Solutions 4 for Oscillations and Waves

Module F12MS3 **2007-08**

1 (a) The differential equation governing the motion of the object is

$$\ddot{x} + \dot{x} + 4x = 2\cos(\omega t). \tag{1}$$

The associated homogeneous equation has the characteristic polynomial $\lambda^2 + \lambda + 4$, with roots $\lambda_\pm = -\frac{1}{2} \pm i\frac{\sqrt{15}}{2}$. Hence the general solution of the homogeneous equation is

$$x^{CF}(t) = e^{-\frac{1}{2}t}\left(A\cos\left(\frac{\sqrt{15}}{2}t\right) + B\sin\left(\frac{\sqrt{15}}{2}t\right)\right).$$

In order to find a particular solution of (1) we solve the complex equation

$$\ddot{x} + \dot{x} + 4x = 2e^{i\omega t} \tag{2}$$

for a complex function x and take the real part of the solution. We try a solution of form $x(t) = Ce^{i\omega t}$. Differentiating and plugging into (2) gives $C = \dfrac{2}{4 - \omega^2 + i\omega}$ (see also lecture notes) or, in modulus-argument form

$$C = R(\omega)e^{-i\phi}, \quad R(\omega) = \frac{2}{\sqrt{(4-\omega^2)^2 + \omega^2}}, \quad \tan\phi = \frac{\omega}{4 - \omega^2} \tag{3}$$

Thus we have the particular solutions

$$x_p(t) = \mathrm{Re}\left(Ce^{i\omega t}\right) = R(\omega)\cos(\omega t - \phi). \tag{4}$$

and the general solution

$$x(t) = e^{-\frac{1}{2}t}\left(A\cos\left(\frac{\sqrt{15}}{2}t\right) + B\sin\left(\frac{\sqrt{15}}{2}t\right)\right) + R(\omega)\cos(\omega t - \phi). \tag{5}$$

(b) The steady state solution is the particular solution (4) found in (a). To find the angular frequency at which the amplitude $R(\omega)$ attains its maximum value, we differentiate $R(\omega)$ with respect to ω:

$$\frac{dR}{d\omega}(\omega) = \frac{14\omega - 4\omega^3}{((4-\omega^2)^2 + \omega^2)^{\frac{3}{2}}}. \tag{6}$$

The condition for a stationary point is $\dfrac{dR}{d\omega} = 0$, which is solved by $\omega_1 = -\sqrt{\dfrac{7}{2}}$, $\omega_2 = \sqrt{\dfrac{7}{2}}$, $\omega_3 = 0$. The only stationary point for positive ω is at

$$\omega_2 = \sqrt{\frac{7}{2}} \approx 1.87.$$

Since $R(\omega_2) = \frac{4}{\sqrt{15}} \approx 1.03 > R(0) = 0.5$ we conclude that R has as maximum at ω_2. For plot see Fig. 1

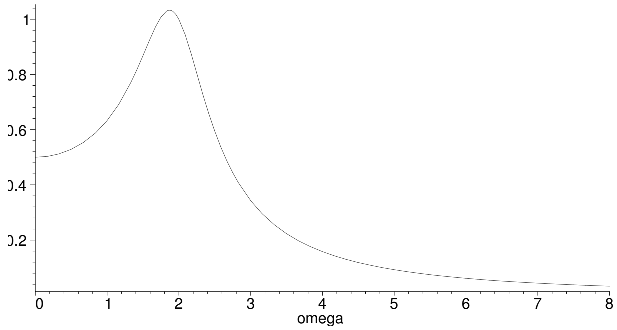

Figure 1: Amplitude R as a function of ω

(c) Resonance occurs when the driving frequency is equal to the characteristic frequency of the spring. In this case the characteristic frequency is $\omega_0 = 2\ \mathrm{s}^{-1}$, and the amplitude at the resonance frequency is $R(2) = 1$.

2 (a) When $r = r_1 = 0.5$ N sec m^{-1}, the amplitude and the phase of the steady state solution are

$$R(\omega) = \frac{2}{\sqrt{(4-\omega^2))^2 + \frac{1}{4}\omega^2}}, \qquad \phi = \tan^{-1}\left(\frac{\frac{1}{2}\omega}{4-\omega^2}\right) \tag{7}$$

When $r = r_2 = 0.1$ N sec m^{-1}, the amplitude and the phase of the steady state solution are

$$R(\omega) = \frac{2}{\sqrt{(4-\omega^2))^2 + \frac{1}{100}\omega^2}}, \qquad \phi = \tan^{-1}\left(\frac{\frac{1}{10}\omega}{4-\omega^2}\right) \tag{8}$$

(b) The Plot of the amplitude and the phase for $r = r_1$ is shown in figure 2. The Plot of the amplitude and the phase for $r = r_2$ is shown in figure 3.

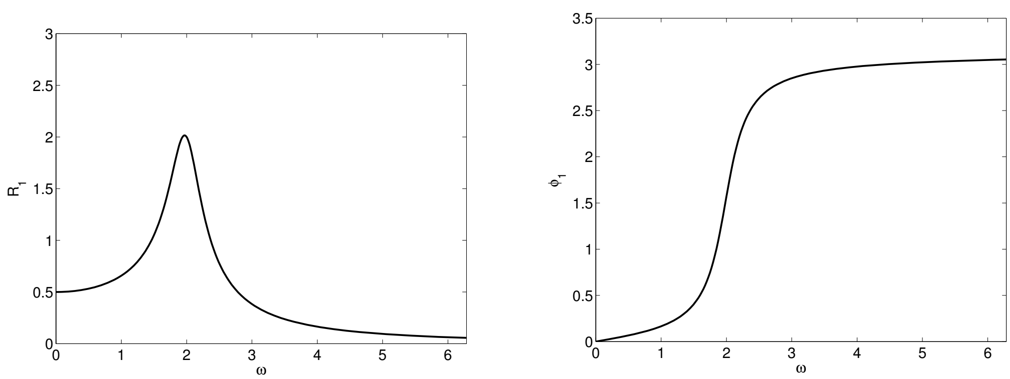

Figure 2: Amplitude and phase as a function of ω for damping r_1

(c) The Plot of the steady solutions (at resonance) and the driving force are shown figure 4.

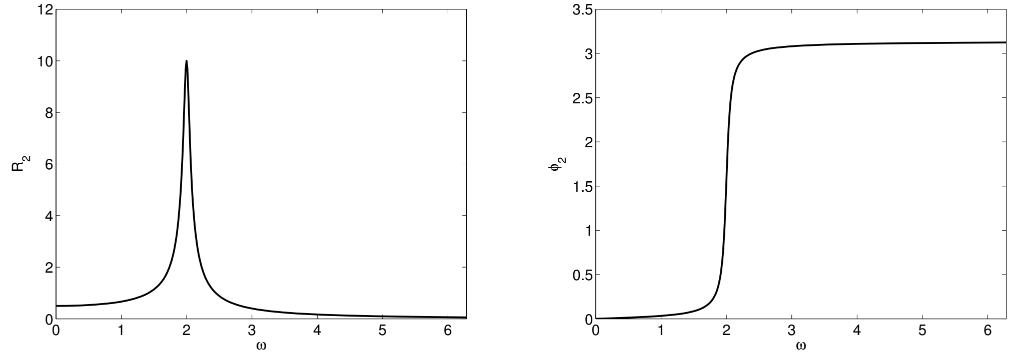

Figure 3: Amplitude and phase as a function of ω for damping r_2

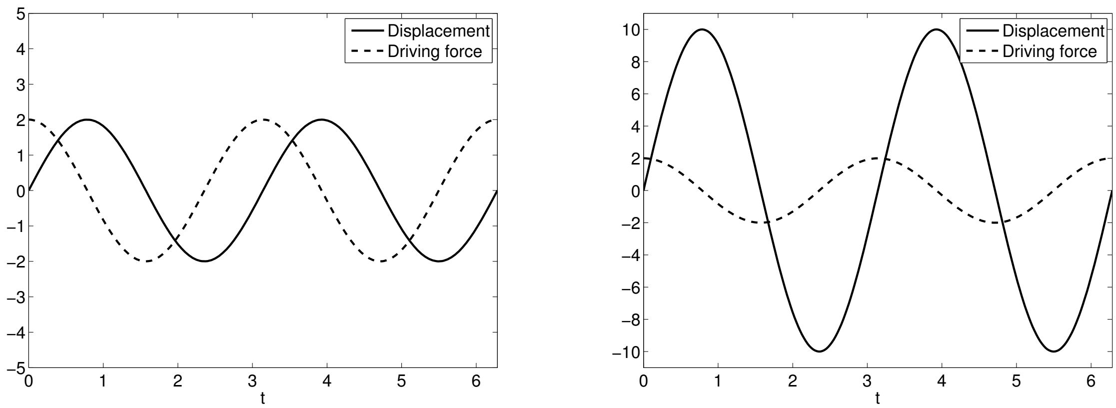

Figure 4: Particular solution and driving force for weak damping r_1 (left) and very weak damping r_2 (right)

3 Let $x(t)$ be the displacement from the equilibrium at time t; The equation of motion for x is

$$\ddot{x} + 16x = \frac{A}{4}\cos^3(pt). \tag{9}$$

To solve this equation, we need to expand $\cos^3(pt)$ in terms of cos. We have

$$\begin{aligned}
\cos^3(pt) &= \cos(pt)\cos^2(pt) && (10)\\
&= \cos(pt)\frac{1+\cos(2pt)}{2} && (11)\\
&= \frac{1}{2}\cos(pt) + \frac{1}{2}\cos(pt)\cos(2pt) && (12)
\end{aligned}$$

Since $\cos(pt)\cos(2pt) = \frac{1}{2}(\cos(pt) + \cos(3pt))$ we finally obtain

$$\cos^3(pt) = \frac{3}{4}\cos(pt) + \frac{1}{4}\cos(3pt) \tag{13}$$

Thus the equation of motion (9) becomes

$$\ddot{x} + 16x = \frac{3A}{16}\cos(pt) + \frac{A}{16}\cos(3pt) \tag{14}$$

As shown on sheet 3, a particular solution of this equation is given by the sum of particular solutions of

$$\ddot{x} + 16x = \frac{3A}{16}\cos(pt) \tag{15}$$

and

$$\ddot{x} + 16x = \frac{A}{16}\cos(3pt). \tag{16}$$

Resonance occurs in the system when the natural angular frequency $\omega_0 = 4\ \mathrm{s}^{-1}$ is equal to the driving frequency ω. There is resonance in (14) is there is resonance in either (15) or (16). This means there is resonance in (14) if $p = 4$ or $3p = 4$ or

$$p = 4 \qquad \text{or} \qquad p = \frac{4}{3}$$

4 Find the eigenvalues from the equation

$$\det\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} = 0 \Leftrightarrow (\lambda - a)^2 = b^2$$

i.e. the eigenvalues are $\lambda_+ = a + b$ and $\lambda_- = a - b$. The defining equation for the eigenvector v_+ for the eigenvalue λ_+ is

$$\begin{pmatrix} -b & b \\ b & -b \end{pmatrix} v_+ = 0,$$

which is solved by $v_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The defining equation for the eigenvector v_- for the eigenvalue λ_- is

$$\begin{pmatrix} b & b \\ b & b \end{pmatrix} v_- = 0,$$

which is solved by $v_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

5 (a) Use the trigonometrical identities from the formula sheet to find

$$\begin{aligned}
\cos(2t) + \sin\left(5t + \frac{\pi}{3}\right) &= \sin\left(\frac{\pi}{2} - 2t\right) + \sin\left(5t + \frac{\pi}{3}\right) \\
&= 2\sin\left(\frac{1}{2}\left(\frac{\pi}{2} - 2t + 5t + \frac{\pi}{3}\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2} - 2t - 5t - \frac{\pi}{3}\right)\right) \\
&= 2\sin\left(\frac{3}{2}t + \frac{5\pi}{12}\right)\cos\left(\frac{7}{2}t - \frac{\pi}{12}\right)
\end{aligned} \tag{17}$$

The plot is shown in Fig. 5.

(b) Again using trigonometrical identities

$$\begin{aligned}
2\cos\left(4t + \frac{\pi}{4}\right) + 2\cos(5t) &= 4\cos\left(\frac{1}{2}\left(4t + \frac{\pi}{4} + 5t\right)\right)\cos\left(\frac{1}{2}\left(4t + \frac{\pi}{4} - 5t\right)\right) \\
&= 4\cos\left(\frac{9}{2}t + \frac{\pi}{8}\right)\cos\left(-\frac{1}{2}t + \frac{\pi}{8}\right).
\end{aligned} \tag{18}$$

The plot is shown in Fig. 6.

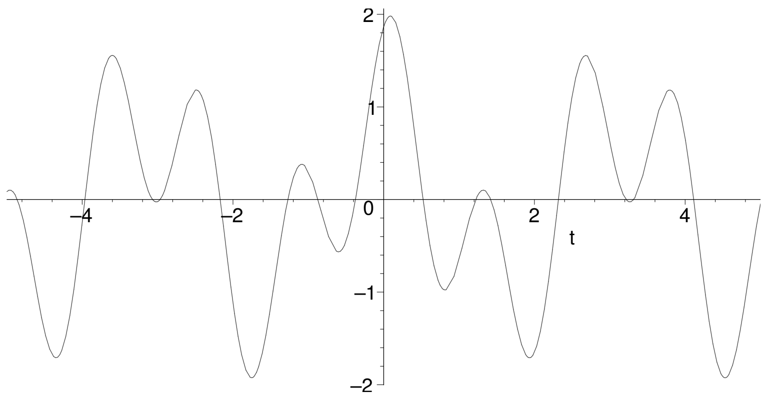

Figure 5: The function $\cos(2t) + \sin\left(5t + \frac{\pi}{3}\right)$

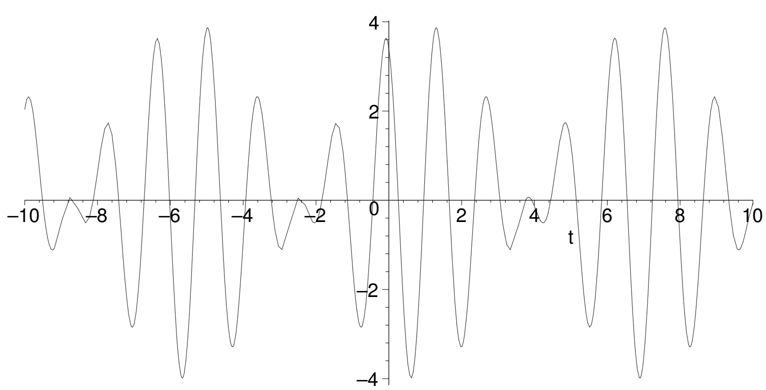

Figure 6: The function $2\cos\left(4t + \frac{\pi}{4}\right) + 2\cos(5t)$