10. As before, we fix a basepoint $x_0 \in X - U$, identify each element g of G with $gx_0 \in X$ and let $\text{dist}(g_1, g_2)$ denote $\text{dist}(g_1x_0, g_2x_0)$ for $g_1, g_2 \in G$.

Lemma 11. [4, Lemma 6.4] (*Cocompact actions of parabolic subgroups on thick horospheres*) *Let B be a horoball of X with G -stabilizer P . For any $M > 0$, P acts cocompactly on $\mathcal{N}_M(B) \cap (X - U)$.*

Lemma 12 (Parabolic Approximation). *Let Q be a subgroup of G and let $\mu > 0$ be a real number. There is a constant $M = M(Q, \mu)$ with the following property. If P is a maximal parabolic subgroup of G stabilizing a horoball B , and $\{1, q\} \subset Q \cap \mathcal{N}_\mu(B)$ then there is $p \in Q \cap P$ such that $\text{dist}(p, q) < M$.*

Proof. By Lemma 11, $\text{dist}(q, P) < M_1$ for some constant $M_1 = M_1(Q, P)$. Then Lemma 9 implies that $\text{dist}(q, Q \cap P) < M_2$ where $M_2 = N(Q, P, M_1)$. Since B is a horoball at distance less than μ from 1, there are only finitely many possibilities for B and hence for the subgroup P . Let M the maximum of all $N(Q, P, \mu)$ among the possible P . $\square$

Definition 13 (Relatively Quasiconvex Subgroup). A subgroup Q of G is *relatively quasiconvex* if there is $\mu \geq 0$ such that for any geodesic c in X with endpoints in Q , $c \cap (X - U) \subset \mathcal{N}_\mu(Q)$.

The choice of horoballs turns out not to make a difference:

Proposition 14. [7] *If Q is relatively quasiconvex in G then for any $L \geq 0$ there is $\mu \geq 0$ such that for any geodesic c in X with endpoints in Q , $c \cap \mathcal{N}_L(X - U) \subset N_\mu(Q)$.*

3. A LEMMA ON GROMOV'S INNER PRODUCT

Let Q and R be relatively quasiconvex subgroups with compatible parabolic subgroups, and let H be a finite index subgroup of $Q \cap R$.

Let Q' and R' be subgroups of Q and R respectively such that $Q' \cap R' = H$. Let $g \in Q'R'$ (or $g \in R'Q'$) such that $g \notin H$. Suppose $g = qr$ (or $g = rq$) with $q \in Q'$, $r \in R'$ and such that $\text{dist}(1, q) + \text{dist}(1, r)$ is minimal among all such products.

Lemma 15. *Suppose that there exists $a \in H$ and a point p at distance at most A from the geodesic segment $[1, g]$ so that $\text{dist}(p, qa) \leq B$. Then*

$$\text{dist}(1, q) + \text{dist}(1, r) \leq \text{dist}(1, g) + 2A + 2B.$$

Proof. Let $p' \in [1, g]$ be such that $\text{dist}(p, p') < A$. Then

$$\begin{aligned} \text{dist}(1, qa) + \text{dist}(1, a^{-1}r) &\leq \\ &\leq \text{dist}(1, p') + \text{dist}(p', qa) + \text{dist}(qa, p') + \text{dist}(p', g) \\ &\leq \text{dist}(1, g) + 2A + 2B \end{aligned}$$

As g can be written as $(qa)(a^{-1}r)$, the minimality assumption implies $\text{dist}(1, q) + \text{dist}(1, r) \leq \text{dist}(1, g) + 2A + 2B$ $\square$

Lemma 16. (*Gromov's Inner Product is Bounded*) *There exists a constant $K = K(Q, R, H)$ with the following property.*

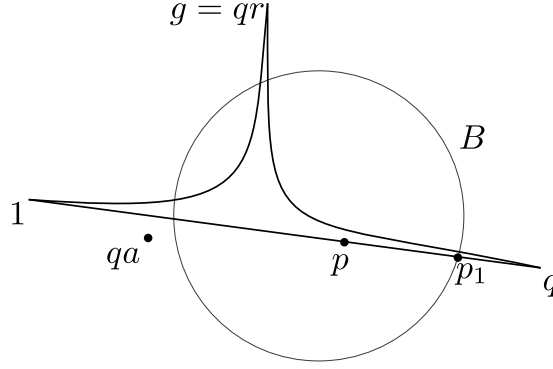
$$\text{dist}(1, q) + \text{dist}(1, r) \leq \text{dist}(1, g) + K.$$

Proof. Constants which depend only on Q , R , H and δ are denoted by M_i , the index counts positive increments of the constant during the proof. Suppose $g = qr$, the other case being symmetric. The constant K of the statement corresponds to M_{13} .

Consider a triangle Δ with vertices $1, q, g$. Let $p \in [1, q]$ be a center of Δ , i.e., the δ -neighborhood of p intersects all sides of Δ .

Suppose that $p \in X - U$. Then $\text{dist}(p, Q), \text{dist}(p, qR) \leq M_1$ by relative quasiconvexity of Q and R . By Lemma 9, there exists $a \in Q \cap R$ so that $\text{dist}(p, qa) \leq M_2$. Since H is a finite index subgroup of $Q \cap R$, there is $b \in H$ such that $\text{dist}(p, qb) \leq M_3$. By Lemma 15, $\text{dist}(1, q) + \text{dist}(1, r) \leq \text{dist}(1, g) + 2M_3 + 2\delta$.

Suppose instead that p is in a horoball B , whose stabilizer is P . We can assume $\text{dist}(q, B) \leq M_8$. Indeed, let p_1 be the entrance point of the geodesic $[q, 1]$ in B ; then $\text{dist}(p_1, Q) < M_4$ by quasiconvexity of Q . Notice that $\text{dist}(p_1, [q, g])$ is at most 2δ since p is a center of Δ and $p_1 \in [q, p]$ (consider a triangle with vertices p, q, p' for $p' \in [q, g]$ so that $d(p, p') \leq \delta$). By quasiconvexity of R , there is $p_2 \in [q, g]$ such that $\text{dist}(p_1, p_2), \text{dist}(p_2, qR) < M_5$. Lemma 9 implies there is $a \in Q \cap R$ such that $\text{dist}(qa, p_1), \text{dist}(qa, p_2) < M_6$. Since H is a finite index subgroup of $Q \cap R$, there is $b \in H$ such that $\text{dist}(qb, p_1), \text{dist}(qb, p_2) < M_7$. Since g can be written as $(qb)(b^{-1}r)$,



by minimality

$$\begin{aligned}
 \text{dist}(1, p_1) + \text{dist}(p_1, q) + \text{dist}(q, p_2) + \text{dist}(p_2, g) &= \\
 &= \text{dist}(1, q) + \text{dist}(1, g) \\
 &\leq \text{dist}(1, qb) + \text{dist}(1, b^{-1}r) \\
 &= \text{dist}(1, p_1) + \text{dist}(p_1, qb) + \text{dist}(qb, p_2) + \text{dist}(p_2, g),
 \end{aligned}$$

and therefore

$$\begin{aligned}
 2 \text{dist}(q, B) &= 2 \text{dist}(p_1, q) \\
 &\leq \text{dist}(p_1, q) + \text{dist}(q, p_2) + \text{dist}(p_1, p_2) \\
 &\leq \text{dist}(p_1, qb) + \text{dist}(qb, p_2) + \text{dist}(p_1, p_2) \\
 &\leq 2M_8.
 \end{aligned}$$

Since Q and R have compatible parabolic subgroups, assume that $Q \cap q^{-1}Pq \leq R \cap q^{-1}Pq$, the other case being symmetric. By quasiconvexity of Q , there is $q_1 \in Q$ at distance M_9 from the entrance point of $[1, q]$ in B . In particular, the distance from q_1 to $[1, g]$ is at most M_{10} . By the parabolic approximation lemma applied to $\{1, q^{-1}q_1\} \subset Q \cap \mathcal{N}_{M_{10}}(q^{-1}B)$, there is an element $a \in Q \cap q^{-1}Pq$ such that $\text{dist}(qa, q_1) \leq M_{11}$. Since $Q \cap q^{-1}Pq \leq R \cap q^{-1}Pq$ it follows that $a \in Q \cap R$. Since H is finite index in $Q \cap R$, by increasing the constant we can assume that $a \in H$ and $\text{dist}(qa, q_1) \leq M_{12}$. Then Lemma 15 implies

$$\text{dist}(1, q) + \text{dist}(1, r) \leq \text{dist}(1, g) + M_{13}. \quad \square$$

4. PROOF OF THEOREM 2

Let Q and R be relatively quasiconvex subgroups with compatible parabolic subgroups, and let H be a finite index subgroups of $Q \cap R$.

Let $K = K(Q, R, H)$ be the constant of Lemma 16. Let M be large enough so that $M > k, \lambda' \mu'$ where k, λ' and μ' are as in Lemma 8 for $\lambda = 1, \mu = K$.

Let Q' and R' be subgroups satisfying the hypothesis of the theorem, in particular $Q' \cap R' = H$. Consider $1 \neq g \in Q' *_{Q' \cap R'} R'$ and suppose that $g \notin Q' \cap R'$. Then $g = g_1 \dots g_n$ where the g_i 's are alternatively elements of $Q' \setminus Q' \cap R'$ and $R' \setminus Q' \cap R'$. Moreover, assume that this product is *minimal* in the sense that $\sum \text{dist}(1, g_i)$ is minimal among all such products describing g .

Lemma 17. *For each i , let $h_i = g_1 \dots g_i$. Then the concatenation $\alpha = \alpha_1 \dots \alpha_{n-1}$ of geodesics α_i from h_i to h_{i+1} is an M -local $(1, K)$ -quasi-geodesic.*

Proof. By the choice of Q' and R' each segment α_i has length at least M . Let $x \in [h_{i-1}, h_i]$ and $y \in [h_i, h_{i+1}]$. By Lemma 16

$$\begin{aligned} \text{dist}(h_{i-1}, x) + \text{dist}(x, y) + \text{dist}(y, h_{i+1}) &\geq \text{dist}(h_{i-1}, h_{i+1}) \geq \\ &\geq \text{dist}(h_{i-1}, h_i) + \text{dist}(h_i, h_{i+1}) - K = \\ &= \text{dist}(h_{i-1}, x) + \text{dist}(x, h_i) + \text{dist}(h_i, y) + \text{dist}(y, h_{i+1}) - K. \end{aligned}$$

Therefore $\text{dist}(x, y) + K \geq \text{dist}(x, h_i) + \text{dist}(h_i, y)$. $\square$

Since $M > k$, Lemma 8 implies that α is a (λ', μ') -quasi-geodesic. Since $M > \lambda'\mu'$, it follows that α has different endpoints. Therefore we have shown that the map $Q' *_{Q' \cap R'} R' \rightarrow G$ is injective.

It is left to prove that if Q' and R' are relatively quasiconvex, then $\langle Q', R' \rangle$ is relatively quasiconvex. Let $g \in \langle Q \cap R \rangle$ and let γ be a geodesic from 1 to g . Since H is quasiconvex, if $g \in H$ then $\gamma \cap (X - U)$ is uniformly close to H and hence to $\langle Q \cap R \rangle$. Suppose that $g \notin H$. By Lemma 7 (Morse Lemma), any (λ', μ') -quasi-geodesic is at Hausdorff distance at most L from any geodesic between its endpoints. In particular, $\gamma \cap (X - U) \subseteq \mathcal{N}_L(\alpha) \cap (X - U)$ where α is the quasi-geodesic constructed above. It is enough to show that $\alpha \cap \mathcal{N}_L(X - U)$ is contained in $\mathcal{N}_\mu(\langle Q' \cup R' \rangle)$. Let $p \in \alpha \cap \mathcal{N}_L(X - U)$ and let i be so that $p \in [h_i, h_{i+1}] \cap \mathcal{N}_L(X - U)$. Assume $g_{i+1} \in Q'$, the other case being symmetric. As Q' is relatively quasiconvex and in view of Proposition 14, there is a constant μ so that $p \in \mathcal{N}_\mu(h_i Q') \subseteq \mathcal{N}_\mu(\langle Q' \cup R' \rangle)$ (as $h_i \in \langle Q' \cup R' \rangle$).

5. SEPARABILITY OF DOUBLE COSETS

We now show Corollary 6. Suppose that all quasiconvex subgroups of G are separable. Let Q and R be quasiconvex subgroups with compatible parabolic subgroups. Let $g \in G$ and suppose that $g \notin QR$. We follow an argument described in [11, 14].

Let $K = K(Q, R, Q \cap R)$ be the constant of Lemma 16. As in the proof of Theorem 2, let M be large enough so that $M > k, \lambda'\mu'$ where k, λ' and μ' are as in Lemma 8 for $\lambda = 1, \mu = K$. In addition, assume that

$$(1) \quad M > \lambda' \text{dist}(1, g) + \lambda'\mu'$$

Lemma 18. *There are finite index subgroups Q' and R' of Q and R respectively such that $g \notin Q\langle Q', R' \rangle R$.*

Proof. Since $Q \cap R$ is separable, there are finite index subgroups Q' and R' of Q and R respectively, such that $Q' \cap R' = Q \cap R$ and $\text{dist}(1, f) \geq 2M$ for any f in $Q' \setminus Q' \cap R'$ or $R' \setminus Q' \cap R'$. By Theorem 2 $\langle Q' \cup R' \rangle$ is a quasiconvex subgroup of G isomorphic to $Q' *_{Q \cap R} R'$.

Suppose that $g \in Q\langle Q', R' \rangle R$. Since $g \notin QR$ it follows that $g = g_1 \dots g_{2n}$ where $g_1 \in Q$, $g_{2n} \in R$, $g_{2i+1} \in Q' \setminus Q \cap R$, $g_{2i} \in R' \setminus Q \cap R$, and $n \geq 2$. Assume that this product is minimal in the sense that $\sum \text{dist}(1, g_i)$ is minimal among all such products describing g .

For each i , let $h_i = g_1 \dots g_i$; let α_i be a geodesic from h_i to h_{i+1} . By the choice of Q' and R' each segment α_i has length at least $2M$ except α_1 and α_{2n-1} .

Notice that $g_2 \dots g_{2n-1}$ represents an element of $Q' *_{Q \cap R} R'$ and such product is minimal in the sense of the previous section, so that by Lemma 17 the concatenation $\alpha_2 \dots \alpha_{2n-1}$ is an M -local $(1, K)$ -quasi-geodesic. Minimality of $g_1 \dots g_{2n}$

and Lemma 16 imply that the concatenations $\alpha_1\alpha_2$ and $\alpha_{2n-1}\alpha_{2n}$ are M -local $(1, K)$ -quasi-geodesics. Since α_2 and α_{2n-1} have both length at least $2M$, it follows that the concatenation $\alpha = \alpha_1 \cdots \alpha_{2n}$ is an M -local $(1, K)$ -quasi-geodesic.

By Lemma 8, it follows that α is a (λ', μ') -quasi-geodesic between 1 and g . It follows that $\text{dist}(1, g) \geq 4M/\lambda' - \mu'$; this is a contradiction with (1) above. $\square$

Since Q' and R' are finite index, there are $q_1, \dots, q_k \in Q$ and $r_1, \dots, r_m \in R$ such that

$$Q\langle Q', R' \rangle R = \bigcup_{q_i, r_j} q_i \langle Q', R' \rangle r_j.$$

Since $\langle Q', R' \rangle$ is quasiconvex, it is closed in the profinite topology. It follows that $Q\langle Q', R' \rangle R$ is a finite union of closed sets. Therefore $Q\langle Q', R' \rangle R$ is a closed set in the profinite topology containing QR and such that $g \notin Q\langle Q', R' \rangle R$. Since g was an arbitrary element of $g \in G$ not in QR , it follows that QR is closed in the profinite topology of G .

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MEMORIAL UNIVERSITY, SAINT JOHN'S, NEWFOUNDLAND, CANADA
E-mail address: emartinezped@mun.ca

MATHEMATICAL INSTITUTE, OXFORD OX1 3LB, UNITED KINGDOM
E-mail address: sisto@maths.ox.ac.uk