Solutions 1 for Oscillations and Waves

Module F12MS3 **2007-08**

1 (a) see Fig. 1

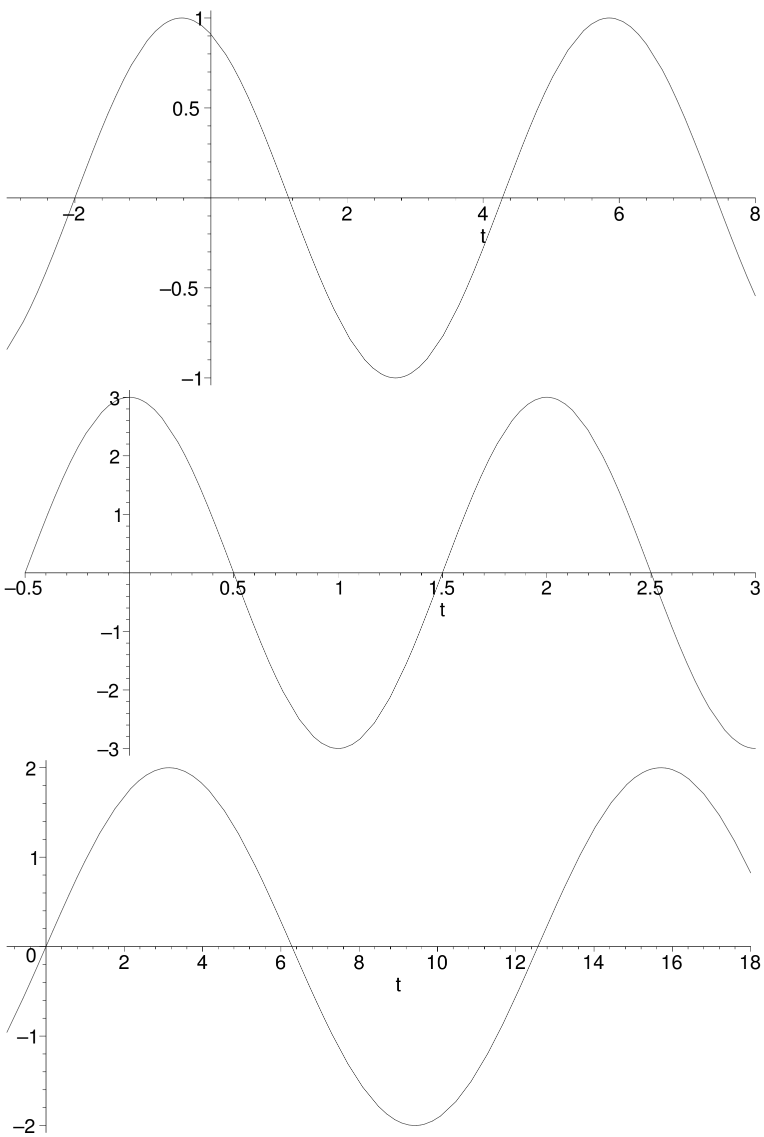

Figure 1: Plots of x_1 (top),x_2 (middle),x_3(bottom) as a function of t

(b) $y_1(t) = 2\sin(3t - \frac{\pi}{6}), \quad y_2(t) = \sqrt{2}\cos(\pi t + \frac{\pi}{4})$

(c) see Fig. 2

(d) $\dot{x_1}(t) = \cos(t+2), \quad \dot{x_2}(t) = -3\pi\sin(\pi t), \quad \dot{x_3}(t) = -\sin(\frac{t}{2} - \frac{\pi}{2}) = \cos(\frac{t}{2}),$
$\dot{y_1}(t) = 3\cos(3t) - 3\sqrt{3}\sin(3t), \quad \dot{y_2}(t) = -\pi\sin(\pi t) - \pi\cos(\pi t),$
$\dot{K}(t) = -4\pi\cos(2\pi t)\sin(2\pi t) = -2\pi\sin(4\pi t)$

(e) $(i)\ \int_0^{2\pi} \sin(t+2)\,dt = 0, \quad (ii)\ \int \cos\left(\frac{1}{2}(t+\pi)\right)\,dt = 2\sin\left(\frac{1}{2}(t+\pi)\right) + c,$
$(iii)\ \int_0^{\frac{1}{2}} \cos^2(2\pi t)\,dt = \frac{1}{2}\int_0^{\frac{1}{2}}(1+\cos(4\pi t))\,dt = \frac{1}{4}$

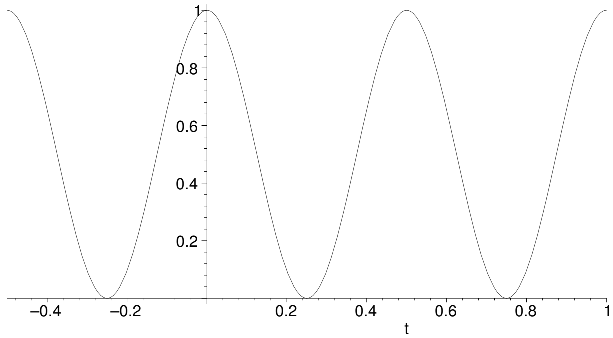

Figure 2: Plots of K as a function of t

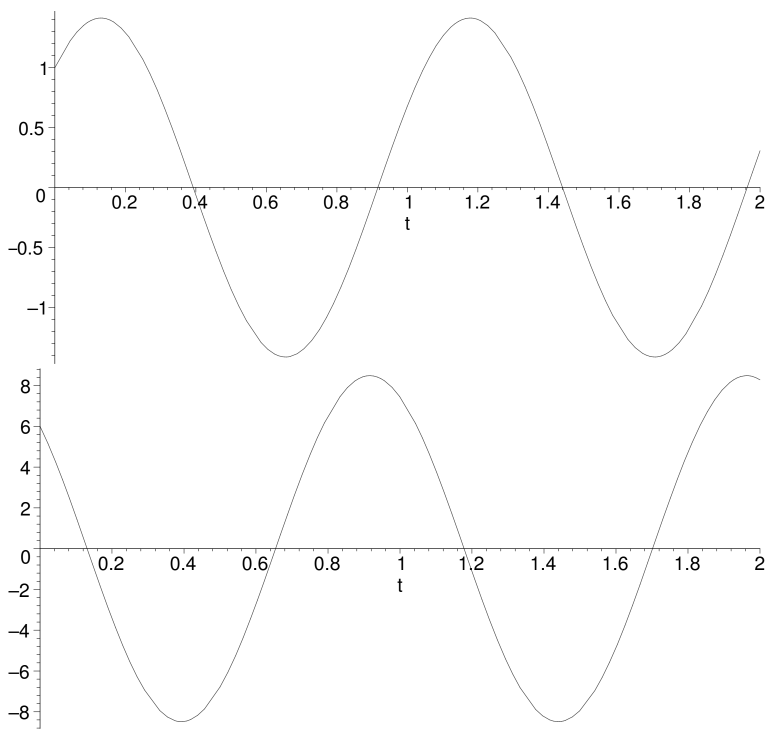

Figure 3: Plots of displacement x (top) and velocity $\dot{x}$ (bottom)

2 (a) The general solution of the equation of motion is $x(t) = A\cos(6t) + B\sin(6t)$, where A and B are two real constants. For the general solution $\dot{x}(t) = -6A\sin(6t) + 6B\cos(6t)$, hence $x(0) = A$ and $\dot{x}(0) = 6B$. To satisfy the initial conditions we have to choose $A = 1$ and $B = 1$. Thus the displacement at time t is given by $x(t) = \cos(6t) + \sin(6t)$, and the velocity is $\dot{x}(t) = -6\sin(6t) + 6\cos(6t)$; the plots are shown in Fig. 3

(b) The largest distance from the origin reached by the particle is the amplitude of the motion, given by $R = \sqrt{A^2 + B^2} = \sqrt{2}$.

(c) The kinetic energy, measured in Joules, is

$$K(t) = \frac{1}{2}\dot{x}^2 = \frac{1}{2}(36\sin^2(6t) - 72\sin(6t)\cos(6t) + 36\cos^2(6t)) = \frac{1}{2}(36 - 36\sin(12t)) = 18(1 - \sin(12t)).$$

The potential energy is

$$V(t) = \frac{k}{2}x^2 = \frac{36}{2}(\cos^2(6t) + 2\sin(6t)\cos(6t) + \sin^2(6t)) = 18(1 + \sin(12t)),$$

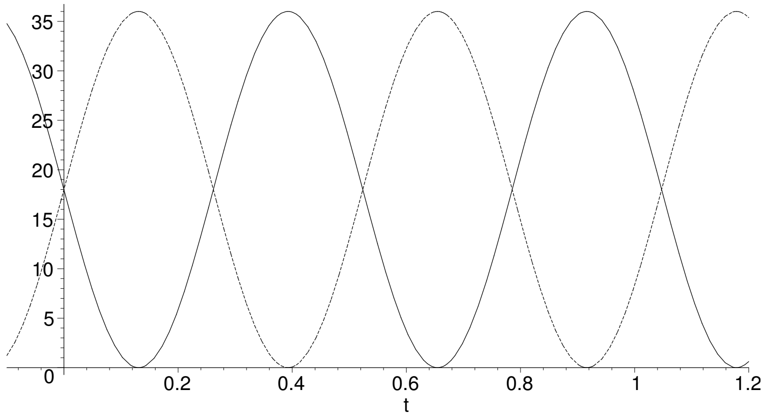

Figure 4: Plots of kinetic energy (solid line) and potential energy (dashed line) as a function of t

again measured in Joules. Hence the total eneryg is $E = K + V = 36$, measured in Joules. The plots are shown in Fig. 4

(d) The period of the motion is $T = \frac{2\pi}{6} = \frac{\pi}{3}$. Hence the average kinetic energy is

$$K_{av} = \frac{3}{\pi} \int_0^{\frac{\pi}{3}} 18(1 - \sin(12t)) = 18$$

and the average potential energy is

$$V_{av} = \frac{3}{\pi} \int_0^{\frac{\pi}{3}} 18(1 + \sin(12t)) = 18$$

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3 (a) According to the Google calculator, $M_E = 5.9742 \times 10^{24}$ kg and $R_E = 6378.1$ kilometres.

(b) Assuming a perfectly spherical shape, the total volume of the earth is $V_E = \frac{4\pi}{3} R_E^3$. The density of the earth is $\rho_E = M_E/V_E$, and therefore the mass inside a ball of radius r is

$$m(r) = \frac{4\pi}{3} r^3 \rho_E = M_E \frac{r^3}{R_E^3}.$$

(c) According to Newton's law of gravitational attraction, two masses M and m separated by a distance r will attract with a force of magnitude $F = GMm/r^2$, where G is the gravitational constant, equal to 6.67300×10^{-11} m^3 kg^{-1} s^{-2} according to Google. Thus, using the rule in italics, and denoting the mass of the capsule by m, the force on the capsule has magnitude $F = GM(r)m/r^2 = GM_E mr/R_E^3$ and is directed towards the centre of the earth.

(d) Hence the equation of motion (Newton's second law) reads

$$m\ddot{r} = -\frac{GmM_E}{R_E^3} r,$$

which is the equation of simple harmonic motion with angular frequency

$$\omega = \sqrt{\frac{GM_E}{R_E^3}}.$$

(e) The general solution of the equation found in (d) is

$$r(t) = A\cos(\omega t) + B\sin(\omega t).$$

If $r(0) = R_E$ (the capsule is released at the surface of the earth) and $\dot{r}(0) = 0$ (the capsule is released from rest) then the unique solution is

$$r(t) = R_E\cos(\omega t).$$

(d) Inserting the values for the various constants into the formula for ω given in (d) we find

$$\omega \approx 1.24 \times 10^{-3}\text{s}^{-1}.$$

so that the period of the motion is

$$T = \frac{2\pi}{\omega} = 5066\,\text{seconds} = \;84.4\,\text{minutes}.$$

The capsule reaches the centre of the earth after a quarter of the period, i.e. 21.1 minutes, and it reaches New Zealand after half a period i.e. 42.2 minutes. The speed at time t is $v(t) = |\dot{r}|(t) = R_E\omega|\sin(\omega t)|$. The speed when the capsule reaches the centre of the earth is

$$v\left(\frac{T}{4}\right) = R_E\omega\sin\left(\frac{\pi}{2}\right) = R_E\omega = 7909\text{m/s} = 2.85\times 10^4\text{km/h}.$$

The speed when the capsule reaches New Zealand is $v(T/2) = R_E\omega\sin(\pi) = 0$. The average speed for the journey is

$$v_{av} = \frac{2}{T}R_E\omega\int_0^{\frac{T}{2}}\sin(\omega t)\,dt = \frac{2}{\pi}R_E\omega = 5035\text{m/s} = 1.81\times 10^4\text{km/h}.$$