Solutions 2 for Oscillations and Waves

Module F12MS3 **2007-08**

1 (a) From Hooke's law applied to a mass m attached to a spring of spring constant k, we have $mg = kl$. With $l = \frac{49}{320}$ m and $m = 1$ kg we deduce $k = \frac{mg}{l} = 64$ N m^{-1}. Hence the equation of motion for the position x of the mass is

$$\ddot{x} = -64x.$$

The general solution of the equation of motion is $x(t) = A\cos(8t) + B\sin(8t)$, where A and B are two real constants. For the general solution $\dot{x}(t) = -8A\sin(8t) + 8B\cos(8t)$, hence $x(0) = A$ and $\dot{x}(0) = 8B$. The initial conditions are $x(0) = \frac{1}{4}$ and $\dot{x}(0) = -0.5$. To satisfy these we have to choose $A = \frac{1}{4}$ and $B = -\frac{1}{16}$. Thus the displacement at time t is given by $x(t) = \frac{1}{4}\cos(8t) - \frac{1}{16}\sin(8t)$

(b) The period of the motion is $T = \frac{2\pi}{8}$s$= \frac{\pi}{4}$s, and the amplitude of the motion is given by $R = \sqrt{A^2 + B^2}$m$= \frac{\sqrt{17}}{16}$ m.

2 The period is related to the characteristic frequency ω and hence the length of the pendulum via

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}.$$

Thus, if $T = 4$s, we deduce $\omega = \frac{\pi}{2}$ and we calculate the length of the pendulum from the period via

$$l = \frac{T^2 g}{4\pi^2} = 3.97\text{m}.$$

3 (a) The modulus is $|z| = \sqrt{3^2 + 4^2} = 5$, the argument is $\phi = \pi - \tan^{-1}(4/3) \approx 2.214$. Thus

$$z = 5e^{i\phi}, \quad \bar{z} = 5e^{-i\phi}, \quad \frac{1}{z} = \frac{1}{5}e^{-i\phi}$$

(b)

$$|z| = 2\sqrt{2} \qquad \arg(z) = -\frac{3\pi}{4}$$
$$|w| = 2 \qquad \arg(w) = \frac{\pi}{3}$$

(i)

$$\begin{aligned} |zw| &= |z||w| = 4\sqrt{2} \\ \arg(zw) &= \arg(z) + \arg(w) = -\frac{5}{12}\pi \end{aligned}$$

(ii)

$$\begin{aligned} \left|\frac{z}{w}\right| &= \frac{|z|}{|w|} = \sqrt{2} \\ \arg\left(\frac{z}{w}\right) &= \arg(z) - \arg(w) = -\frac{13}{12}\pi, \\ &\text{hence principal argument is } \frac{11}{12}\pi \qquad (1) \end{aligned}$$

These numbers are displayed in the Argand diagram 1

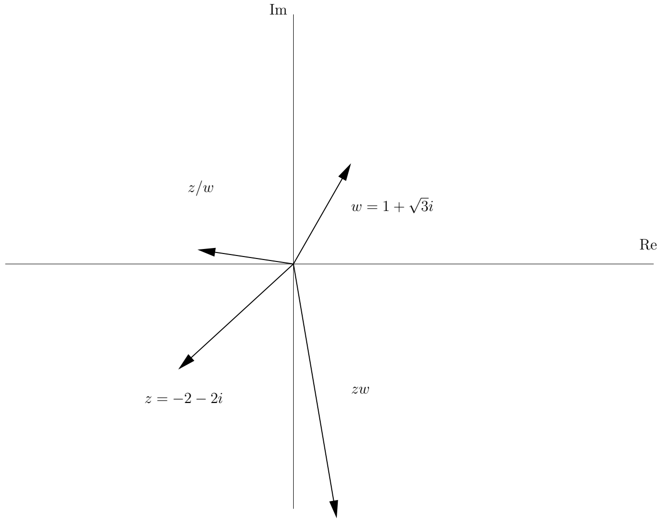

Figure 1: Argand diagram for (b)

(c) If $(1 - i/2)$ is one root of a quadratic equation with real coefficients then the other root must be $(1 + i/2)$. Hence equation is

$$\begin{aligned} (z - (1 - i/2))\,(z - (1 + i/2)) &= 0 \\ \Leftrightarrow z^2 - 2z + 5/4 &= 0 \\ \Leftrightarrow 4z^2 - 8z + 5 &= 0. \end{aligned}$$

(d) See Fig. 2.

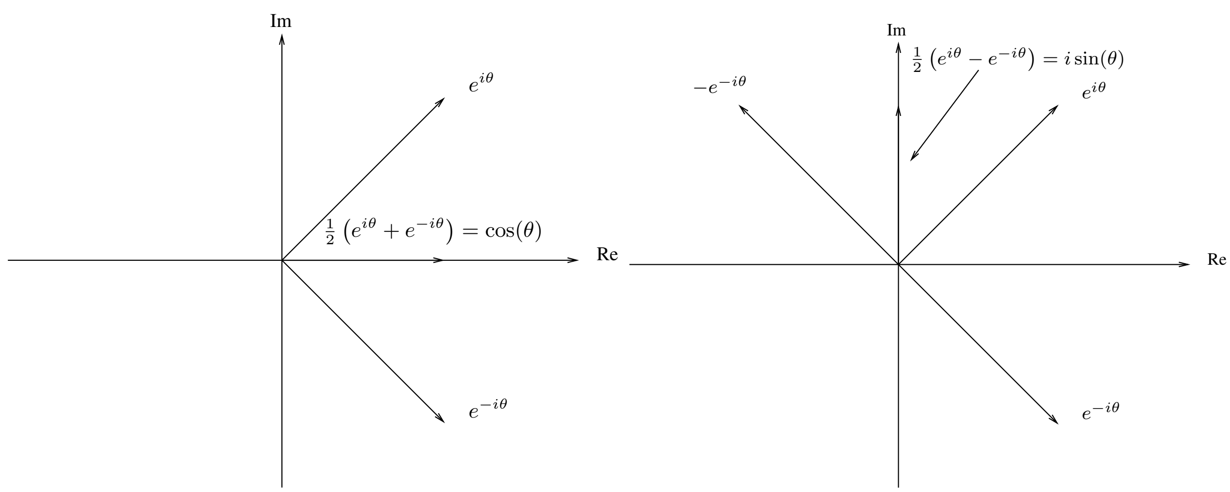

Figure 2: Argand diagrams for (d)

(e) $C = 5\sqrt{2}e^{-i\pi/4} \Rightarrow Cz(t) = 5\sqrt{2}e^{i(3t-\pi/4)}$. Differentiating $\dot{z}(t) = 3ie^{3it} = 3e^{i(3t+\pi/2)}$ and hence $C\dot{z}(t) = 15\sqrt{2}e^{i(3t+\pi/4)}$. Thus we find real and imaginary parts:

$$\mathrm{Re}[Cz(t)] = 5\sqrt{2}\cos(3t - \pi/4) \qquad \mathrm{Im}[Cz(t)] = 5\sqrt{2}\sin(3t - \pi/4)$$

and

$$\mathrm{Re}[C\dot{z}(t)] = 15\sqrt{2}\cos(3t + \pi/4) \qquad \mathrm{Im}[C\dot{z}(t)] = 15\sqrt{2}\sin(3t + \pi/4)$$

4 (a) The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = -2.$$

$\lambda_1 = \lambda_2$, roots are real and equal. Hence general solution is

$$x(t) = (A + Bt)e^{-2t},$$

where A and B are constants.

(b) The characteristic equation is

$$\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda_1 = 1 + i, \lambda_2 = 1 - i.$$

Roots are complex, hence the general solution is

$$x(t) = e^t \left(A\cos(t) + B\sin(t)\right),$$

where A and B are constants.

(c) Characteristic equation for the homogeneous equation is

$$\lambda^2 + 2\lambda + 4 = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = -2.$$

Therefore the complementary function (solution of the homogeneous equation) is

$$x(t)^{CF} = (A + Bt)e^{-2t},$$

where A and B are constants.
Particular solution: $f(t) = t^2$ so try

$$x(t) = a_2t^2 + a_1t + a_0.$$

On substitution this gives

$$2a_2 + 4(2a_2t + a_1) + 4(a_2t^2 + a_1t + a_0) = t^2.$$

By matching coefficients

$$x_p(t) = \frac{t^2}{4} - \frac{t}{2} + \frac{3}{8}.$$

The general solution is therefore

$$x(t) = (A + Bt)e^{-2t} + \frac{t^2}{4} - \frac{t}{2} + \frac{3}{8}.$$

(d) Characteristic equation for the homogeneous equation is

$$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 3$$

Therefore the complementary function is

$$x(t)^{CF} = Ae^t + Be^{3t}$$

where A and B are constants.
Particular solution: $f(t) = 2e^t$, e^t is in the CF so try

$$x(t) = ate^t.$$

Substituting and matching coefficients gives $a = -1$ and so

$$x_p(t) = -te^t.$$

The general solution is therefore

$$x(t) = Ae^t + Be^{3t} - te^t.$$

(e) Characteristic equation for the homogeneous equation is

$$\lambda^2 + 2\lambda + 5 = 0 \;\Rightarrow\; \lambda_1 = -1 + 2i,\, \lambda_2 = -1 - 2i.$$

Therefore, the complementary function is

$$x(t)^{CF} = e^{-t}\left(A\cos(2t) + B\sin(2t)\right),$$

where A and B are constants.
Particular solution: $f(t) = \cos(2t)$, so try $x(t) = Ce^{2it}$. Substituting into the complex equation

$$\ddot{x} + 2\dot{x} + 5x = e^{2it}$$

and matching coefficients gives $-4C + 4iC + 5C = 1$, or $C = \frac{1-4i}{17}$, so that

$$x(t) = \frac{1}{17}\left(1 - 4i\right)\left(\cos(2t) + i\sin(2t)\right),$$

the real part of which is the particular solution:

$$x_p(t) = \frac{1}{17}\left(\cos(2t) + 4\sin(2t)\right).$$

The general solution is therefore

$$x(t) = e^{-t}\left(A\cos(2t) + B\sin(2t)\right) + \frac{1}{17}\left(\cos(2t) + 4\sin(2t)\right).$$