Question 1 (20 Marks)

(a) A particle of mass $M = 5$ kg has position vector $\vec{r}(t) = [(3 + 5t + 2t^2)\vec{i} + 3t\vec{k}]$m relative to a fixed origin O at time t seconds. Compute the

(i) velocity, (ii) speed, (iii) momentum, (iv) acceleration

of the particle relative to O. State the units in each case.

(b) A particle of mass $M = 3$ kg is attachted to a spinning wheel of radius 3 m. Its position vector at time t seconds relative to the fixed centre O of the wheel is $\vec{r}(t) = [3\cos(2t)\vec{i} + 3\sin(2t)\vec{j}]$ m. Compute the

(i) velocity, (ii) kinetic energy, (iii) angular momentum

of the particle relative to the centre of the wheel, stating the units in each case.

Question 2 (20 Marks)

A particle of mass M is projected from a point O on a horizontal plane with initial speed $u = 30$ m/s at an angle of 45° to the horizontal. Choose a basis $\{\vec{i}, \vec{j}, \vec{k}\}$ located at O so that $\vec{k}$ points vertically upwards, and the initial velocity $\vec{u}$ lies in the $\vec{i}\vec{k}$-plane.

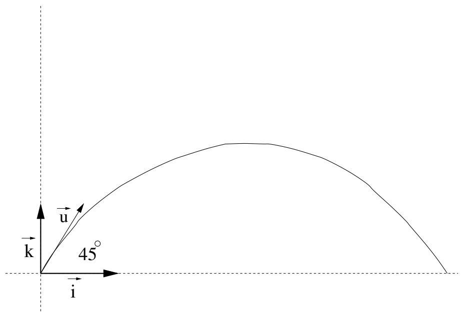

Figure 1: Projectile motion

(a) Taking the acceleration due to gravity to be $g = 10\text{m s}^{-2}$ and ignoring air resistance, write down the equation of motion governing the particle's position vector $\vec{r}(t)$.

(b) What are the particle's position and velocity vector at the initial time $t = 0$? Solve the equation of motion subject to these initial conditions to show that the particle's position at time t seconds is

$$\vec{r}(t) = [\frac{30}{\sqrt{2}}\, t\,\vec{i} + \left(\frac{30}{\sqrt{2}}\, t - 5t^2\right)\vec{k}]\text{ m}.$$

(c) Find the time when the particle reaches its greatest height and the particle's position at that point.

(d) Find the time when the particle again reaches the horizontal plane and calculate its speed at that point.

continued overleaf

Question 3 (10 Marks)

(a) Consider two bodies with masses m_1 and m_2, moving with velocities $\vec{v}_1$ and $\vec{v}_2$. What is (i) the total momentum and (ii) the total kinetic energy of the two bodies?

Figure 2: Colliding spheres

(b) A smooth sphere of mass M moving with velocity $\vec{v}$ hits a second smooth sphere of the same mass M which is at rest. Assuming the collision to be perfectly elastic, show that the velocities of the spheres after the collision are orthogonal.

Question 4 (15 Marks)

(a) Consider a body of mass M falling under gravity near the earth's surface. Its position vector at time t is $\vec{r}(t) = z(t)\vec{k}$, where $\vec{k}$ is a constant unit vector pointing "up". The gravitational force is $\vec{F} = -Mg\vec{k}$, where $g = 10$ m s^{-2}.

(i) State the equation of motion for the body.

(ii) Show that the total energy

$$E = \frac{1}{2}M\dot{z}^2 + Mgz$$

is constant during the fall.

(iii) If the body is released from rest at a height of $h = 20$ m above sea level, use energy conservation to find its speed when it reaches sea level.

(b) Consider now an asteroid of mass m falling towards the earth. The asteroid's distance r from the centre of the earth obeys the equation of motion

$$m\ddot{r} = -\frac{GM_E m}{r^2},$$

where G is Newton's constant and M_E the mass of the earth. Show that the total energy

$$E = \frac{1}{2}m\dot{r}^2 - \frac{GM_E m}{r}$$

is constant during the fall.

continued overleaf

Question 5 (20 Marks)

At time t seconds, a particle of mass m kg has position vector $\vec{r}(t)$ metres relative to some fixed origin O. The particle moves under the influence of the force

$$\vec{F} = 4\dot{\vec{r}} \times \vec{k}\,\text{Newtons},$$

where $\vec{k}$ is a constant unit vector.

(a) Write down the equation of motion for the particle's position.

(b) Show that kinetic energy $E = \frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}}$ is conserved during the motion.

(c) Suppose that the particle's mass is 1 kg, its initial position is $\vec{r}(0) = 4\vec{i}$ and its initial velocity $\dot{\vec{r}}(0) = -16\vec{j} + \vec{k}$. Use these initial conditions to find the numerical values for the constants R, ω and v when

$$\vec{r}(t) = R\cos(\omega t)\vec{i} + R\sin(\omega t)\vec{j} + vt\vec{k}. \quad (1)$$

(d) Show that the equation of motion is satisfied by (1) with the values for R, ω and v determined in (c).

(e) If the particle's position at time t is given by (1), sketch the particle's trajectory and desribe its shape.

Question 6 (15 Marks)

(a) An inertial frame S' is moving relative to an inertial frame S with speed v in the positive x-direction. State the Lorentz transformations that express the coordinates (x', y', z', t') of the frame S' in terms of the coordinates (x, y, z, t) of the frame S and *vice-versa*

(b) Two events, A and B, are observed in two different inertial frames, S and S'. Event A occurs at the spacetime origin in both frames i.e. has coordinates $x_A = y_A = z_A = t_A = 0$ in S and $x'_A = y'_A = z'_A = t'_A = 0$ in S'. Event B occurs at

$$x_B = 10, \quad y_B = z_B = 0, \qquad ct_B = 2$$

as observed in S and at

$$y'_B = z'_B = ct'_B = 0$$

in frame S' (all distances are measured in metres). Find the velocity of S' with respect to S and, compute the spatial separation of the two events in S'.

End of paper