Question 1 (20 Marks) At time t seconds, the position of a particle of mass $M = 5$kg relative to a fixed origin O is $\boldsymbol{r}(t) = (1+3t)\,\boldsymbol{i} + 2\boldsymbol{j} - 5t^2\boldsymbol{k}$ metres.

(a) Compute the particle's velocity, momentum, acceleration, kinetic energy and speed relative to O at time t. For each quantity, state the unit in which it is measured.

(b) Sketch the particle's trajectory and describe it geometrically.

(c) The point O' has the position vector $\boldsymbol{R}(t) = 3t\boldsymbol{i}$ metres relative to O at time t seconds. Compute the particle's position, velocity and angular momentum relative to O', stating units in each case.

Question 2 (15 Marks)

(a) State Newton's first and second law of motion.

(b) Consider a car of mass M moving in the positive $\boldsymbol{i}$-direction, so that its position at time t seconds is $\boldsymbol{r}(t) = x(t)\boldsymbol{i}$ and its velocity is $\boldsymbol{v} = v(t)\boldsymbol{i}$, where $v(t) = \dot{x}(t)$. The car is moving at constant velocity $\boldsymbol{v} = v_0\boldsymbol{i}$ for times $t \leq 0$, but at time $t = 0$ seconds the driver brakes. As soon as the brakes are on, the car experiences a constant force $\boldsymbol{F} = -\lambda M\boldsymbol{i}$, where λ is a constant. Show that the equation of motion of the car during the braking process is

$$\ddot{x} = -\lambda$$

and find the solution satisfying the intial conditions $x(0) = 0, \dot{x}(0) = v_0$.

(c) Show that the quantity

$$E = \frac{1}{2}M\dot{x}^2 + \lambda M x \qquad (1)$$

is constant during the braking process.

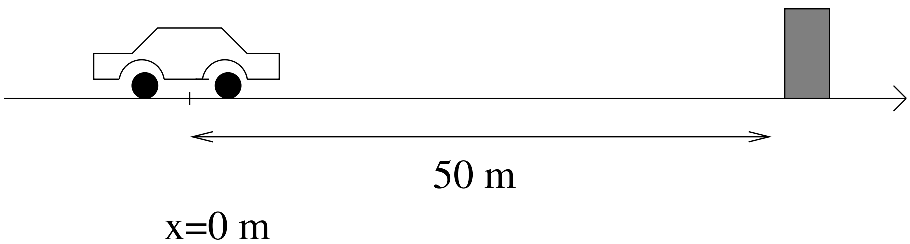

Figure 1: Braking car

(d) Suppose there is an obstacle on the road at $x = 50$ metres, as shown in Fig. 1. If the car approaches with initial speed $v_0 = 80$ m s^{-1} and starts braking when it passes $x = 0$ it comes to a halt precisely at the obstacle. If the initial speed of the car is increased to $u_0 = 100$ m s^{-1} what is the speed with which it will hit the obstacle, supposing that it again starts braking when it passes $x = 0$? (*Hint: use the conservation law* (1))

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Question 3 (15 Marks)

(a) Consider two bodies with masses m_1 and m_2, with body 1 exerting a force $\boldsymbol{F}_{21}$ on body 2, and body 2 exerting a force $\boldsymbol{F}_{12}$ on body 1. State Newton's third law for this situation

(b) If body 1 is moving with velocity $\boldsymbol{v}_1$ and body 2 with velocity $\boldsymbol{v}_2$, deduce from Newton's third law that the total momentum

$$\boldsymbol{P} = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2$$

is constant.

Figure 2: Colliding spheres

(c) A smooth sphere of mass M moving with velocity $\boldsymbol{v} = 2\,\boldsymbol{i}$ m s^{-1} hits a second sphere of mass $2M$ which is at rest, see Fig. 2. The motion takes place along the $\boldsymbol{i}$ direction. Assuming the collision to be elastic, compute the velocities of both spheres after the collision.

Question 4 (20 Marks)

At time t seconds, a particle of mass $M = 1$ kg has position vector $\boldsymbol{r}(t)$ metres relative to some fixed origin O. The particle moves under the influence of a central force $\boldsymbol{F} = -\dfrac{\boldsymbol{r}}{r^3}$ Newtons.

(a) Write down the equation of motion for the particle's position.

(b) Show that the total energy $E = \dfrac{1}{2}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} - \dfrac{1}{r}$ is conserved during the motion.

(c) Show that the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$ is conserved during the motion.

(d) If the particle's initial position is given by $\boldsymbol{r}(0) = \boldsymbol{i}$ m and its initial velocity is $\dot{\boldsymbol{r}}(0) = -2\boldsymbol{j}$ m s^{-1}, compute E and $\boldsymbol{L}$. Show that the motion takes place in the $\boldsymbol{ij}$-plane.

(e) In terms of polar coordinates (r, θ) in the $\boldsymbol{i}, \boldsymbol{j}$-plane the position vector can be written $\boldsymbol{r} = r\cos\theta\,\boldsymbol{i} + r\sin\theta\,\boldsymbol{j}$ where both r and θ are functions of t. Show that the angular momentum can be expressed as $\boldsymbol{L} = r^2\dot{\theta}\boldsymbol{k}$ and that $\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} = (\dot{r}^2 + r^2\dot{\theta}^2)$.

(f) Use the results of (e) to derive the formula $E = \dfrac{1}{2}\dot{r}^2 + \dfrac{L^2}{2r^2} - \dfrac{1}{r}$ for the total energy ($L = |\boldsymbol{L}|$).

continued overleaf

Question 5 (20 Marks)

(a) An inertial frame S' is moving relative to an inertial frame S with speed v in the positive x-direction. State the Lorentz transformations that express the coordinates (x', y', z', t') of the frame S' in terms of the coordinates (x, y, z, t) of the frame S and *vice-versa*

(b) Also state the Galilean transformations expressing the coordinates (x', y', z', t') of the frame S' in terms of the coordinates (x, y, z, t). Show that, for speeds v much smaller than the speed of light c, the Lorentz transformations reduce to the Galilean transformations.

(c) Two events occur at the same place in an inertial frame S', and are separated by a time interval of 2 seconds there. Using Lorentz transformations, find the spatial separation between these two events in an inertial frame S, in which the events are separated by a time interval of 2.3 seconds.

Question 6 (10 Marks)

A muon is produced by cosmic ray bombardment of the atmosphere at a height H =14km above the earth and travels towards the earth with a speed of $v = 0.999c$ in the earth's frame of reference. The lifetime of the muon in its rest frame is $\tau = 2.2 \times 10^{-6}$ s. You may assume $c = 3 \times 10^8$ m s^{-1} in this question.

(a) Find the lifetime of the muon in the earth frame and show that it can just reach the earth before decaying.

(b) Reach the same conclusion by using a reference frame in which the muon is at rest.

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