Module F12MR2: Mathematics of Motion

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The first part of the course, described in Sects. 1-4, deals with **classical mechanics**. This is the theory of motion based on the work of *Isaac Newton* 1643-1727 and valid when all speeds are small compared to the speed of light $c = 2.998 \times 10^8$ m s^{-1}. The theory is sufficient for a very accurate description of the motion of the planets around the sun. Newton himself showed that his theory explains Kepler's laws of planetary motion, and we will repeat his analysis in modern language in Sect. 4. The last part of the course, consisting of Sects. 5 and 6, is an introduction to **relativistic mechanics**. This is the modification of Newtonian mechanics made necessary by Einstein's special theory of relativity, described in Sect. 5.

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1 Kinematics

1.1 Position and frames

Kinematics is the study of the motion of points in space. A point is an idealised particle which has no extent and which one can be characterise completely by giving its position P in space. In order to develop a mathematical model of motion, we therefore need a mathematical model of space. The model of space used in classical mechanics (but not in relativistic mechanics) is called **Euclidean space**. By definition, a point P in Euclidean space can be described by choosing an origin O and then giving a **vector** which gives the displacement of P relative to O:

$$\vec{r} = \vec{OP}. \tag{1.1}$$

A key feature of Euclidean space is the **inner product** $\vec{r}_1 \cdot \vec{r}_2$ for any two position vectors $\vec{r}_1$ and $\vec{r}_2$. This allows us to compute the **length** of the vector $\vec{r} = \vec{OP}$ via $|\vec{r}| = \sqrt{\vec{r} \cdot \vec{r}}$, which in turn gives the **distance** between the points O and P, denoted $|\vec{OP}|$. The inner product also allows us to define (and compute) the angle ϕ between two vectors $\vec{r}_1$ and $\vec{r}_2$ via

$$\vec{r}_1 \cdot \vec{r}_2 = |\vec{r}_1||\vec{r}_2| \cos\phi. \tag{1.2}$$

For actual computations we need to describe vectors in terms of numbers. We do this by introducing a basis $\{\vec{i}, \vec{j}, \vec{k}\}$ of vectors based at O. Then we can expand

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \qquad x, y, z \in \mathbb{R}. \tag{1.3}$$

The origin O together with the basis $\{\vec{i}, \vec{j}, \vec{k}\}$ is called a **reference frame** or simply **frame**. The real numbers x, y, z are called **coordinates** of the point P relative to the frame $(O, \{\vec{i}, \vec{j}, \vec{k}\})$. In the following we assume that $\{\vec{i}, \vec{j}, \vec{k}\}$ is an **orthonormal** basis i.e. that

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1, \qquad \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{i} \cdot \vec{k} = 0. \tag{1.4}$$

We often use the abbreviation "ON basis" for " orthonormal basis". Using the defining properties of an ON basis, the inner product of two vectors $\vec{r}_1 = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$ and $\vec{r}_2 = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$ can be expressed in terms of their coordinates with respect to the ON basis $\{\vec{i}, \vec{j}, \vec{k}\}$:

$$\vec{r}_1 \cdot \vec{r}_2 = (x_1\vec{i} + y_1\vec{j} + z_1\vec{k}) \cdot (x_2\vec{i} + y_2\vec{j} + z_2\vec{k}) = x_1x_2 + y_1y_2 + z_1z_2. \tag{1.5}$$

Similarly, the length of $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}. \tag{1.6}$$

In these notes vectors are denoted by a roman letter with an arrow on top. Often we denote the length of a vector by the same letter withouth the arrow, so r is the length of $\vec{r}$, v the length of $\vec{v}$ and so on.

The final step required to describe the real world in terms of numbers is a choice of **units**. We measure length in units of meters in this course. Thus, having chosen a frame and units, the mathematical information $\vec{r} = 4\vec{i} + 2\vec{j}$ for the position vector of P relative to O means that to get from O to P you need to walk four meters in the direction of $\vec{i}$ and two meters in the direction of $\vec{j}$. More generally we use SI (Système International) units: length is measured in metres (m), time in seconds (s) and mass in kilograms (kg). In general it is good practice to write down the unit of every physical quantity explicitly. However, in simple calculations it is often convient to state the units at beginning and omit them in the calculation.

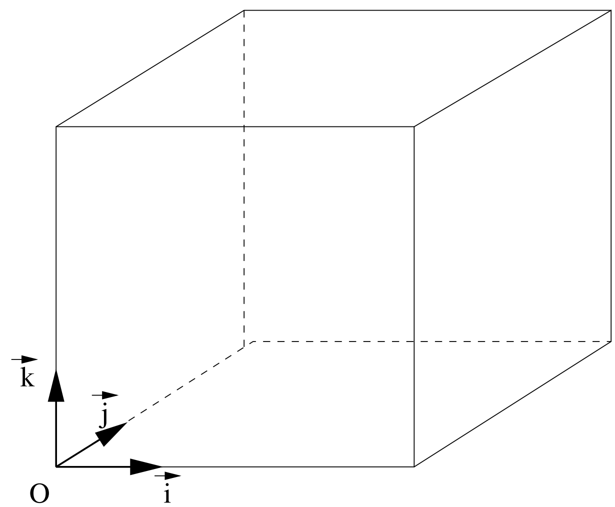

Figure 1: Coordinates for vertices of a cube

Example 1.1 *Consider a solid cube with edgelength* 4 *m. Pick one of its vertices as the origin O for a frame whose basis vectors $\vec{i}, \vec{j}, \vec{k}$ are aligned with the cube's edges emmanating form O and having length* 1 *m, see Fig. 1. Give the coordinates of all the cube's vertices and its centre of mass in this frame.*

The vertex which we singled out as the origin has the coordinates $(x, y, z) = (0, 0, 0)$. The adjacent vertices have coordinates $(4, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 4)$ and the remaining vertices have coordinates $(4, 4, 0)$, $(0, 4, 4)$,$(4, 0, 4)$ and $(4, 4, 4)$. The centre of mass has coordinates $(2, 2, 2)$, see the figure. $\square$.

Example 1.2 *(***Distance between points***) The vector giving the position P relative to O is $\vec{r} = 3\vec{i} - 2\vec{j} + 4\vec{k}$, where $\{\vec{i}, \vec{j}, \vec{k}\}$ is an orthonormal basis. Compute the distance between O and P*

Using the formula $|\vec{OP}| = \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{(3\vec{i} - 2\vec{j} + 4\vec{k})\cdot(3\vec{i} - 2\vec{j} + 4\vec{k})} = \sqrt{9 + 4 + 16} = \sqrt{29}$.
$\square$

Once the origin O and the basis $\{\vec{i}, \vec{j}, \vec{k}\}$ is chosen, the position P is completely characterised by the three **coordinates** x, y, z. Coordinates necessarily refer to a frame. If we had chosen a different origin O' or a different basis $\{\vec{i}', \vec{j}', \vec{k}'\}$ the same point P would be described by different coordinates.

Example 1.3 *(***Changing frame** *) Suppose the vector giving the position of P relative to O is*

$$\vec{r} = 2\vec{i} - 3\vec{j}, \tag{1.7}$$

where $\{\vec{i}, \vec{j}, \vec{k}\}$ is an orthogonal basis. Let

$$\vec{i}' = \vec{j}, \qquad \vec{j}' = -\vec{i}, \qquad \vec{k}' = \vec{k}. \tag{1.8}$$

Check that $\{\vec{i}', \vec{j}', \vec{k}'\}$ is also an orthogonal basis, and find the coordinates x', y', z' in the expansion

$$\vec{r} = x'\vec{i}' + y'\vec{j}' + z'\vec{k}'. \tag{1.9}$$

The orthogonality of $\{\vec{i}', \vec{j}', \vec{k}'\}$ follows directly from the orthogonality conditions (1.4) for $\{\vec{i}, \vec{j}, \vec{k}\}$. To find the coordinates x', y', z' we insert (1.8) into (1.9) to find

$$\vec{r} = x'\vec{j} - y'\vec{i} + z'\vec{k}. \tag{1.10}$$

Comparing coefficients with (1.7) we deduce $x' = -3$, $y' = -2$ and $z' = 0$. □

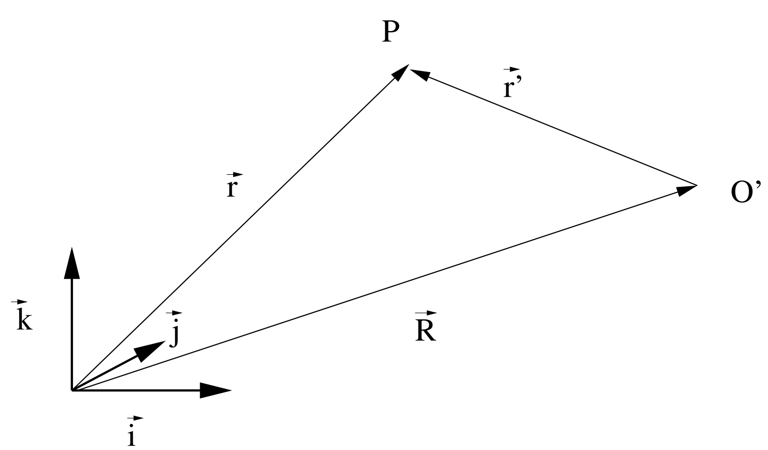

Figure 2: Position vectors relative to O and O'

Example 1.4 *(***Change of origin***) Suppose the vector giving the position of P relative to O is $\vec{r} = 2\vec{i} + \vec{j} + \vec{k}$ and the vector giving the position of O' is $\vec{R} = 4\vec{k}$. What is the vector giving the position of P relative to O'? What is the cosine of the angle ϕ between $\vec{r}$ and $\vec{R}$?*

The general formula for the relative position is

$$\vec{O'P} = \vec{OP} - \vec{OO'} = \vec{r} - \vec{R}. \tag{1.11}$$

In the example we therefore have $\vec{O'P} = 2\vec{i} + \vec{j} - 3\vec{k}$. To work out the angle between $\vec{r}$ and $\vec{R}$ we use

$$\vec{r}\cdot\vec{R} = |\vec{r}||\vec{R}|\cos\phi \tag{1.12}$$

to find

$$\cos\phi = \frac{4}{4\sqrt{6}} = \frac{1}{\sqrt{6}}.$$

□

1.2 Velocity, speed and acceleration

Suppose now that the position P of a particle varies with time. At time t seconds the position vector relative to O is now

$$\vec{r}(t) = \vec{OP}(t). \tag{1.13}$$

Hence the position of P relative to O is characterised by a vector-valued function of time. We can visualise it by plotting the **trajetory** of the particle, which is the set of all points in space occupied by the particle at some time - think of it like the vapour trail of a plane, which marks all the points in space through which the plane has passed (assuming there is no wind). In practice we expand the postion vector $\vec{r}(t)$ in terms of a fixed ON basis, which we assume to be time-independent. Then the position relative to O is characterised by three real functions, the coordinates x, y and z as as function of time:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}. \tag{1.14}$$

In order to plot a trajectory we draw the basis vectors $\vec{i}, \vec{j}$ and $\vec{k}$ in a diagram, evaluate the coordinates $x(t), y(t)$ and $z(t)$ for various values of the time t, mark the corresponding point in the diagram and then join the dots - see the example sheets. In Fig. 3 we show a typical trajectory, and also the difference vector

$$\Delta\vec{r}(t) = \vec{r}(t + \Delta t) - \vec{r}(t) \tag{1.15}$$

between two nearby points on the trajectory which are passed at times t and $t+\Delta t$. Note that only the coordinates x, y, and z change with time - we assume the basis $\{\vec{i}, \vec{j}, \vec{k}\}$ to be constant. Dividing by Δt and taking the limit $\Delta t \to 0$ we obtain the derivative of the vector-valued function $\vec{r}(t)$:

$$\frac{d\vec{r}}{dt}(t) = \lim_{\Delta t\to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \tag{1.16}$$

Geometrically, the derivative $\frac{d\vec{r}}{dt}(t)$ is a vector tangent to the trajectory $\vec{r}(t)$ at t, as shown in Fig. 3. Note that, since the basis $\{\vec{i}, \vec{j}, \vec{k}\}$ is independent of t we have

$$\begin{aligned}
\lim_{\Delta t\to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} &= \lim_{\Delta t\to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}\vec{i} + \lim_{\Delta t\to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t}\vec{j} \\
&+ \lim_{\Delta t\to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t}\vec{k} \\
&= \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k},
\end{aligned} \tag{1.17}$$

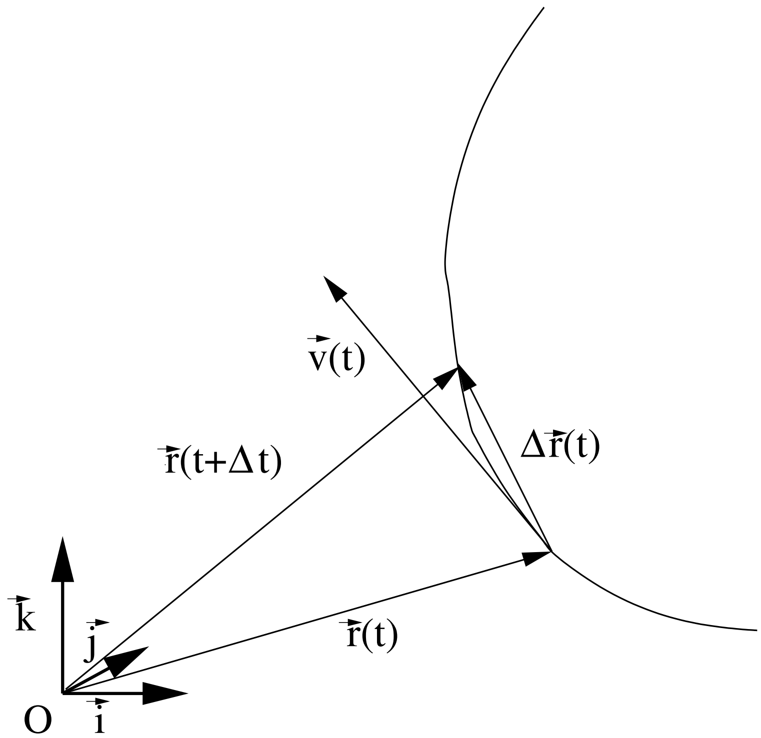

Figure 3: Trajectory and velocity of a particle

so that the differentiation of one vector-valued function simply amounts to the differentiation of three ordinary functions.

Definition 1.5 *We define the* **velocity** *to be the derivative of $\vec{r}(t)$ with respect to t:*

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \tag{1.18}$$

The **speed** *is defined as the magnitude of the velocity:*

$$v = |\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} \tag{1.19}$$

We sometimes denote derivatives with respect to time with a dot over the function being differentiated. Thus

$$\vec{v} = \dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} \tag{1.20}$$

and

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{1.21}$$

The unit of velocity and speed is m s^{-1}.

Example 1.6 *If the position vector of a particle with respect to a fixed origin O at time t seconds is $\vec{r}(t) = 6t\vec{i} + 2\vec{j} - 4t^2\vec{k}$, where $\{\vec{i}, \vec{j}, \vec{k}\}$ is a fixed orthonormal basis at O, find the velocity and the speed of the particle at time t seconds.*

$\vec{v} = 6\vec{i} - 8t\vec{k}; v = \sqrt{36 + 64t^2}$. $\square$

Example 1.7 *Suppose a particle's position vector, measured in metres, at time $t = 0$ seconds is $\vec{r}_0 = 3\vec{i} + 2\vec{j}$. If the particle's velocity, in metres per second, is $\vec{v} = 2\vec{i}$, find the particle's position at time t s.*

Expanding $\dot{\vec{r}} = \vec{v}$ in the basis $\{\vec{i}, \vec{j}, \vec{k}\}$ we have

$$\dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} = 2\vec{i} \quad \Leftrightarrow \quad \dot{x} = 2, \dot{y} = 0, \dot{z} = 0, \tag{1.22}$$

where we compared coefficients of the basis vectors $\vec{i}, \vec{j}, \vec{k}$. Thus we obtain three (very simple) differential equations for the coordinate functions x, y, z, whose general solutions are

$$x(t) = x_0 + 2t, \quad y(t) = y_0, \quad z(t) = z_0 \tag{1.23}$$

with constants x_0, y_0, z_0. Comparing with the initial position $\vec{r}_0 = 3\vec{i} + 2\vec{j}$ we conclude $x_0 = 3$, $y_0 = 2$, $z_0 = 0$, so that

$$\vec{r}(t) = (3 + 2t)\vec{i} + 2\vec{j}. \tag{1.24}$$

$\square$

Definition 1.8 *We define the* **acceleration** *to be the derivative of the velocity $\vec{v}(t)$ with respect to the time t:*

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}. \tag{1.25}$$

Sometimes we use a double dot to denote the second derivative, i.e. $\vec{a} = \ddot{\vec{r}}$. The unit of acceleration is m s^{-2}.

Example 1.9 *Find the acceleration for the motion of the previous example.*

$\vec{a} = -8\vec{k}$. $\square$

Example 1.10 *(***Uniform circular motion** *) Suppose the position vector of a particle with respect to a fixed origin O at time t seconds is*

$$\vec{r}(t) = r\cos(\omega t)\vec{i} + r\sin(\omega t)\vec{j}, \tag{1.26}$$

where r and ω are constant. Describe the motion geometrically and find the velocity, speed and acceleration of the particle.

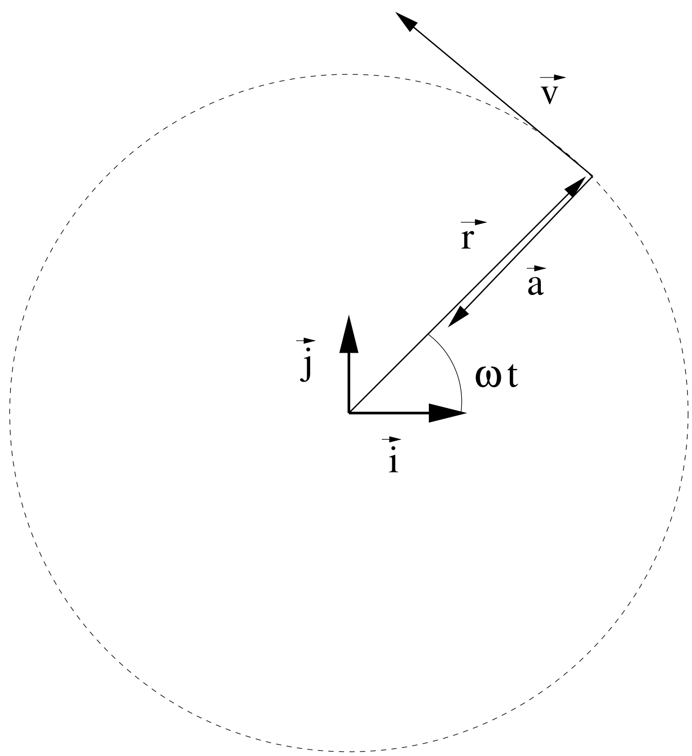

Figure 4: Position vectors, velocity and acceleration for uniform circular motion

Since $|\vec{r}| = \sqrt{r^2\cos^2(\omega t) + r^2\sin^2(\omega t)} = r$=constant the particle moves on a circle of radius r in the $\vec{i}\vec{j}$ plane. Differentiating we find

$$\vec{v} = -r\omega\sin(\omega t)\vec{i} + r\omega\cos(\omega t)\vec{j} \tag{1.27}$$

and hence $v = |\vec{v}| = \omega r$. The speed of the particle is constant, even though the direction of the velocity changes all the time. The particle thus moves on a circle with uniform speed. We can think of the position vector $\vec{r}$ as the pointer on a watch: it rotates at a constant rate. To make the notion of "rate of rotation" more precise, we note that the angle, in radians, between the vector $\vec{r}(t)$ and the vector $\vec{r}(t+\Delta)$ is $\omega\Delta T$. Hence the vector $\vec{r}(t)$ sweeps out an angle $\omega\Delta t$ in the time interval Δt. Thus the angle, in radians, swept out per unit time is ω. The quantity ω characterises the rate of rotation; it is called the **angular speed** or **angular frequency** of the particle and its unit is s^{-1}. The time required for $\vec{r}$ to complete one revolution is called the **period** of the motion, and usually denoted T. The angle swept out in a complete revolution is 2π; therefore T satsifies the relations

$$T\omega = 2\pi \Leftrightarrow T = \frac{2\pi}{\omega}. \tag{1.28}$$

Note that $\vec{r}\cdot\vec{v} = 0$ so that the velocity $\vec{v}$ is at right angles to the particle's position vector $\vec{r}$ at all times. Differentiating again we compute

$$\vec{a}(t) = -\omega^2 r\cos(\omega t)\vec{i} - \omega^2 r\sin(\omega t)\vec{j} = -\omega^2\vec{r}(t),$$

with (constant) magnitude

$$a = \omega^2 r = \frac{v^2}{r}.$$

The acceleration is non-zero for circular motion, and points in the opposite direction to $\vec{r}$. □

1.3 Relative velocity and acceleration

Suppose the position of an observer O' relative to the fixed origin O varies with time and, at time t seconds, is given by $\vec{R}(t)$ and its velocity by

$$\vec{V} = \frac{d\vec{R}}{dt} \tag{1.29}$$

and its acceleration by

$$\vec{A} = \frac{d\vec{V}}{dt} \tag{1.30}$$

If the position vector of particle P relative to O is $\vec{r}(t)$, the position of P relative to O' is, according to (1.11),

$$\vec{r}'(t) = \vec{r}(t) - \vec{R}(t). \tag{1.31}$$

We define the **velocity relative to** O' via

$$\vec{v}' = \frac{d\vec{r}'}{dt} \tag{1.32}$$

and immediately deduce the relation

$$\vec{v}' = \vec{v} - \vec{V} \tag{1.33}$$

between the velocities $\vec{v} = \dot{\vec{r}}$ and $\vec{v}' = \dot{\vec{r}}'$. Similarly, the **acceleration relative to** O' is defined via

$$\vec{a}' = \frac{d\vec{v}'}{dt} \tag{1.34}$$

and satisfies

$$\vec{a}' = \vec{a} - \vec{A}. \tag{1.35}$$

Example 1.11 *A train is passing the platform of a station at a constant speed of* 60 *km/h. A passenger walks towards the rear of the train, at a speed of 1 m/s relative to the train. What is the speed of the passenger relative to the station?*

Choose a basis $\{\vec{i}, \vec{j}, \vec{k}\}$ attached to the platform, and let $\vec{i}$ be parallel to the tracks, in the direction of the train's motion. Then the train's velocity is $\vec{V} = 60\vec{i}$ km/h and the passenger's velocity relative to the train is $\vec{v}' = -1\vec{i}$ m/s. Hence $\vec{v} = \vec{V} + \vec{v}'$ is the passenger's velocity relative to the platform. It has magnitude v=(60,000/3600 -1) m/s =(100/6-1) m/s =15.7 m/s. □

2 Dynamics

Dynamics is the study of how the motion of a body is related to its environment (through applied forces) and its properties (such as its mass). Isaac Newton (1643-1727) was the first to construct a comprehensive mathematical theory of dynamics. In order to formulate it he needed to invent calculus. Newton's theory is traditionally summarised in the from of three basic laws.

2.1 Newton's first two laws of motion

Physical Law 2.1 (**Newton I**) *A body remains in a state of rest or uniform motion (constant velocity) unless acted upon by an external force:*

$$\vec{F} = 0 \Rightarrow \vec{a} = 0 \tag{2.1}$$

Definition 2.2 *The* **momentum** *of a particle of mass m moving with velocity $\vec{v}$ is defined via*

$$\vec{p} = m\vec{v}. \tag{2.2}$$

Physical Law 2.3 (**Newton II**) *The rate of change of momentum of a particle is equal to the force acting on it:*

$$\frac{d\vec{p}}{dt} = \vec{F}. \tag{2.3}$$

In particular if the mass m is constant,

$$m\vec{a} = \vec{F}. \tag{2.4}$$

The unit of force is the Newton. From Newton II it follows that 1 Newton = 1 kg m s^{-2}.

Note that **Newton I** follows from **Newton II**. It is stated mainly for pedagogical reasons, in order to emphasise the difference with the pre-Newtonian assumption that in the absence of forces any object will eventually come to rest. Newton's second law is also sometimes referred to as the **equation of motion** for classical mechanics. More precisely, using $\vec{a} = \ddot{\vec{r}}$, the equation (2.4) is a second order differential equation for the position vector $\vec{r}$ of a particle of mass m. Note that Newton's second law does not specify the force $\vec{F}$. It only states that accelerated motion is always related to a force. Finding that force is a separate, and often difficult, task. Newton himself famously found the force that governs the gravitational interaction of bodies.

2.2 Free fall in a gravitational field

Physical Law 2.4 (**Newton's law of universal gravitation**) *A particle of mass M attracts another particle of mass m with a force*

$$\vec{F} = -\frac{GMm}{r^2}\hat{\vec{r}}, \tag{2.5}$$

where $\vec{r}$ is the vector pointing from M to m, $r = |\vec{r}|$ and $\hat{\vec{r}} = \vec{r}/r$. G is a constant, with value $G = 6.67 \times 10^{-11}$ Newton m^2 kg $^{-2}$.

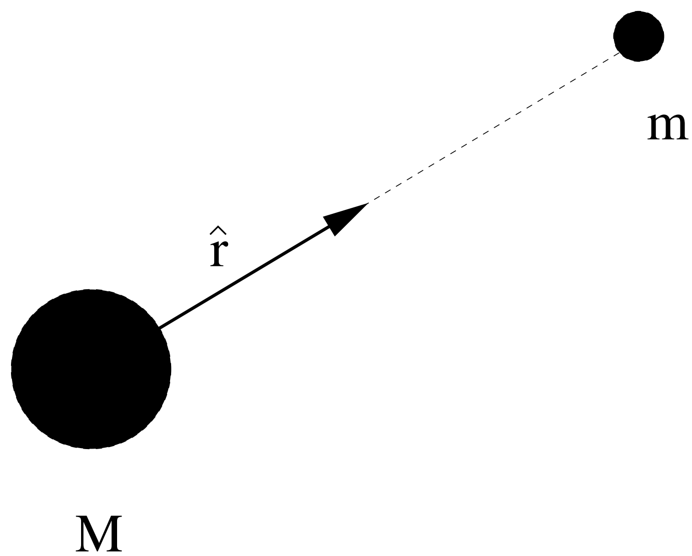

Figure 5: Gravitational force between two particles

We say that a **gravitational field** exists at a point in space if a particle placed at that point experiences a gravitational force. If the particle's mass is m and the force $\vec{F}$ then gravitational field is given mathematically by the ratio $\vec{F}/m$. The magnitude of the field is called gravitational field strength.

Newton's second law combined with Newton's law of universal gravitation gives the equation of motion

$$m\ddot{\vec{r}} = -\frac{GM_E m}{r^2}\hat{\vec{r}} \tag{2.6}$$

for the position vector $\vec{r}$ of a body of mass m relative to the centre of the earth (M_E is the mass of the earth, G is the gravitational constant as before). This equation is both difficult and very important. Understanding it and describing its solutions will take up most of this course! We begin by deriving a simplified version of the equation and studying its solutions.

Example 2.5 *Compute the force exerted by the earth on a particle of mass m near the earth's surface. In computing the force treat the earth like a point particle with all its mass $M = 5.97 \times 10^{24}$ kg concentrated at its centre and assume that the earth's circumference is $C = 40,075km$. What is the gravitational field near the earth's surface?*

Denote the radius of the earth by R_{earth}. Since $C = 2\pi R_{\text{earth}}$ we deduce that $R_{\text{earth}} = 6378$ km. Inserting this value for r in Newton's law of universal gravitation (2.5) and using the value of G given there we compute $\vec{F} = -m \times 9.8$ m s$^{-2}\hat{\vec{r}}$, where $\hat{\vec{r}}$ is a unit vector pointing upwards. The gravitational field is therefore -9.8 m s$^{-2}\hat{\vec{r}}$. □

Often we use the notation $g = 9.8$ m s^{-2} for the gravitational field strength near the earth's surface. Note that the vector $\hat{\vec{r}}$ has constant magnitude and that its direction varies very slowly as we change position on the surface of the earth. This is because the radius of the earth is so large that the earth's surface is, to a good approximation, flat. Thus the direction "upwards" in Edinburgh does not differ very much from "upwards" in Glasgow. In many situations we therefore assume that $\hat{\vec{r}}$ is a constant unit vector, which we call $\vec{k}$. Then we write the gravitational force as $\vec{F} = -mg\vec{k}$. By Newton II we have the equation of motion

$$m\ddot{\vec{r}} = -mg\vec{k}, \tag{2.7}$$

for the position vector $\vec{r}$ or a particle moving in the gravitational field near the earth's surface. Expanding the position vector in a basis $\{\vec{i}, \vec{j}, \vec{k}\}$ attached to the earth's surface with $\vec{i}, \vec{j}$ horizontal and $\vec{k}$ vertical, we have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and hence

$$\ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k} = -10\vec{k}. \tag{2.8}$$

Comparing coefficients we get three second order differential equations

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = -g. \tag{2.9}$$

The equation of motion (2.7) or, equivalently, its component form (2.9) is the promised simplification of the equation (2.6).

Example 2.6 *A ball of mass m is thrown vertically upwards from ground level with initial speed v. Neglecting air resistance, find its position at time t seconds. If $v = 5$ m s^{-1} find the time when the ball hits the ground. You may assume $g \approx 10$ m s^{-2}.*

The general solution of $\ddot{x} = 0$ is $x = x_0 + u_0 t$ and similarly for y. With the initial condition $x(0) = y(0) = 0$ and $\dot{x}(0) = \dot{y}(0) = 0$ we deduce $x = y = 0$ for all t. The general solution of $\ddot{z} = -10$ is $z = z_0 + v_0 t - 5t^2$. With the initial condition $z(0) = 0$ and $\dot{z}(0) = 5$ we conclude

$$z(t) = 5t - 5t^2.$$

The ground level is characterised by $z = 0$ and this occurs when $t = 0$ and $t = 1$ s. Hence the balls returns to ground level after one second. □

In certain situations one can save a lot of time by using conservation laws instead of solving the equations of motion. We will study this systematically in later lectures, but consider two examples here.

Proposition 2.7 *(***Energy of a falling body near the earth's surface***) For body of mass m falling under gravity at a height z above the earth's surface the quantity*

$$E = \frac{1}{2}m\dot{z}^2 + mgz \tag{2.10}$$

is constant during the fall.

Proof: Differentiating, and using the chain rule, we find

$$\begin{aligned} \frac{dE}{dt} &= m\frac{d}{d\dot{z}}\left(\frac{1}{2}\dot{z}^2\right)\frac{d\dot{z}}{dt} + mg\frac{dz}{dt} \\ &= m\dot{z}\ddot{z} + mg\dot{z} \\ &= 0 \end{aligned} \tag{2.11}$$

by the equation of motion $m\ddot{z} = -mg$. □

Example 2.8 *A ball is dropped from a height h above sea level. What is its speed v when it reaches sea level?*

At the beginning of the motion $z = h$ and $\dot{z} = 0$, so $E = mgh$. At the end of the motion $z = 0$, and $\dot{z} = v$, so $E = \frac{1}{2}mv^2$. From the conservation of energy we conclude $v = \sqrt{2hg}$. □

We can illustrate the power of conservation laws further by returning to the (difficult) equation (2.6). When we restrict ourselves to vertical motion $\vec{r} = r\vec{k}$ we obtain the equation

$$m\ddot{r} = -\frac{GM_E m}{r^2} \tag{2.12}$$

for the distance r of the body from the centre of the earth. This is is still a tricky equation to integrate, but deduce some information about the motion from the following conservation law.

Proposition 2.9 *(***Energy of a falling body***) For body of mass m falling under gravity at a distance r from the earth's surface, the total energy*

$$E = \frac{1}{2}m\dot{r}^2 - \frac{GM_e m}{r} \tag{2.13}$$

is constant during the fall.

Proof: As in the proof of the constancy of (2.10) we differentiate and use the chain rule. This time we find

$$\begin{aligned} \frac{dE}{dt} &= m\frac{d}{d\dot{r}}\left(\frac{1}{2}\dot{r}^2\right)\frac{d\dot{r}}{dt} - \frac{d}{dr}\left(\frac{GM_e m}{r}\right)\frac{dr}{dt} \\ &= m\dot{r}\ddot{r} + \frac{GM_e m}{r^2}\dot{r} \\ &= 0 \end{aligned} \tag{2.14}$$

by the equation of motion (2.12). □

Example 2.10 *(***Escape velocity***) What is the minimum speed v with which a rocket must be fired from the earth's surface in order to escape the earth's gravitational field?*

The energy when the rocket is launched is $E = \frac{1}{2}mv^2 - \frac{GM_E m}{R_E}$, where R_E is the radius of the earth. "Escaping the gravitational field" means reaching $r = \infty$, and to compute the minimal launch speed we can assume that the rocket's speed is zero when it reaches $r = \infty$. Hence the total energy is $E = 0$ when the rocket has escaped, and we deduce from energy conservation

$$0 = \frac{1}{2}mv^2 - \frac{GM_E m}{R_E} \Rightarrow v = \sqrt{\frac{2GM_E}{R_E}}. \tag{2.15}$$

Inserting values one finds $v \approx 11181\text{m / s} \approx 40250$ km/h. This is extremely fast - recall that commercial airplanes with jet engines travel at about 800 km/h! No projectile could be fired at this speed in the earth's atmosphere, so one has to proceed differently in order to send rockets into outer space. □

2.3 Projectiles: two-dimensional motion with constant acceleration

In this subsection we study the motion of point-like, massive objects which are fired or thrown near the earth's surface and then allowed to move under the influence of gravity alone. Such objects are called projectiles and their study has numerous application. (most of them are nasty and have to do with damaging or destroying structures or living creatures from a safe distance). We will neglect air resistance and treat the subject through a sequence of examples. The equation governing the motion of projectiles is (2.7). However, unlike in previous examples where the motion was only up or down, we now consider two-dimensional motion.

Example 2.11 *(***Projectile trajectory***) At time $t = 0$ seconds, a particle of mass m is projected from a (small) height h above the earth's surface with speed u at an angle θ to the horizontal. Find the position of the particle at time $t > 0$ seconds and describe the geometry of its trajectory.*

The indication that the height is small suggests that we may assume that we can treat the gravitational force as constant, and given by $-mg\vec{k}$, where $\vec{k}$ is the unit vector pointing upwards. We can choose our basis $\{\vec{i}, \vec{j}, \vec{k}\}$ in such a way that the initial velocty is

$$\vec{u} = u\cos\theta\vec{i} + u\sin\theta\vec{k} \tag{2.16}$$

and the initial position vector is $\vec{r}_0 = h\vec{k}$. Thus we need to solve

$$m\ddot{\vec{r}} = -mg\vec{k} \tag{2.17}$$

with $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ and $\vec{r}(0) = h\vec{k}$ and $\dot{\vec{r}}(0) = \vec{u}$. In components

$$\begin{aligned} \ddot{x} &= 0, \quad \dot{x}(0) = u\cos\theta, \quad x(0) = 0 && (2.18)\\ \ddot{y} &= 0, \quad \dot{y}(0) = 0, \quad y(0) = 0 && (2.19)\\ \ddot{z} &= -g, \quad \dot{z}(0) = u\sin\theta, \quad z(0) = h && (2.20) \end{aligned}$$

The equation for y is trivially solved by $y(t) = 0$ for all t. The equation and initial coniditions for x have the unique solution

$$x(t) = u\cos\theta \ t \tag{2.21}$$

and the equation and initial conditions for z have the unique solution

$$z(t) = h + u\sin\theta \ t - \frac{1}{2}gt^2. \tag{2.22}$$

The motion thus takes place entirely in the $\vec{i}\vec{k}$ plane. To understand the geometry of the trajectory, we eliminate t and express z as function of x. Solving (2.21) for t and substituting into (2.22) we find

$$z(x) = h + \tan\theta \ x - \frac{g}{2u^2\cos^2\theta}x^2. \tag{2.23}$$

This is the equation of a downward sloping parabola, shown in Fig. 6 for h=0.

□

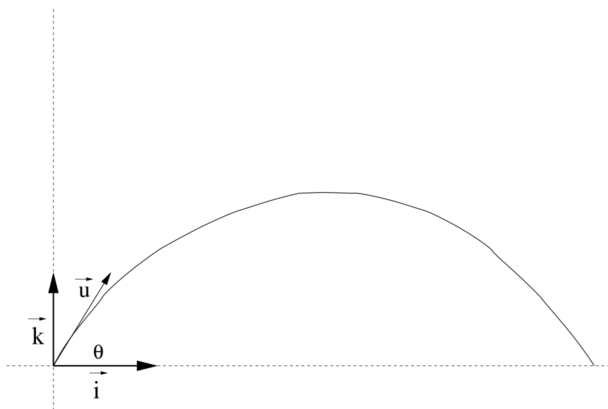

Figure 6: Parabolic trajetory of a projectile

Example 2.12 *(***Greatest range for given speed***) A particle is projected from sea level with speed u at an angle $\theta \in [0, \pi/2]$ to the horizontal. Find the time when the particle returns to sea level, and give the range (i.e. horizontal distance travelled) as a function of u and θ. Keeping u fixed, for which angle θ is the range maximal?*

We can use the same basis $\{\vec{i}, \vec{j}, \vec{k}\}$ as in the previous example, and obtain the same solution, with $h = 0$:

$$x(t) = u\cos\theta \ t, \qquad z(t) = u\sin\theta \ t - \frac{1}{2}gt^2. \tag{2.24}$$

with y again being zero for all t. The time T when the particle is at sea level is characterised by the equation

$$z(T) = 0 \Leftrightarrow T = 0 \quad \text{or} \quad T = \frac{2u\sin\theta}{g}, \tag{2.25}$$

so that that the particle returns to sea level at time $T = 2u\sin\theta/g$. The horizontal distance travelled at that point is

$$x(T) = \frac{2u^2\cos\theta\sin\theta}{g} = \frac{u^2\sin 2\theta}{g} \tag{2.26}$$

The range R as a function of the θ is thus

$$R(\theta) = \frac{u^2\sin 2\theta}{g}. \tag{2.27}$$

The maximal value of the function sin is 1 which is reached when its argument is $\pi/2$. Thus R is maximal when $\theta = \pi/4$. For given initial speed, the projectile flies furthest when fired at 45 degrees to the horizontal. □

Example 2.13 *(***A worked example with numbers***) A projectile is fired from a cliff of height 25 m above sea level at an angle of* 30 *degrees to the horizontal with speed* 40 *ms^{-1}. Find the time taken until the projectile hits the sea and the horizontal distance travelled by the projectile until that moment. Also find the time take for the projectile to reach its greatest height and the numerical value of the greatest height. You may assume $g = 10$ m s^{-2}.*

Using again the basis $\{\vec{i}, \vec{j}, \vec{k}\}$ introduced in example 2.11, and inserting the numerical values of the parameters we have the following initial velocity, in m/s:

$$\vec{u} = 20\sqrt{3}\vec{i} + 20\vec{k},$$

where we used $\cos(\pi/6) = \sqrt{3}/2$ and $\sin(\pi/6) = 1/2$. The equations of motion and initial conditions (2.18) therefore take the following form

$$\begin{aligned} \ddot{x} &= 0, \quad \dot{x}(0) = 20\sqrt{3}, \quad x(0) = 0 & (2.28)\\ \ddot{z} &= -10, \quad \dot{z}(0) = 20, \quad z(0) = 25, & (2.29) \end{aligned}$$

where we omitted the (trivial) equations for y. The unique solution is

$$x(t) = 20\sqrt{3}t, \quad z(t) = 25 + 20t - 5t^2. \tag{2.30}$$

The condition for the projectile to be at sea level is $z(t) = 0$ or $t^2-4t-5 = 0$ which is solved by $t = -1$ and $t = 5$. Since the projectile was fired at $t = 0$ we can discard the negative value and conclude that the projectile hits the sea after 5 seconds. The horizontal distance travelled in those five seconds is $x(5)$ m$= 100\sqrt{3}$m ≈ 173m. A necesssary condition for the height to be maximal is $\dot{z}(t) = 0$ or $20 - 10t = 0$ which is solved by $t = 2$. Since $\ddot{z}(2) < 0$ this is indeed a maximum. Thus the projectile reaches the greatest height after 2 seconds. The height reached is $z(2)$ m $= 45$ m. □

2.4 Resisted motion

It is found experimentally that the magnitude of air resistance experienced by a body depends mainly on three factors.

1. The area of the body perpendicular ot the direction of motion.
2. The speed of the body.
3. The density of air.

The direction of the resistance $\vec{F}_r$ is parallel but opposite to the velocity of the body, so

$$\vec{F}_r = -f(|\vec{v}|)\vec{v}, \tag{2.31}$$

where f is a positive function of the speed. The equation of motion for a body of mass m moving near the earth's surface, taking into account air resistance is therefore

$$m\ddot{\vec{r}} = -mg\vec{k} + \vec{F}_r. \tag{2.32}$$

Example 2.14 (**parachutist**) *A parachutist experiences a resistive force $\vec{F} = -\alpha|\vec{v}|\vec{v}$, where α is a positive constant. Find the equation for the speed of the parachutist when falling near the earth's surface, and check that it has a constant solution $v(t) \equiv u$.*

We assume the motion to be entirely vertical, so $\vec{r} = z\vec{k}$ and $\dot{\vec{r}} = \dot{z}\vec{k}$. For downward motion with speed $v(t)$ we have $\dot{z} = -v$, so finally $\ddot{\vec{r}} = -\dot{v}\vec{k}$. The equation of motion (2.32) is therefore equivalent to

$$m\frac{dv}{dt} = mg - \alpha v^2. \tag{2.33}$$

For a constant solution we have $\frac{dv}{dt} = 0$. Hence $v(t) \equiv u$ is a solution if

$$u = \sqrt{mg/\alpha}. \tag{2.34}$$

The speed (2.34) is called the *terminal speed*. When an object is released from rest its speed will increase and tend to (though never quite reach) the terminal speed. Note that the terminal speed increases with the square root of the mass. Our model predicts that, when air resistance is taken into accout, heavier objects fall faster than lighter objects, in agreement with experience.

3 Conservation laws

We saw in our study of particles moving in a gravitational field that the conservation laws for the total energy provide shortcuts when the analysis of the equation of motion would be longer or more difficult. The mathematical reason for this is that conservation laws only involve the position and the velocity, whereas the equation of motion involves the position, the velocity and the acceleration. In this section we explore conservation laws for momentum, energy and angular momentum.

3.1 Newton's third law

So far we have concentrated on the motion of a single particle. However, all forces in nature arise as a result of an interaction between bodies. The earth exerts a force on the falling particle, but the falling particle also exerts a force on the earth. Since the earth is so much heavier than the particle it is justified to neglect the motion of the earth when studying the interaction between particle and earth. We will now consider the interaction between several particles of comparable masses and therefore will need to keep track of the motion of all particles involved. We mostly consider two bodies, although much of what we will say can be generalised to n bodies. Suppose the two bodies have masses m_1 and m_2 and position vectors $\vec{r}_1(t)$ and $\vec{r}_2(t)$ relative to a fixed origin O. We only consider forces which the bodies exert on each other, so body 1 exerts a force $\vec{F}_{21}$ on body 2, and body 2 exerts a force $\vec{F}_{12}$ on body 1. Each of the bodies move according to Newton's second law:

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = F_{12}(\vec{r}_1, \vec{r}_2), \qquad m_2 \frac{d^2\vec{r}_2}{dt^2} = F_{21}(\vec{r}_1, \vec{r}_2). \tag{3.1}$$

Newton's third law imposes a restriction on the possible forms of F_{12} and F_{21}.

Physical Law 3.1 (**Newton III**) *If body 1 exerts a force $\vec{F}_{12}$ on body 2, then body 2 exerts a force $\vec{F}_{21} = -\vec{F}_{12}$ on body 1.*

3.2 Momentum conservation

Definition 3.2 *We define the* **total momentum** *of the two particles as*

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2, \tag{3.2}$$

where $\vec{v}_1 = \dfrac{d\vec{r}_1}{dt}$ *and* $\vec{v}_2 = \dfrac{d\vec{r}_2}{dt}$

Theorem 3.3 *The total momentum is conserved.*

This is a direct consequence of Newton's third law:

$$\frac{d\vec{P}}{dt} = m_1\frac{d^2\vec{r}_1}{dt^2} + m_2\frac{d^2\vec{r}_2}{dt^2} = \vec{F}_{12} + \vec{F}_{21} = 0.$$

$\square$

One important consequence of momentum conservation for bodies of constant mass is a particularly simply motion of their centre-of-mass.

Definition 3.4 *The* **centre-of-mass** *of the two bodies with position vectors* $\vec{r}_1$ *and* $\vec{r}_2$ *relative to some origin* O *has position vector*

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{3.3}$$

relative to the same origin 0.

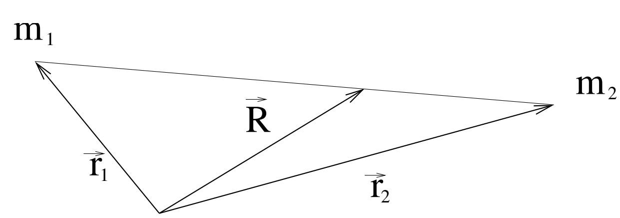

Figure 7: The centre-of-mass of two particles

Physically one can determine the centre-of-mass of two bodies by connecting them with a thin rod: the centre-of-mass is where you have to hold the rod in order for it to balance.

Example 3.5 *A particle of mass 1 kg has position vector* $\vec{r}_1 = 3\vec{i}$ *and a particle of mass 2 kg has position vector* $\vec{r}_2 = 6\vec{i}$. *Find the position vector of the centre-of-mass.*

Inserting the numbers into the definition (3.3) we find $\vec{R} = \frac{1}{3}(3\vec{i} + 2 \times 6\vec{i}) = 5\vec{i}$ $\square$.

Returning to the general situation, and allowing the position vectors $\vec{r}_1$ and $\vec{r}_2$ to vary with time, we define the centre-of-mass velocity as $\vec{V} = \dot{\vec{R}}$. Note that the total momentum is related to the centre-of-mass velocity via $\vec{P} = (m_1 + m_2)\vec{V}$. Since the masses m_1 and m_2 are constant it follows from the conservation of the the total momentum that

$$\frac{d^2\vec{R}}{dt^2} = 0. \tag{3.4}$$

The most general solution of this equation is

$$\vec{R} = \vec{V}t + \vec{R}_0, \tag{3.5}$$

for constant vectors $\vec{V}$ and $\vec{R}_0$. Hence we have

Theorem 3.6 *The centre-of-mass moves uniformly i.e. with constant velocity.*

3.3 Collision problems

As an illustration of the power of conservation laws we study situations were two particles collide. Examples are collisions of smooth balls in a game of pool, but also, more dramatically, the collision of two cars in an accident. In studying such collisions we need to keep track of both momentum and energy. More precisely, we have the following definition for the energy stored in the motion of a particle.

Definition 3.7 *The* **kinetic energy** *of a particle of mass m moving with velocity $\vec{v}$ is*

$$K = \frac{1}{2}m\,\vec{v}\cdot\vec{v} = \frac{1}{2}m\,|\vec{v}|^2. \tag{3.6}$$

The total kinetic energy of two particles with masses m_1 and m_2 and velocities $\vec{v}_1$ and $\vec{v}_2$ is

$$K = \frac{1}{2}m_1|\vec{v}_1|^2 + \frac{1}{2}m_2|\vec{v}_2|^2. \tag{3.7}$$

When two particles are fired towards each other it is often a good approximation to say that they only interact when they are very close to each other. Think for example of two billiard balls colliding on a horizontal billiard table. In that situation the kinetic energy is, to a good approximation, the same before and after the collision. In practice, friction may convert some of the energy into heat. An idealised collision, where both the kinetic energy and the total momentum is conserved, is called an **elastic collision**. Denoting the velocities of the particles before the collision by $\vec{v}_1$ and $\vec{v}_2$ and after the collision by $\vec{u}_1$ and $\vec{u}_2$ we therefore have

$$\begin{aligned} m_1\vec{v}_1 + m_2\vec{v}_2 &= m_1\vec{u}_1 + m_2\vec{u}_2 && (3.8)\\ \frac{1}{2}m_1|\vec{v}_1|^2 + \frac{1}{2}m_2|\vec{v}_2|^2 &= \frac{1}{2}m_1|\vec{u}_1|^2 + \frac{1}{2}m_2|\vec{u}_2|^2. && (3.9) \end{aligned}$$

The first equation is an equation for a three component vector, so we have a total of four relations between $\vec{v}_1, \vec{v}_2, \vec{u}_1$ and $\vec{u}_2$. If, for example, $\vec{v}_1$ and $\vec{v}_2$ are given this is not enough to determine all six components of $\vec{u}_1$ and $\vec{u}_2$, but we nonetheless deduce much information about the outcome of the collision.

Example 3.8 *(***Elastic collision of equal spheres***) A smooth sphere of mass m moving with velocity $\vec{v}$ hits a second smooth sphere of the same mass m which is at rest. Assuming the collision to be perfectly elastic show that the velocities of the spheres after the collision are orthogonal.*

Momentum conservation gives $\vec{v} = \vec{u}_1 + \vec{u}_2$ and kinetic energy conservation $\frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{1}{2}m(\vec{u}_1\cdot\vec{u}_1 + \vec{u}_2\cdot\vec{u}_2)$ Inserting $\vec{v} = \vec{u}_1 + \vec{u}_2$ into the kinetic energy conservation we deduce

$$(\vec{u}_1 + \vec{u}_2)\cdot(\vec{u}_1 + \vec{u}_2) = \vec{u}_1\cdot\vec{u}_1 + \vec{u}_2\cdot\vec{u}_2 \quad \Leftrightarrow \quad \vec{u}_1\cdot\vec{u}_1 + \vec{u}_2\cdot\vec{u}_2 + 2\vec{u}_1\cdot\vec{u}_2 = \vec{u}_1\cdot\vec{u}_1 + \vec{u}_2\cdot\vec{u}_2$$

hence $\vec{u}_1 \cdot \vec{u}_2 = 0$. $\square$

Collisions simplify significantly if they take place entirely along the line joining the colliding bodies. We can then introduce a basis vector $\vec{i}$ in that direction and write $\vec{v}_1 = v_1 \vec{i}, \vec{v}_2 = v_2 \vec{i}, \vec{u}_1 = u_1 \vec{i}, \vec{u}_2 = u_2 \vec{i}$. Then we obtain:

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2, \tag{3.10}$$
$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2. \tag{3.11}$$

Suppose the massess and intial velocities are given, and we want to know the final velocities. We now have two equations for two unkowns - enough to compute the final velocities.

Example 3.9 *(***One dimensional elastic collision***) A smooth sphere of mass M moving with velocity $\vec{v} = 2\vec{i} m\ s^{-1}$, hits a second smooth sphere of mass $2M$ which is at rest. The motion takes place along the $\vec{i}$ direction. Compute the velocities of both spheres after the collision.*

The conservation laws give the following set of equations for u_1 and u_2:

$$2 = u_1 + 2u_2, \qquad 4 = u_1^2 + 2u_2^2$$

Substituting the linear into the quadratic equation, we obtain $\vec{u}_1 = -2/3\,\vec{i}\mathrm{m\ s}^{-1}$ and $\vec{u}_2 = 4/3\,\vec{i}\mathrm{m\ s}^{-1}$. $\square$

The position vector $\vec{R}$ (3.3) of the centre-of-mass of two particles defines the origin of a special frame of reference, called the centre-of-mass frame. In studying collision problems it is often useful to study the collision in that frame. If the particles have velocities $\vec{v}_1$ and $\vec{v}_2$ in the laboratory frame, their velocities relative to the centre-of-mass frame are $\vec{v}'_1 = \vec{v}_1 - \vec{V}$ and $\vec{v}'_2 = \vec{v}_2 - \vec{V}$. It follows that the total momentum in the centre-of-mass frame is zero:

$$m_1(\vec{v}_1 - \vec{V}) + m_2(\vec{v}_2 - \vec{V}) = \vec{P} - (m_1 + m_2)\vec{V} = 0. \tag{3.12}$$

Example 3.10 *(***Elastic collision in centre-of-mass frame***) Two spheres of masses m_1 and m_2 have initial velocities $\vec{v}_1$ and $\vec{v}_2$ such that the total momentum is zero. Show that after the collision their speeds are unchanged, i.e. $v_1 = u_1$ and $v_2 = u_2$.*

Before the collision we have total momentum zero and therefore $m_1\vec{v}_1 = -m_2\vec{v}_2$; by momentum conservation we also have $m_1\vec{u}_1 = -m_2\vec{u}_2$ after the collision. Hence the kinetic energy before the collision can be expressed in terms of v_1:

$$\frac{1}{2} m_1 |\vec{v}_1|^2 + \frac{1}{2} m_2 |\vec{v}_2|^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_1^2,$$

and the kinetic energy afer the collision can similarly be expressed in terms of u_1:

$$\frac{1}{2} m_1 |\vec{u}_1|^2 + \frac{1}{2} m_2 |\vec{u}_2|^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) u_1^2.$$

Kinetic energy conservation then implies $v_1 = u_1$ (speeds are positive by definition). Similarly we find $v_2 = u_2$. $\square$

Finally, we consider one example of a collision where kinetic energy is not conserved; such collisions are called inelastic.

Example 3.11 *(***Inelastic collision***) Consider the collision of two particles of masses m_1 and m_2 which stick together after the collision. If particle 1 has initial velocity $\vec{v}$ and particle 2 is at rest before the collision compute the fraction of the kinetic energy which is lost in the collision. Describe the same process in the centre-of-mass frame and compute the fraction of the energy lost in that frame.*

Let $\vec{V}$ be the common speed of the particles after the collision. Momentum conservation implies

$$\vec{V} = \frac{m_1}{m_1 + m_2}\vec{v}.$$

The kinetic energy before the collision is $K_i = \frac{1}{2}m_1|\vec{v}|^2$ and the kinetic energy after the collision is $K_f = \frac{1}{2}(m_1 + m_2)|\vec{V}|^2 = \frac{1}{2}\frac{m_1^2}{m_1+m_2}|\vec{v}|^2$. Hence the fraction of the energy lost is

$$\frac{K_i - K_f}{K_i} = \frac{m_2}{m_1 + m_2}.$$

The centre-of-mass frame moves with velocity $\vec{V}$, so the initial velocities are $\vec{v}_1' = \vec{v} - \vec{V} = \frac{m_2}{m_1+m_2}\vec{v}$ and $\vec{v}_2' = -\frac{m_1}{m_1+m_2}\vec{v}$. The kinetic energy before the collision is

$$K_i' = \frac{1}{2}m_1|\vec{v}_1'|^2 + \frac{1}{2}m_2|\vec{v}_2'|^2 = \frac{1}{2}\left(\frac{m_1 m_2^2}{(m_1+m_2)^2} + \frac{m_1^2 m_2}{(m_1+m_2)^2}\right)|\vec{v}|^2 = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}|\vec{v}|^2.$$

The kinetic energy after the collision is zero, so the energy loss is 100 %. $\square$

3.4 Rocket motion

Consider a rocket moving in outer space. To describe the motion we pick an origin O and restrict attention to motion in one direction, which we take to be the $\vec{k}$ direction. Hence the rocket's position relative to the origin O at time t is $z(t)\vec{k}$ and the velocity is $v(t)\vec{k}$, with $v(t) = \dot{z}(t)$. We will not consider motions where the rocket reverses its direction, so we assume $v > 0$ in the following, so that v is the rocket's speed.

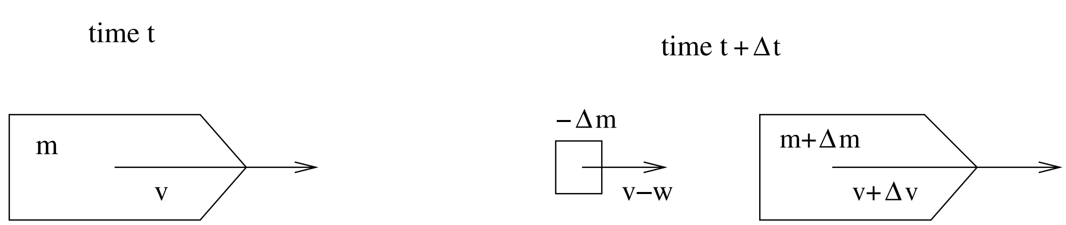

Figure 8: Rocket motion

There is no external force acting on the rocket, which propels itself by ejecting fuel. Suppose the fuel is ejected at a constant speed w relative to the rocket. At time t the rocket is moving with speed $v(t)$ and has mass $m(t)$; a a short time interval Δt later it is moving with speed $v(t+\Delta t) = v(t) + \Delta v$ and has mass $m(t+\Delta t) = m(t) + \Delta m$. Since the rocket's mass decreases with time, Δm is negative and $-\Delta m = -(m(t+\Delta t) - m(t))$ is the amount of fuel is ejected by the rocket. By momentum conservation we have

$$\begin{aligned} (m(t)+\Delta m)(v(t)+\Delta v) + (-\Delta m)(v(t)-w) &= m(t)v(t) \\ m\Delta v + \Delta m w + \Delta m \Delta v &= 0. \end{aligned} \tag{3.13}$$

Dividing by $m\Delta t$ and taking the limit $\Delta t \to 0$ we obtain the differential equation for rocket motion without external forces:

$$m\frac{dv}{dt} + w\frac{dm}{dt} = 0. \tag{3.14}$$

Writing the equation as

$$\frac{dv}{dt} = -w\frac{d\ln m}{dt} \tag{3.15}$$

and integrating from $t = 0$ to t we have

$$(v(t) - v_0) = -w\ln\frac{m(t)}{m_0}, \tag{3.16}$$

where we introduced the constants $v_0 = v(0)$ and $m_0 = m(0)$. Hence

$$v(t) = v_0 + w\ln\frac{m_0}{m(t)}. \tag{3.17}$$

Example 3.12 *A rocket of total mass M contains fuel of mass ϵM. The parameter ϵ is called the* **fuel ratio**, $0 < \epsilon < 1$. *When ignited the fuel burns at a constant rate k, ejecting exhaust gas at a constant speed w. In a force-free environment, find the distance travelled at burn-out, assuming the rocket starts from rest.*

Start by looking at

$$\frac{dm}{dt} = -k,$$

which is solved by $m(t) = M - kt$. Hence the burn-out time T is reached when $m(t) = M - \epsilon M$ i.e. when $T = \epsilon M/k$. Now (3.14) gives us a differential equation for v:

$$\frac{dv}{dt} = \frac{wk}{M - kt} \tag{3.18}$$

Integrating once, using

$$\int \frac{1}{M-kt}dt = -\frac{1}{k}\ln(M - kt) + c$$

we find

$$v(t) - v(0) = -w\ln(M-kt) + w\ln M. \tag{3.19}$$

Since $v(0) = 0$ we conclude

$$v(t) = -w\ln(M-kt) + w\ln M = w\ln\frac{M}{M-kt}. \tag{3.20}$$

Recalling that distance travelled $z(t)$ is related to speed via $\dot{z} = v$, we integrate again to find the distance travelled at time t. Using $\int \ln x dx = x\ln x - x + c$ we find

$$z(t) - z(0) = wt\ln M + \frac{w}{k}\left((M-kt)\ln(M-kt) - (M-kt)\right) - \frac{w}{k}\left(M\ln M - M\right) \tag{3.21}$$

Assuming that the rocket starts at $z = 0$, and re-arranging terms we conclude

$$z(t) = wt + \frac{w}{k}(M-kt)\ln\frac{M-kt}{M}. \tag{3.22}$$

Burn out time is $T = \epsilon M/k$ and hence the distance travelled at that time is

$$z(T) = \frac{wM}{k}(\epsilon + (1-\epsilon)\ln(1-\epsilon)) \tag{3.23}$$

□

For a rocket travelling upwards in the constant gravitational field near the earth's surface, we need to modify the analysis which led to (3.14). The rate of change of momentum at time t is now equal to the gravitational force $-m(t)g\vec{k}$ acting on the rocket. Assuming, as above, that the rocket motion takes place entirely in the $\vec{k}$-direction, we obtain the following equation for the upward velocity v:

$$m\frac{dv}{dt} + w\frac{dm}{dt} = -mg. \tag{3.24}$$

This equation is not much more difficult to integrate than the rocket equation in free space, but we will not do this here.

3.5 Angular momentum

The vector product $\vec{a} \times \vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ is defined geometrically as follows.

Definition 3.13 (**Vector product**) *The vector product $\vec{a}\times\vec{b}$ is the vector orthogonal to $\vec{a}$ and $\vec{b}$ of length $|\vec{a}||\vec{b}||\sin(\angle(\vec{a},\vec{b}))$, pointing in the direction into which a screw moves when rotated from $\vec{a}$ to $\vec{b}$ through the smallest angle required to bring $\vec{a}$ and $\vec{b}$ into congruence.*

It follows from the defintion that

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}, \tag{3.25}$$

since the rotation from $\vec{a}$ to $\vec{b}$ has the opposite direction to that from $\vec{b}$ to $\vec{a}$. Since $\vec{a} \times \vec{b}$ is defined to be orthogonal to $\vec{a}$ and $\vec{b}$ we also have

$$(\vec{a} \times \vec{b})\cdot\vec{a} = (\vec{a} \times \vec{b})\cdot\vec{b} = 0. \tag{3.26}$$

Finally one can show, using trigonometry, that the vector product obeys the distributite law:

$$\vec{a} \times (\vec{b}_1 + \vec{b}_2) = \vec{a} \times \vec{b}_1 + \vec{a} \times \vec{b}_2. \tag{3.27}$$

This rule is very useful for calculations in an orthonormal basis $\{\vec{i}, \vec{j}, \vec{k}\}$. We assume the basis $\{\vec{i}, \vec{j}, \vec{k}\}$ to be in standard orientation, which means that we have the following cross products between basis vectors:

$$\begin{aligned} &\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0 \\ &\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}. \end{aligned} \tag{3.28}$$

Using these rules, the vector product of two vectors $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ can be expressed in terms of the coordinates a_1, a_2, a_3 and b_1, b_2, b_3 as

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}, \tag{3.29}$$

which is often written as a determinant:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{3.30}$$

Definition 3.14 *Consider a particle of mass m whose position at time t relative to an origin O is given by the position vector $\vec{r}(t)$. The angular momentum of the particle is defined by*

$$\vec{L} = m\,\vec{r} \times \dot{\vec{r}} \tag{3.31}$$

Note that the unit of angular momentum is kg m^2 s^{-1}.

Example 3.15 *At time t seconds, the position vector of a particle of mass $M = 5$ kg relative to a fixed origin is $\vec{r}(t) = 6t\vec{i} + 3\vec{j} - 5t^2\vec{k}$ metres. Compute the particle's angular momentum relative to O, assuming that $\{\vec{i}, \vec{j}, \vec{k}\}$ is an ON basis in standard orientation.*

We find $\vec{L} = M\vec{r} \times \dot{\vec{r}} = 5(6t\vec{i} + 3\vec{j} - 5t^2\vec{k}) \times (6\vec{i} - 10t\vec{k})$kg m^2 s^{-1} $= -150t\vec{i} + 150t^2\vec{j} - 90\vec{k}$ kg m^2 s^{-1}. □

So far we have described the position of a particle by giving its coordinates x, y, z relative to an orthonormal basis. Such coordinates are also sometimes called cartesian coordinates. In certain two-dimensional situations it is convenient to trade the cartesian coordinates

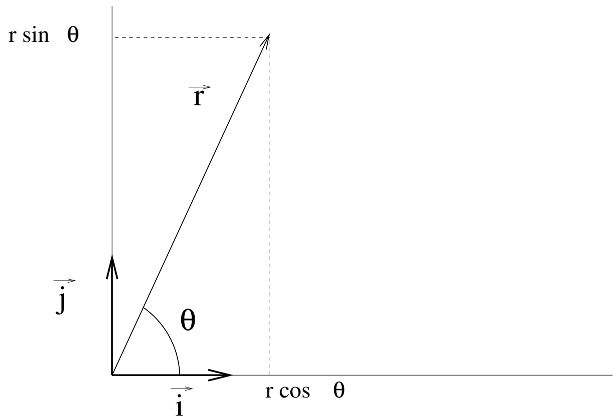

Figure 9: Polar coordinates in the plane

x and y for polar coordinates r, θ which describe the position in terms of the distance r from the origin, and the angle θ with respect to the $\vec{i}$, see 9. The position vector is given in terms of polar coordinates as

$$\vec{r} = r\cos\theta\vec{i} + r\sin\theta\vec{j}. \tag{3.32}$$

Example 3.16 *(***Angular momentum in polar coordinates***) At time t seconds, the position vector of a particle of mass M relative to a fixed origin is $\vec{r}(t) = r(t)\cos\theta(t)\vec{i} + r(t)\sin\theta(t)\vec{j}$, where r and θ are arbitrary functions of time. Compute the particle's angular momentum.*

We use

$$\dot{\vec{r}} = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\vec{i} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\vec{j}. \tag{3.33}$$

Multiplying out the vector product $\vec{r} \times \dot{b}r$ and using $\cos^2\theta + \sin^2\theta = 1$, we find

$$\vec{L} = M\vec{r} \times \dot{\vec{r}} = Mr^2\dot{\theta}\vec{k}. \tag{3.34}$$

Note that angular momentum is in the $\vec{k}$ direction i.e. orthogonal to the $\vec{i}, \vec{j}$-plane in which the motion takes places. The magnitude of the angular momentum is

$$L = |\vec{L}| = Mr^2\dot{\theta}. \tag{3.35}$$

□

On sheet 5 you are asked to check the following rule for the differentiation of a vector product.

Lemma 3.17 *Suppose $\vec{a}$ and $\vec{b}$ are two vector-valued functions of time. Then*

$$\frac{d}{dt}(\vec{a} \times \vec{b}) = \dot{\vec{a}} \times \vec{b} + \vec{a} \times \dot{\vec{b}}.$$

Using it, we deduce in particular the rule for differentiation of the angular momentum:

$$\frac{d\vec{L}}{dt} = m\frac{d}{dt}(\vec{r}\times\dot{\vec{r}}) = m\dot{\vec{r}}\times\dot{\vec{r}} + m\vec{r}\times\frac{d\dot{\vec{r}}}{dt} = m\vec{r}\times\ddot{\vec{r}} \tag{3.36}$$

Theorem 3.18 *(***Angular momentum conservation***) Suppose a particle moves according to Newton's second law, and let its position vector at time t relative to a fixed origin O be given by $\vec{r}(t)$. If the force acting on the particle can be written as $\vec{F}(\vec{r}) = f(|\vec{r}|)\vec{r}$ for some function f the angular momentum is conserved during the motion:*

$$\frac{d\vec{L}}{dt} = 0. \tag{3.37}$$

We prove the theorem by direct calculation. Newton's second law

$$m\ddot{\vec{r}} = \vec{F}$$

implies

$$\frac{d\vec{L}}{dt} = m\vec{r}\times\ddot{\vec{r}} = \vec{r}\times\vec{F} = f(|\vec{r}|)\vec{r}\times\vec{r} = 0.$$

$\square$

Forces which are parallel to the position vector $\vec{r}$ are called **central** forces.

An example which we have already come across is the gravitational force experienced by a particle of mass m moving the in the earth's gravitational field. The force

$$\vec{F} = -\frac{GmM_E}{r^3}\vec{r} \tag{3.38}$$

is clearly of the form $\vec{F}(\vec{r}) = f(|\vec{r}|)\vec{r}$.

It follows from the conservation of angular momentum that motion in central forces is always planar: the position vector $\vec{r}$ satisfies $\vec{r}\cdot\vec{L} = 0$ and and therefore remains in the plane orthogonal to the constant vector $\vec{L}$ at all times.

Example 3.19 *A particle of mass m moves in a central force. At time $t = 0$ its position and velocity vectors are $\vec{r}(0) = \vec{j}$ and $\vec{v}(0) = -5\vec{i}$. Compute the total angular momentum and find the plane of the particle's motion.*

$\vec{L} = m\vec{j}\times(-5\vec{i}) = 5m\vec{k}$. Therefore the plane of the particle's motion is the $\vec{i}\vec{j}$ plane. $\square$

3.6 Energy conservation

When a particle of mass m moves under the influence of a force $\vec{F}$ its kinetic energy $K = \frac{1}{2}m|\dot{\vec{r}}|^2 = \frac{1}{2}m\,\dot{\vec{r}}\cdot\dot{\vec{r}}$ changes with time according to

$$\frac{dK}{dt} = m\dot{\vec{r}}\cdot\ddot{\vec{r}} = \dot{\vec{r}}\cdot\vec{F}, \tag{3.39}$$

where we used Newton's second law in the second equality. Integrating both sides between t_1 and t_2, and using the fundamental theorem of calculus we deduce

$$\begin{aligned} \int_{t_1}^{t_2} \frac{dK}{dt} &= \int_{t_1}^{t_2} \dot{\vec{r}}\cdot\vec{F}dt \\ K(t_2) - K(t_1) &= \int_{t_1}^{t_2} \dot{\vec{r}}\cdot\vec{F}dt. \end{aligned} \tag{3.40}$$

Definition 3.20 *(**Work**) The quantity*

$$W[t_1, t_2] = \int_{t_1}^{t_2} \dot{\vec{r}}\cdot\vec{F}dt \tag{3.41}$$

is called the **work** *done by the force $\vec{F}$ on the particle between t_1 and t_2*

It follows from (3.40) that the kinetic energy of a particle moving under the influence of a force changes by an amount equal to the work done by the force. In particular, the kinetic energy itself is therefore not generally constant during the motion of a particle. We now show that in certain circumstances it is possible to define the potential energy of a particle in such a way that the total energy - the sum of the kinetic and the potential energy - is conserved.

Definition 3.21 *(**Gradient**) In terms of our orthogonal basis $\{\vec{i}, \vec{j}, \vec{k}\}$ the gradient of a function $f : \mathbb{R}^3 \to \mathbb{R}$ is defined as*

$$\nabla f(\vec{r}) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}. \tag{3.42}$$

Examples of gradients are computed on sheet 5! Here we need the gradient for the following definition:

Definition 3.22 *A force $\vec{F}$ is called* **conservative** *if it can be written*

$$\vec{F} = -\nabla V \tag{3.43}$$

where V is a function of $\vec{r}$ called the **potential energy** *or simply* **potential**.

Example 3.23 *Show that the force* (3.38) *can be written as $-\nabla V$ with $V(r) = -GmM_E/r$*

Answer: You did this on problem sheet 5! □

Example 3.24 *An attractive central force is directed towards the origin O and has magnitude r where $r = |\vec{r}|$ is the distance from O. Find the potential of the force.*

Answer: With $\vec{F}(\vec{r}) = -\vec{r}$ we can again use exercise (5) on problem sheet 4 to conclude that $\vec{F}(\vec{r}) = -\frac{1}{2}\nabla(r^2)$ so that $V(\vec{r}) = \frac{1}{2}r^2$. □

Theorem 3.25 (**Energy conservation**) *For a particle of mass m with position vector $\vec{r}(t)$ moving according to Newton's third law under the influence of conservative force with potential $V(\vec{r})$ the total energy*

$$E = \frac{1}{2}m|\dot{\vec{r}}|^2 + V(\vec{r}) \tag{3.44}$$

is conserved during the motion.

For the proof we need the chain rule for a function of several variables: if f is a function of $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{r}$ a function of t then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot \dot{\vec{r}}. \tag{3.45}$$

Applying the chain rule to the energy we find

$$\begin{aligned} \frac{dE}{dt} &= m\dot{\vec{r}}\cdot\ddot{\vec{r}} + \nabla V \cdot \dot{\vec{r}} \\ &= \dot{\vec{r}}\cdot(m\ddot{\vec{r}} - \vec{F}) \\ &= 0 \end{aligned} \tag{3.46}$$

by Newton's second law. □

The unit of energy is the **Joule**. One Joule = 1 kg m^2 s^{-2}.

We are particularly interested in situations where both the energy and the angular momentum is conserved. Angular momentum conservation implies that the motion takes place in a plane, and we can then assume that plane to be the $\vec{i}\vec{j}$ plane without loss of generality. Thus we parametrise the position vector in terms of unkonwn functions $r(t)$ and $r(\theta)$ as

$$\vec{r} = r(t)\cos(\theta(t))\vec{i} + r(t)\sin(\theta(t))\vec{j}. \tag{3.47}$$

Then we can use the expression (3.33) for $\dot{\vec{r}}$ to express the inner product $\dot{\vec{r}}\cdot\dot{\vec{r}}$ in terms of polar coordinates. Multiplying out and using $\cos^2\theta + \sin^2\theta = 1$, we find the following expression for the kinetic energy

$$K = \frac{1}{2}m\,\dot{\vec{r}}\cdot\dot{\vec{r}} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2). \tag{3.48}$$

Recalling the expression (3.35) for the magnitude $L = mr^2\dot{\theta}$ of the angular momentum we can write the kinetic energy as

$$K = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} \tag{3.49}$$

and the total energy as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r). \tag{3.50}$$

Example 3.26 *A particle of mass 5 kg moves in a gravitational field with potential $-20/r$ Joules. The particle's inital position and velocity are $\vec{r}(0) = 6\vec{i}$ meters and $\vec{v}(0) = -3\vec{i}+5\vec{j}$ m s $^{-1}$. Compute the total energy and angular momentum of the particle.*

Using (3.44) we find

$$E = \frac{1}{2}\ 5kg\ (3^2 + 5^2)\ m^2 s^{-2} - \frac{20}{6} Joules = 81\frac{2}{3} Joules \tag{3.51}$$

and from (3.31) we have

$$\vec{L} = 5kg(6 \times 5)m^2 s^{-1}\vec{k} = 150\vec{k}\ kg\ m^2 s^{-1} \tag{3.52}$$

□

Energy conservation and angular momentum conservation together give the following differential equation for the unkown functions $r(t)$ and $\theta(t)$:

$$\begin{aligned} \left(\frac{dr}{dt}\right)^2 &= \frac{2}{m}\left(E - \frac{L^2}{2mr^2} - V(r)\right) \\ \frac{d\theta}{dt} &= \frac{L}{mr^2} \end{aligned} \tag{3.53}$$

These are complicated coupled differential equations for two functions. However, we can elmininate the variable t and obtain a manageable differential equation for one function $r(\theta)$ if we divide the first equation in (3.53) by the square of the second:

$$\begin{aligned} \left(\frac{dr}{d\theta}\right)^2 &= \frac{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - V(r)\right)}{\frac{L^2}{m^2 r^4}} \\ &= \frac{2mE}{L^2} r^4 - r^2 - \frac{2mr^4 V(r)}{L^2} \end{aligned} \tag{3.54}$$

Knowing the function $r(\theta)$ will enable us to reconstruct the spatial trajectory or orbit of the particle but will not tell us at what time the particle passes a particular point on the orbit.

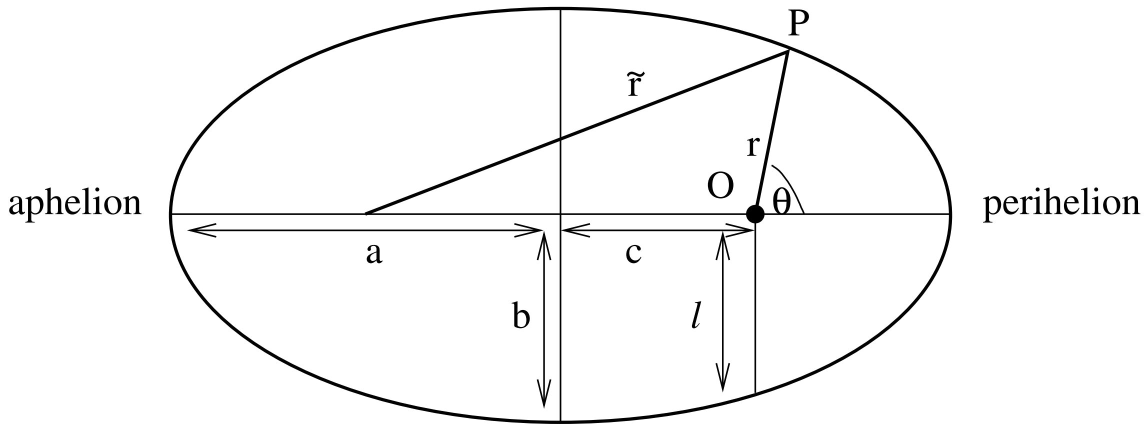

Figure 10: Ellipse parameters and properties

4 The Kepler problem

The German astronomer Johnnes Kepler (1571-1630) arrived at three laws governing the motion of planets through a careful analysis of empirical data collected by the Danish astronomer Tycho Brahe (1546-1601). In this section we will derive Kepler's three laws from Newton's equation of motion and law of unversal gravitation. Kepler's first law states that the planets orbit the sun in elliptical orbits with the sun at one focus. We therefore begin this section with a review of the geometry of ellipses.

4.1 Geometry of the ellipse

In the "Gardener's construction" of the ellipse one takes a rope of fixed length $2a$ and ties it around two poles separated a distance $2c$. The ellipse is locus of all points P in the plane the sum of whose distances from the fixed poles (the foci) is $2a$:

$$r + \tilde{r} = 2a. \tag{4.1}$$

The point on the ellipse which is closest to O is called the perihilion, the point furthest away from O is called aphelion. The Gardener's construction and ellipse parameters are shown in Fig. 10. The parameter a is called the length of the semi-major axis of the ellipse and b the length of the semi-minor axis; ℓ is called the semi-latus rectum. By considering the point P on the minor axis, and using that $r = \tilde{r} = a$ at that point, the parameters a and b can be related to half the distance between the focal points c via the theorem of Pythogoras. We find, from Fig. 11, that

$$a^2 = b^2 + c^2. \tag{4.2}$$

We define the eccentricity

$$\epsilon = \frac{c}{a}, \tag{4.3}$$

and note from Fig. 4.10 that $0 \leq c < a$ and therefore $0 \leq \epsilon < 1$. In the case $\epsilon = 0$ we recover the circle. From (4.2) we deduce

$$b^2 = (1 - \epsilon^2)a^2. \tag{4.4}$$

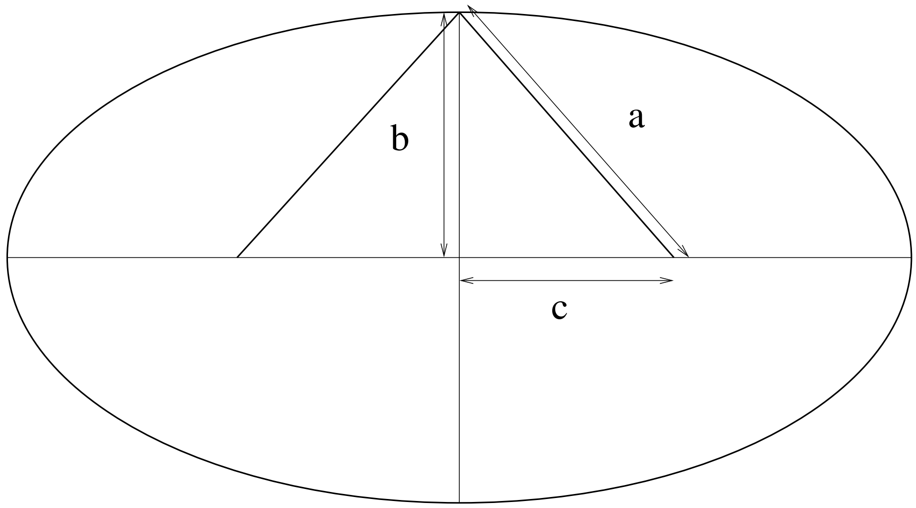

Figure 11: Pythagoras in the ellipse

Applying again Pythagoras to the picture, we deduce a formula for the semi-latus rectum:

$$\begin{aligned} \ell^2 + (2c)^2 &= (2a - \ell)^2 \\ \Rightarrow \ell &= a - \frac{c^2}{a}. \end{aligned} \tag{4.5}$$

Hence also

$$\ell = (1 - \epsilon^2)a = \frac{b^2}{a}. \tag{4.6}$$

For later use, we note that the area of the ellipse is given by

$$A = \pi ab. \tag{4.7}$$

Now introduce a basis $\vec{i}, \vec{j}$ at the right-hand focus O and polar coordinates (r, θ). Write the position vector for a point on the ellipse as

$$\vec{r} = r(\theta)\cos(\theta)\vec{i} + r(\theta)\sin(\theta)\vec{j}.$$

Then, from the picture by the cosine theorem

$$\tilde{r}^2 = 4c^2 - 4cr\cos(\pi - \theta) + r^2. \tag{4.8}$$

Combining this with (4.1) and using $\cos(\pi - \theta) = -\cos(\theta)$ we find

$$\begin{aligned} (2a - r)^2 &= 4c^2 - 4cr\cos(\pi - \theta) + r^2 \\ \Leftrightarrow a^2 - ar &= c^2 + cr\cos(\theta) \\ \Leftrightarrow a^2 - c^2 &= r(a + c\cos(\theta)). \end{aligned} \tag{4.9}$$

With the definitions (4.3) and (4.5) of the eccentricity and semi-latus rectum we arrive at the following functional dependence of the distance r from O on the angle θ:

$$r(\theta) = \frac{\ell}{1 + \epsilon\cos(\theta)}. \tag{4.10}$$

The ellipse is an example of a **conic section**. The other conic sections are the parabola and the hyperbola. They can also be parametrised in the form (4.10). By definition, the parabola is obtained for $\epsilon = 1$ and the hyperbola for $\epsilon > 1$.

4.2 Kepler's laws

Physical Law 4.1 *(***Kepler's first law***) The planets move on ellipses with the sun at one focus of the ellipse*

Physical Law 4.2 *(***Kepler's second law***) The position vectors of the planets relative to the sun sweep out equal areas in equal times*

Physical Law 4.3 *(***Kepler's third law***) The ratio of the square of the orbital period to the cube of the semi-major axis is the same constant for all planets*

We will derive Kepler's laws from the conservation of angular momentum and total energy for a particle of mass m moving in the gravitational potential of sun $V(r) = -\alpha/r$, where $\alpha = GM_{sun}m$. We will neglect the sun's motion and treat the sun's position O as fixed for now.

Proof of Kepler's first law

Our strategy is to check that (4.10) satisfies the equation for the orbit (3.54) with $V(r) = -\alpha/r$. To do so we will have to find suitable expressions for the constants ℓ and ϵ in the parametristation of the ellipse in terms of the physical quantities E, L, m and α. In principle this is simply an exercise in differentiation. In practice we can make life slightly easier by changing coordinates to

$$s(\theta) = \frac{1}{r(\theta)}. \tag{4.11}$$

Substituting

$$\frac{ds}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

into (3.54) we find

$$\left(\frac{ds}{d\theta}\right)^2 = \frac{2mE}{L^2} - s^2 + \frac{2\alpha m}{L^2}s \tag{4.12}$$

Completing the square, this can be written as

$$\left(\frac{ds}{d\theta}\right)^2 + \left(s - \frac{\alpha m}{L^2}\right)^2 = \frac{2mE}{L^2} + \frac{\alpha^2 m^2}{L^4}. \tag{4.13}$$

On the other hand, using (4.11), the parametrisation (4.10) of the ellipse becomes

$$s(\theta) = \frac{1}{\ell} + \frac{\epsilon}{\ell}\cos(\theta). \tag{4.14}$$

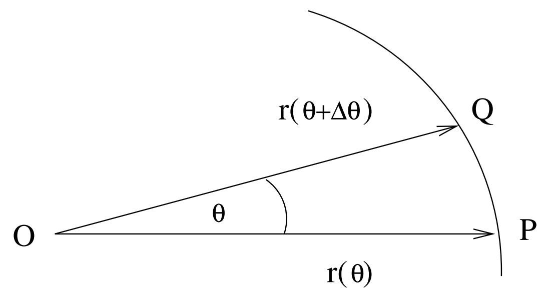

Figure 12: Area element in polar coordinates

so that

$$\frac{ds}{d\theta} = -\frac{\epsilon}{\ell}\sin(\theta) \tag{4.15}$$

and

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{\epsilon}{\ell}\right)^2 (1-\cos^2\theta) = \left(\frac{\epsilon}{\ell}\right)^2 - \left(s - \frac{1}{\ell}\right)^2. \tag{4.16}$$

Now write this as

$$\left(\frac{ds}{d\theta}\right)^2 + \left(s - \frac{1}{\ell}\right)^2 = \left(\frac{\epsilon}{\ell}\right)^2 \tag{4.17}$$

and compare with (4.13). We obtain agreement if we set

$$\ell = \frac{L^2}{m\alpha} \tag{4.18}$$

and

$$\left(\frac{\epsilon}{\ell}\right)^2 = \frac{2mE}{L^2} + \frac{\alpha^2 m^2}{L^4}. \tag{4.19}$$

Using (4.18) to eliminate ℓ we obtain a formula for the eccentricity

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\alpha^2 m}}. \tag{4.20}$$

Hence, with the semi-latus rectum and the eccentricity given by (4.18) and (4.20) the orbit (4.10) does indeed satisfy the orbital equation (3.54). Note that, in order to obtain an ellipse, i.e. $\epsilon < 1$ we require $E < 0$. If the energy is zero, the orbit is a parabola and if the energy is positive the orbit is a hyperbola. □

Proof of Kepler's second law

The proof of the law is based entirely on angular momentum conservation; the law therefore holds for motion under the influence of any central force. We start with the formula for an area element in polar coordinates. Consider the area inside the shape OPQ in Fig. 12. The side OP has length $r(\theta)$ and the side PQ has approximate length $r(\theta)\Delta\theta$.

In the limit $\Delta\theta \to 0$, the shape OPQ becomes a triangle of area $\Delta A = \frac{1}{2} r(\theta) r(\theta) \Delta\theta$. Hence the rate at which area is swept out by the planet's position vector $\vec{r}$ is

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2m} L. \tag{4.21}$$

Since angular momentum is conserved for motion in a central force, the area swept out per unit time by the planet's position vector is constant. □

Proof of Kepler's third law

Integrating equation (4.21) over one revolution we obtain the area of the ellipse on the left-hand side

$$A = \pi ab. \tag{4.22}$$

From (4.21) we have

$$A = \int_0^T \frac{1}{2m} L dt \; = \frac{1}{2m} LT. \tag{4.23}$$

Hence

$$T = \frac{2\pi abm}{L}. \tag{4.24}$$

Now square both sides and use $\ell = b^2/a$ and formula (4.18) for ℓ to find

$$T^2 = 4\pi^2 \frac{m}{\alpha} a^3. \tag{4.25}$$

Since $\alpha/m = GM_{sun}$ we arrive at the promised relation between of T^2 and a^3

$$T^2 = \frac{4\pi^2}{GM_{sun}} a^3, \tag{4.26}$$

with the constant of proportionality independent of the mass of the orbiting planet. □

5 Special Relativity

5.1 The relativity principle

At the beginning of this course, we thought carefully about how we model space mathematically. Our model required a choice of frame: an origin O together with a basis $\{\vec{i}, \vec{j}, \vec{k}\}$ attached to that origin. Once the frame is chosen, a point P in space can be described mathematically by three real numbers, the coordinates x, y and z, which specify the position of P relative to O according to $\vec{r} = \vec{OP} = x\vec{i} + y\vec{j} + z\vec{k}$. By contrast, our mathematical description of time was naive: we simply assumed the existence of a single real parameter t, called time, which is universally accessible and which can be used to parmatrise the trajectory of any moving particle. This assumption is called the assumption of absolute time in Newtonian mechanics. Newton was aware that he needed to make this assumption in order to formulate his laws, but saw no way of avoiding it. In this section we follow Einstein's reasoning, which leads to the abolition of absolute time. The basic argument is simple: the combination of two very general and simple physical facts - the relativity principle and the constancy of the speed of light - leads to a contradiction which can be resolved if one abandons the notion of absolute time and replaces it with a "coordinate" time associated to every observer.

We begin by revisiting the coordinate transformations between frames which are moving relative to each other. Suppose frame S' is moving relative to frame S with velocity $\vec{v}$ and that at time $t = 0$ the frames coincide. Then the position of the origin O' of S' relative to the origin O of S is given by the vector

$$\vec{R}(t) = \vec{v}t, \tag{5.1}$$

and the transformation rule for going from position vectors $\vec{r}$ relative to O, to position vectors $\vec{r}'$ relative to O' is, according to Sect. 1.1,

$$\vec{r}' = \vec{r} - \vec{v}t. \tag{5.2}$$

If the frame S' is moving along the $\vec{i}$-axis we can expand in coordinates and get, with $\vec{v} = v\vec{i}$,

$$x' = x - vt, \quad y' = y, \quad z' = z. \tag{5.3}$$

The transformation rules (5.2) and (5.3) are called **Galilean transformations**.

Recall Newton's first law 2.1 according to which a body remains in a state of rest or uniform motion (constant velocity) unless acted upon by an external force. Frames in which Newton's first law holds are called **inertial frames**. It follows from (5.2) that if a body moves with constant velocity $\dot{\vec{r}} = \vec{u}$ in the inertial frame S, then its velocity $\vec{u}' = \dot{\vec{r}}'$ in frame S' is

$$\vec{u}' = \vec{u} - \vec{v} \tag{5.4}$$

and hence also constant. We conclude that all frames related to a given inertial frame S by a Galilean transformation are also inertial. In fact, one can show more:

Theorem 5.1 *Newton's laws of mechanics are invariant under Galilean transformations: if they hold in a frame S they hold in all frames S' related to S by a Galilean transformation.*

The observation that one cannot detect uniform motion by physical experiments precedes the formulation of Newton's laws. In a famous passage from "Dialogue Concerning the Two Chief World Systems" Galileo Galilei describes the following thought experiment:

Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though there is no doubt that when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.

The failure to detect uniform motion leads to the general

Physical Law 5.2 *(***Relativity principle***) The basic laws of physics are identical in all reference frames that move with uniform velocity relative to each other.*

This principle does not make any assumptions about how we change coordinates when going form one frame to another. It only says that the coordinate change should be such that physical laws take the same form in frames which move uniformly relative to each other. So, by theorem 5.1, Galilean transformations implement the relativity principle for Newtonian mechanics.

5.2 The Lorentz transformations

In the early 1900's there was increasingly sophisticated experimental evidence that the speed of light is the same in all inertial frames.

Physical Law 5.3 *(***Constancy of the speed of light***) The speed of light is $c = 299792458\ m\ s^{-1}$ in all inertial frames*

A particularly famous experiment that you may want to read up about is the **Michelson-Morley** experiment.

The Galilean transformation rule for going from an inertial frame S to another one S' moving relative to the first with velocity $\vec{v}$ would predict that the velocity of light $\vec{c}'$ in frame S' should be related to the velocity of light in S via

$$\vec{c}' = \vec{c} - \vec{v}. \tag{5.5}$$

This rule is just the "common sense" rule for working out relative velocities that we applied to the man walking in a train in example 1.11. However, (5.5) is clearly in contradiction to Physical Law 5.3: if $c = |\vec{c}| = 2.998 \times 10^8$ m s^{-1} in frame S then the value $c' = |\vec{c}'|$ in S' will be different if $\vec{v} \neq 0$. Physicists at the beginning of the 20th century struggled to understand this apparent contradiction, until Albert Einstein proposed a radical solution in 1905: keep the relativity principle (Physical Law 5.2) and the constancy of the speed of light (Physical Law 5.3), but modify the Galilean transformation rule (5.3) for going from one frame to another. The modification proposed by Einstein amounts to giving up the notion of absolute time. Since absolute time is an essential requirements for Newtonian mechanics, this meant abandoning Newtonian mechanics!

A fundamental concept for understanding Einstein's proposal is that of an **event**. Events are specified by specifying both a place and a time (e.g. a party, at no 12 Newton Alley, 8pm today). The Galilean rule tells us how the postion coordinates of an event changes when we change frames, but assumes that the time coordinates remains unchanged. Einstein proposed to use transformation rules under which both space and time coordinates change. The rules had previously been invented by the Dutch physicist Anton Lorentz, but Einstein was the first to interpret them correctly.

Definition 5.4 *(***Lorentz transformations***) Suppose a frame S' is moving relative to a frame S with speed v in the positive x-direction. Denote the coordinates of an event in S by (x, y, z, t) and the coordinates of the same event in S' by (x', y', z', t'). The Lorentz transformation relating the coordinates (x', y', z', t') in S' to the coordinates (x, y, z, t) is*

$$
\begin{aligned}
x' &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x - vt), \\
y' &= y, \\
z' &= z, \\
t' &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(t - \frac{v}{c^2}x).
\end{aligned}
\tag{5.6}
$$

We introduce the abbreviation

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{5.7}$$

so that the Lorentz transformation take the form

$$
\begin{aligned}
x' &= \gamma(x - vt), \\
y' &= y, \\
z' &= z, \\
t' &= \gamma(t - \frac{v}{c^2}x).
\end{aligned}
\tag{5.8}
$$

For some calculations it is useful to write the transformation in matrix form:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma \frac{v}{c^2} & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}. \tag{5.9}$$

In particular, on sheet 7 you are asked to check that the inverse of the Lorentz transformation (5.9) with parameter v is the Lorentz transformation

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v \\ \gamma \frac{v}{c^2} & \gamma \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix} \tag{5.10}$$

with parameter $-v$.

Note the following properties of the Lorentz transformations:

Lemma 5.5 *The Lorentz transformation reduce to the Galilean transformations when the speed v is small compared to the speed of light.*

To prove this proposition note that, by Taylor's theorem,

$$\frac{1}{\sqrt{1-h}} = (1-h)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}h$$

for $h \ll 1$. Thus, with $h = \frac{v^2}{c^2}$,

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \tag{5.11}$$

The Galilean limit is therefore attained if we neglect all terms containing $(v/c) \ll 1$.

Theorem 5.6 *If (x, y, z, t) and (x', y', z', t') are related by the Lorentz transformation* (5.6) *then*

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2. \tag{5.12}$$

This follows by inserting (5.8) on the left hand side, working out the squares and collecting terms. Note that in particular that a light front $x(t) = ct$ satisfies $x^2 - c^2t^2 = 0$ and will therefore satisfy the same equation in the primed coordinate: $x'^2 - c^2t'^2 = 0$. The speed of light is invariant under Lorentz transformations! □

The Lorentz transformations are compatible with the constancy of the speed of light and reduce to the Galilean transformations in the limit of speeds which are small relative to the speed of light (this includes all terrestrial speeds, like those of a cyclist, car, train or airplane!). However, this is achieved at the expense of introducing a new time coordinate for each reference frame. The physical significance and some consequences of this are explored in the next subsection.

5.3 Relativity of simultaneity, time dilation and Lorentz contraction

An **event** is specified by both space coordinates (x, y, z) and a time coordinate t in a frame S. Perhaps the most important lesson of special relativity - and certainly the most important rule to keep in mind when solving problems in relativistic physics - is that one needs to treat spatial and temporal aspects of a physical problem together: always describe the physical situation in terms of **events**, with space **and** time coordinates. Once this is carried out in one frame, we can compute the space and time coordinates of the **same event** in any other inertial frame using Lorentz transformations. Keeping track of space and time coordinates is the only way to understand many of the counter-intuitive features and "paradoxes" of special relativity.

Example 5.7 *The origins of two inertial frames S and S' coincide at $t = t' = 0$. The origin of S' moves at speed $v = 6 \times 10^7$ m s^{-1} along the positive x-axis in S. An event occurs at $x' = 2m$,$y' = 8m$,$z' = 3m$ and $t' = 16$ s. Where and when does the event occur in S?*

We find $\gamma = 1/\sqrt{1-(0.2)^2} \approx 1.01$ and therefore

$$x = 1.01(2 + 16 \times 6 \times 10^7)\text{m} \approx 9.70 \times 10^8\text{m}.$$

The y and z coordinates are unchanged $y = 8$m, $z = 3$m, and for the time coordinate we find

$$t = 1.01(16 + 0.2 \times (2/3)10^{-8})\text{s} \approx 16.16\text{s}.$$

Even for huge speeds like the one considered in the example the difference in the time assigned to the event in the two frames is relatively small. □

Example 5.8 *(**Relativity of simultaneity**) An observer in the inertial frame S has synchronised clocks, distributed at one meter intervals along the x-axis. Let the clocks be numbered, so that the n-th clock is n metres away from the origin in S. Another frame S' is moving with speed v in the positive x-direction relative to frame S. Find the time shown by synchronised clocks in the frame S' when all the clocks in S strike midnight ($t = 0$).*

The event "n-th clock strikes midnight" has the coordinates $x_n = n$ metres and $t_n = 0$ in frame S. Using the Lorentz transformation

$$t' = \gamma(t - \frac{v}{c^2}x') \tag{5.13}$$

we find that the time t' of the same event in frame S' is

$$t'_n = -\gamma\frac{v}{c^2}n. \tag{5.14}$$

Thus, while the clocks strike simultaneously in S, the events of the clocks striking midnight are not simultaneous in S'. The observer in S' finds that the clock at $n = 1$ struck early relative to the clock at $n = 0$ and the clock at $n = -1$ strikes late. So clocks along the positive x-axis in S will be perceived as going fast and those along the negative x-axis will be perceived as going slow by the obsever in S'. $\square$

Most interesting applications of the Lorentz transformations have to do with time and space intervals, and how they transform. For the following examples note that, if

$$\begin{aligned} x_1' &= \gamma(x_1 - vt_1), \\ t_1' &= \gamma(t_1 - \frac{v}{c^2}x_1) \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} x_2' &= \gamma(x_2 - vt_2), \\ t_2' &= \gamma(t_2 - \frac{v}{c^2}x_2). \end{aligned} \tag{5.16}$$

then the differences $\Delta x = x_2 - x_1, \Delta x' = x_2' - x_1', \Delta t = t_2 - t_1, \Delta t' = t_2' - t_1'$ are also related by

$$\begin{aligned} \Delta x' &= \gamma(\Delta x - v\Delta t), \\ \Delta t' &= \gamma(\Delta t - \frac{v}{c^2}\Delta x). \end{aligned} \tag{5.17}$$

Inverting we also have

$$\begin{aligned} \Delta x &= \gamma(\Delta x' + v\Delta t'), \\ \Delta t &= \gamma(\Delta t' + \frac{v}{c^2}\Delta x'). \end{aligned} \tag{5.18}$$

Example 5.9 *(***Time dilation***) An observer in frame S' finds that it takes a time T to boil a kettle at rest at the origin of the frame S'. The frame S' is moving relative to an inertial frame S with speed v in the positive x-direction. Compute the time that elapsed between switching on and reaching boiling point as measured by clocks which are stationary in frame S.*

In frame S' the space-time coordinates of the events "switching kettle on" and "reaching boiling point" are $x_1' = 0, t_1'$ and $x_2' = 0, t_2' = t_1' + T$. So the space and time intervals in S' are

$$\Delta x' = 0, \qquad \Delta t' = T. \tag{5.19}$$

By (5.18), we have

$$\Delta x = \gamma vT \qquad \Delta t = \gamma T. \tag{5.20}$$

The second formula captures the effect of **time dilation**. Since $\gamma > 1$, the time interval T' required for the moving kettle to boil appears longer (dilated) to the stationary observer. $\square$

Summary 5.10 *(***Time dilation***) A time interval between events which happen in the same place in the frame S' moving relative to S is found to be dilated by a factor γ when measured by clocks in the frame S:*

$$\Delta t = \gamma T. \tag{5.21}$$

Example 5.11 *(***Lorentz contraction***) A rod of length L is at rest and lying along the x-axis in a frame S' which is moving relative to an inertial frame S with speed v in the x-direciton. If the endpoints of the rod are measured simultaneously in S what is their separation in S?*

The rod's end points x'_1 and x'_2 are at rest in S' and we can measure their positions at arbitrary times t'_1 and t'_2. We always get $\Delta x' = L$. The observer in S measures the rod's endpoints simultaneously in **his** frame, i.e. the events "measurement of end point 1" and "measurement of end point 2" have coordinates (x_1, t_1) and (x_2, t_1) , and therefore $\Delta t = 0$. However, then the first equation in (5.17) tells us that

$$\Delta x' = \gamma \Delta x \tag{5.22}$$

or

$$\Delta x = \frac{1}{\gamma} L. \tag{5.23}$$

This formula describes the **Lorentz contraction** of a rod in the direction of motioin: since $1/\gamma = \sqrt{1-(v/c)^2} < 1$ the rod seems contracted when the positions of the end points are measured simultaneously in S. Note that a rod at a right angle to the direction of motion would not appear contracted. □

Summary 5.12 *(***Lorentz contraction***) The spatial separation between endpoints of a rod moving relative to a frame S is found to be contracted by a factor γ when the positions of the endpoints are measured simultaneously in the frame S.*

The dependence of space and time intervals on the reference frames means that we have to be careful when talking about "the length" of an object, or "the duration" of a process. However, there is a distinguished frame for objects like rods and clocks, namely the frame in which they are at rest.

Definition 5.13 *(***Proper time and length***) The proper length of a rod is its length measured in a frame in which the rod is at rest. The proper time of a clock is the time measured in the frame in which the clock is at rest.*

5.4 Relativistic velocity addition

Suppose a particle is moving with velocity $\vec{u} = u_x\vec{i} + u_y\vec{j} + u_z\vec{k}$ in a frame S, i.e. the rate of change of the particle's position $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ with respect to the time t is

$$\frac{d\vec{r}}{dt} = \vec{u} = u_x\vec{i} + u_y\vec{j} + u_z\vec{k} \tag{5.24}$$

or, in components,

$$\frac{dx}{dt} = u_x, \quad \frac{dy}{dt} = u_y, \quad \frac{dz}{dt} = u_z. \tag{5.25}$$

Now consider the same particle from the point of view of an observer in a frame S' moving relative to S with speed v in the positive x-direction. We would like to compute the velocity of the particle as seen by an observer at the origin in S', i.e. the rate of change of position as measured in S' with respect to time as measured in S'. Recalling that the particle's position and time coordinates in S' are

$$\begin{aligned} x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \\ t' &= \gamma(t - \frac{v}{c^2}x). \end{aligned} \tag{5.26}$$

we need to compute

$$\frac{dx'}{dt'} = u'_x, \quad \frac{dy'}{dt'} = u'_y, \quad \frac{dz'}{dt'} = u'_z. \tag{5.27}$$

Using the chain rule and

$$\frac{dx'}{dt'} = \frac{dx'}{dt} \Big/ \frac{dt'}{dt}, \tag{5.28}$$

we first compute

$$\frac{dx'}{dt} = \gamma(u_x - v), \quad \frac{dt'}{dt} = \gamma(1 - \frac{vu_x}{c^2}), \tag{5.29}$$

and deduce

$$u'_x = \frac{\gamma(u_x - v)}{\gamma(1 - \frac{vu_x}{c^2})} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}. \tag{5.30}$$

Proceeding similarly for $\frac{dy'}{dt'}$ and $\frac{dz'}{dt'}$ we find

Theorem 5.14 (**Relativistic velocity transformation**) *If a frame S' is moving relative to a frame S with speed v in the positive x-direction, the velocity components (u'_x, u'_y, u'_z) of a moving particle in S' are related to the velocity components in (u_x, u_y, u_z) in S via*

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \quad , \quad u'_y = \frac{u_y}{\gamma(1 - \frac{u_x v}{c^2})} \quad , \quad u'_z = \frac{u_z}{\gamma(1 - \frac{u_x v}{c^2})} \tag{5.31}$$

Inverting these relations to find u_x, u_y, u_z in terms of (u'_x, u'_y, u'_z) one finds, after some algebra,

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \quad , \quad u_y = \frac{u'_y}{\gamma(1 + \frac{u'_x v}{c^2})} \quad , \quad u_z = \frac{u'_z}{\gamma(1 + \frac{u'_x v}{c^2})}. \tag{5.32}$$

These equations are sometimes referred to as the relativitic velocity addition rule: if (u'_x, u'_x, u'_z) are the components of the velocity $\vec{u}' = \frac{d\vec{r}'}{dt'}$ of a particle relative to S', and $(v, 0, 0)$ are the components of the velocity $\vec{v}$ of S' relative to S, then the formulae (5.32) tell us how to "add" these velocities in order to work out the components of the velocity $\vec{u}$ of the particle relative to S.

Example 5.15 *Assume frame S' is moving at a speed $0.8c$ relative to S in the x-direction, and a particle is moving at speed $0.9c$ in the x-direction when measured by an observer in S'. What is the speed of the particle when measured by an observer in S?*

With $v = 0.8c$ and $u'_x = 0.9c$ we find $u_x = 1.7/(1 + 0.72) \times c = 0.988c$. □

Example 5.16 *(***Constancy of the speed of light***) Light travels at speed c in the x-direction in the frame S', which is moving with speed v in the x-direction relative to frame S. Find the speed of light in S.*

Now we have $u'_x = c$ so $u_x = (c + v)/(1 + (cv)/c^2) = c$. The speed of light is equal to c in S is as well - as it should be according to 5.3! □

Example 5.17 *(*c **is the limit***) With the notation of formula* (5.32) *show that, if $0 \leq u'_x \leq c$ and $0 \leq v \leq c$ then $0 \leq u_x \leq c$*

To prove this, first note that for any two numbers $0 \leq \alpha, \beta \leq 1$ we have $0 \leq (1-\beta)(1-\alpha) \leq 1$ and therefore

$$\begin{aligned} & \alpha + (1 - \alpha) = 1 \\ \Rightarrow\ & 0 \leq \alpha + (1 - \alpha) - (1 - \beta)(1 - \alpha) \leq 1 \\ \Rightarrow\ & 0 \leq \alpha + \beta(1 - \alpha) \leq 1 \\ \Rightarrow\ & 0 \leq \alpha + \beta \leq 1 + \alpha\beta. \end{aligned} \tag{5.33}$$

So recalling that $u_x = (u'_x+v)/(1+(u'_x v)/c^2)$ the claim follows with $\alpha = u'_x/c$ and $\beta = v/c$:

$$0 \leq \frac{u'_x}{c} + \frac{v}{c} \leq 1 + \frac{u'_x}{c}\frac{v}{c} \Leftrightarrow 0 \leq \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \leq c \tag{5.34}$$

□

Example 5.18 (**Fizeau experiment**) *Light moves at a slower speed in a material medium than in vacuum. Let c_w be the speed of light in water. Assuming that the water is flowing at a speed v relative to an inertial frame S and that the light is propagating in the same direction as the water find an expression for the speed of light c_S in the frame S. Show that, if v is small, then*

$$c_S \approx c_w + v(1 - \frac{c_w^2}{c^2}). \tag{5.35}$$

The relativistic velocity addition for collinear velocities gives

$$c_S = \frac{c_w + v}{1 + \frac{c_w v}{c^2}}. \tag{5.36}$$

If $v \ll c$ we use Taylor's theorem to expand

$$\frac{1}{1 + \frac{c_w v}{c^2}} \approx 1 - \frac{c_w v}{c^2} \tag{5.37}$$

and obtain (5.35) by neglecting terms which are quadratic in (v/c). This result was verified experimentally already in 1851 by Fizeau. □

More generally we note that if $v \ll c$ we can applying the expansions (5.11) and (5.37) to the general formula for relativistic velocity addition (5.32). Neglecting *all* terms in v/c we obtain the Galilean rule (5.4) for adding collinear velocities

$$u_x = u'_x + v, \quad u_y = u'_y, \quad u_z = u'_z. \tag{5.38}$$

6 Outlook: Momentum and energy in relativistic physics

Energy and momentum are important physical quantities because they are conserved in many physical processes. We shall now discuss one collisions viewed from two different inertial frames; we will see that, in order to have momentum conservation in both frames, we need to allow the mass of the colliding particles to vary with their speed. Consider the following elastic collision of two equal particles, displayed in Fig. 13.

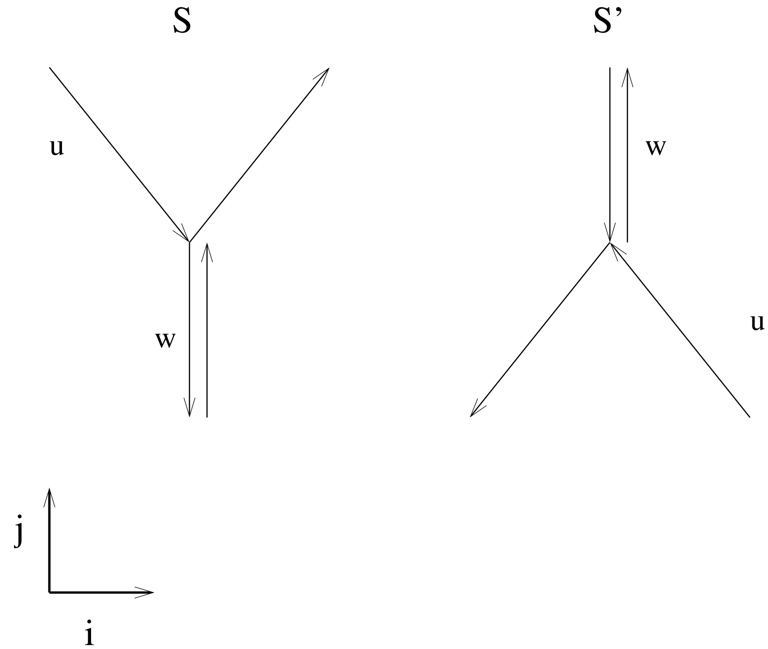

Figure 13: Momentum conservation in a relativistic collision, viewed in two inertial frames

Viewed in frame S, particle 1 (in the upper half of the plane) has velocity $\vec{u} = u_x\vec{i} - u_y\vec{j}$ before the collision, with u_x and u_y both positive; particle 2 (in the lower half of the plane) moves in positive $\vec{j}$-direction with speed w, so its velocity before the collision is $w\vec{j}$. The total momentum before the collision is thus

$$\vec{P} = m(u)u_x\vec{i} + (-m(u)u_y + m(w)w)\vec{j}, \tag{6.1}$$

where $u = \sqrt{u_x^2 + u_y^2}$ is the speed of particle 1, and $m(u)$ and $m(w)$ are the (speed-dependent) masses of the particles. Let us now further assume that our frame is chosen so that, after the collision, particle 2 moves in the negative $\vec{j}$ direction with speed w, i.e. its velocity is then $-w\vec{j}$. It then follows form the conservation of kinetic energy (even with speed-dependent masses!) that the speed of particle 1 is also unchanged. Furthermore, momentum conservation in the $\vec{i}$-direction implies that the $\vec{i}$-component of the first particle's velocity does not change. For the speed to remain unchanged, its $\vec{j}$-component can therefore only change sign in the collision. Thus, after the collision particle 1 has velocity $u_x\vec{i} + u_y\vec{j}$. Momentum conservation in the $\vec{j}$-direction requires

$$-m(u)u_y + m(w)w = m(u)u_y - m(w)w \Leftrightarrow m(u)u_y = m(w)w, \tag{6.2}$$

so that, in particular, the total momentum in the $\vec{j}$-direction is zero before and after the collision.

Now consider the same collision from the point of view of a frame S' which is moving with speed $v = u_x$ in the positive $\vec{i}$-direction. The collision as seen in S' is also shown in Fig. 13. Now particle 1 only moves in the $\vec{j}$-direction, but the velocity for particle 2 has a component in the $\vec{i}$-direction. What are the velocities of the particles in frame S'? We can work them out by applying the relativistic velocity addition; in particular, the $\vec{j}$-component of the first particle's velocity after the collision must be

$$u'_y = \frac{u_y}{\gamma(1 - \frac{u_x v}{c^2})} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v^2}{c^2}} u_y = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} u_y. \tag{6.3}$$

However, we can find the velocities in S' also by the following trick. Recall that in frame S particle 2 comes in along the negative $\vec{j}$ direction, whereas in frame S' particle 1 enters the collision along the positive $\vec{j}$ direction. However, there is no way of physically distinguishing between the positive and the negative $\vec{j}$-direction: there is no total momentum in this direction, and the particles are identical. Therefore, the pictures on the left and the right in Fig. 13 must be the same, only rotated by 180 degrees relative to each other! It follows that the velocity of particle 1 must be $-w\vec{j}$ before the collision and $w\vec{j}$ after the collision in frame S'. Comparing this with (6.3) we deduce

$$w = u'_y = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} u_y. \tag{6.4}$$

Inserting this result into (6.2) we find

$$\frac{m(u)}{m(w)} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{6.5}$$

Finally consider the limit where w tends to zero. In that limit $u = v$, so we find the following speed dependence of the mass:

$$m(u) = \frac{M}{\sqrt{1 - \frac{u^2}{c^2}}} = M\gamma(u), \tag{6.6}$$

where $M = m(0)$ is the mass of the particle when at rest, also called the rest mass. It is instructive to plot this function. For $u = 0$ we have $m(0) = M$ and for $0 < u < c$, m is increasing function which tends to infinity as $u \to c$. The mass of a particle approaches infinity as its speed approaches that of light! This is displayed in Fig. 14 for the case where the rest mass is 1.

With this new definition of the mass, the momentum of a particle moving with velocity $\vec{u}$ becomes

$$\vec{p} = m(u)\vec{u}. \tag{6.7}$$

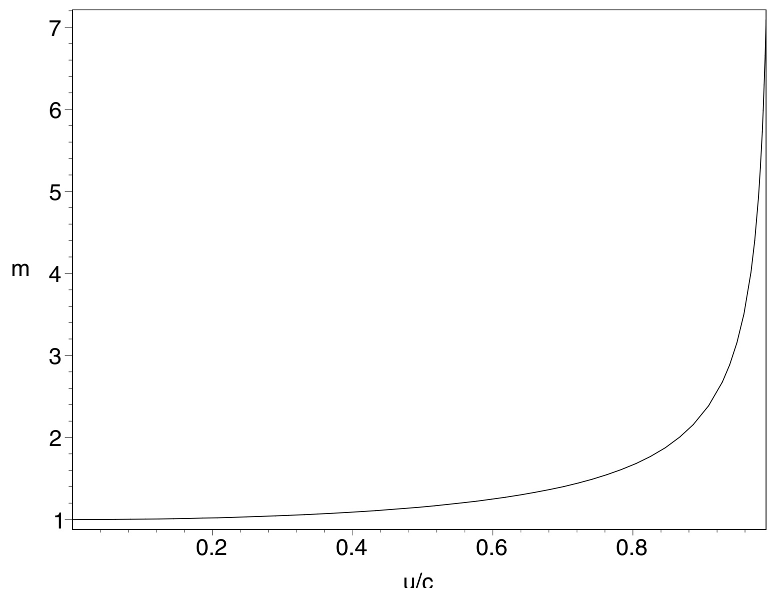

Figure 14: Relativistic mass as a function of $\frac{u}{c}$

In order to compute the kinetic energy of a particle moving with speed u we recall from (3.39) that in Newtonian physics the kinetic energy K and the momentum are related by the equation

$$\frac{dK}{dt} = \vec{u} \cdot \dot{\vec{p}} \tag{6.8}$$

We now use this formula to compute the relativistic kinetic energy K from the relativistic expression for the momentum i.e. we want to find K in (6.8) when $\vec{p} = m(u)\vec{u}$. First we note

$$\begin{aligned} \frac{d}{dt}K &= \frac{1}{m}\vec{p} \cdot \dot{\vec{p}} \\ &= \frac{1}{2m}\frac{d}{dt}\left(p^2\right), \end{aligned} \tag{6.9}$$

where $p = |\vec{p}|$ and the explicit dependence of m on u is suppressed. Before integrating this equation observe that

$$p^2 - m^2c^2 = m^2(u)(u^2 - c^2) = -M^2c^2. \tag{6.10}$$

Hence

$$m = \left(M^2 + \frac{p^2}{c^2}\right)^{\frac{1}{2}} \tag{6.11}$$

so that we can rewrite (6.9) as

$$\begin{aligned} \frac{d}{dt}K &= \frac{1}{2}\left(M^2 + \frac{p^2}{c^2}\right)^{-\frac{1}{2}} \frac{d}{dt}\left(p^2\right) \\ &= c^2\frac{d}{dt}\left(M^2 + \frac{p^2}{c^2}\right)^{\frac{1}{2}}. \end{aligned} \tag{6.12}$$

Integrating both sides we find

$$K = c^2 \left(M^2 + \frac{p^2}{c^2} \right)^{\frac{1}{2}} + B = c^2 m + B, \tag{6.13}$$

where B is a constant which we determine by requiring that the kinetic energy vanishes when the momentum vanishes. This yields $B = -c^2 M$. Using (6.11) we arrive at the following formula for relativistic kinetic energy:

$$K = c^2(m - M) = c^2 M(\gamma - 1), \tag{6.14}$$

or, equivalently,

$$m = M + \frac{K}{c^2}, \tag{6.15}$$

showing that the kinetic energy of a moving body contributes to its mass. In his second paper on special relativity, also written in 1905, Einstein went on to argue that **all** energy E, regardless of its form, contributes an amount E/c^2 to the mass of body. Since energy is only defined up to an additive constant we can therefore relate the mass of body with its energy content E according to the famous formula

$$E = mc^2. \tag{6.16}$$

The most dramatic evidence for the correctness of this formula came more than thirty years later in the context of nuclear fission. The uranium nucleus was found to break into consituents whose total mass was smaller than the mass of the original uranium nucleus. The missing mass had been converted into energy, precisely according to (6.16). Since c^2 is a huge number, the energy released in such a fission process is large even when the mass differences are tiny.