Solutions 5 for Oscillations and Waves

Module F12MS3 **2007-08**

1 (a) With the numerical values for the parameters given in the question, the equations of motion are

$$\begin{aligned}\ddot{x}_1 &= -9x_1 - 8(x_1 - x_2)\\ \ddot{x}_2 &= -9x_2 - 8(x_2 - x_1).\end{aligned} \tag{1}$$

(b) The normal modes are

$$y_1 = x_1 + x_2, \qquad y_2 = x_1 - x_2. \tag{2}$$

By adding and subtracting the equations in (1) we deduce the normal mode equations

$$\begin{aligned}\ddot{y}_1 &= -9y_1\\ \ddot{y}_2 &= -(9+16)y_2 = -25y_2,\end{aligned} \tag{3}$$

which are solved by

$$y_1(t) = A_1\cos(3t) + B_1\sin(3t), \quad y_2(t) = A_2\cos(5t) + B_2\sin(5t). \tag{4}$$

The normal mode angular frequencies are 3 s^{-1} for the first and 5 s^{-1} for the second normal mode.

(c) The initial condition

$$x_1(0) = 1, \qquad x_2(0) = 0, \qquad \dot{x}_1(0) = 0, \qquad \dot{x}_2(0) = 0 \tag{5}$$

imply the following initial conditions for the normal modes

$$y_1(0) = 1, \qquad y_2(0) = 1, \qquad \dot{y}_1(0) = 0, \qquad \dot{y}_2(0) = 0. \tag{6}$$

Imposing these conditions on the general solution (4) we deduce

$$y_1(t) = \cos(3t), \qquad y_2(t) = \cos(5t), \tag{7}$$

and, by inverting (2)

$$x_1(t) = \frac{1}{2}(\cos(3t) + \cos(5t)) \qquad x_2(t) = \frac{1}{2}(\cos(3t) - \cos(5t)). \tag{8}$$

The displacements are plotted in in Fig. 1.

(d) The initial condition

$$x_1(0) = 0, \qquad x_2(0) = 0, \qquad \dot{x}_1(0) = 0, \qquad \dot{x}_2(0) = 5 \tag{9}$$

imply the following initial conditions for the normal modes

$$y_1(0) = 0, \qquad y_2(0) = 0, \qquad \dot{y}_1(0) = 5, \qquad \dot{y}_2(0) = -5. \tag{10}$$

Imposing these conditions on the general solution (4) we deduce

$$y_1(t) = \frac{5}{3}\sin(3t), \qquad y_2(t) = -\sin(5t), \tag{11}$$

and, by inverting (2)

$$x_1(t) = \frac{1}{2}(\frac{5}{3}\sin(3t) - \sin(5t)) \qquad x_2(t) = \frac{1}{2}(\frac{5}{3}\sin(3t) + \sin(5t)). \tag{12}$$

The displacements are plotted in Fig. 2.

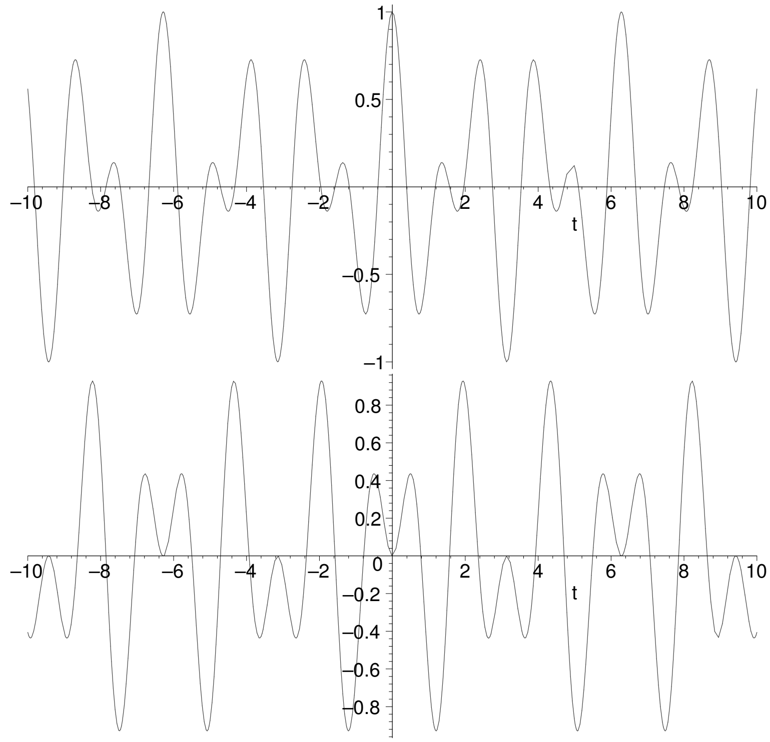

Figure 1: The displacements x_1 and x_2 for the solution of 1(c)

2 (a) The equations of motion are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - K(x_1 - x_2) \\ m\ddot{x}_2 &= -kx_2 - K(x_2 - x_1) \end{aligned} \tag{13}$$

(b)

$$\begin{aligned} \frac{dE}{dt} &= m\dot{x}_1\ddot{x}_1 + m\dot{x}_2\ddot{x}_2 + kx_1\dot{x}_1 + kx_2\dot{x}_2 \\ &\quad + K(x_1 - x_2)(\dot{x}_1 - \dot{x}_2) \\ &= \dot{x}_1(m\ddot{x}_1 + kx_1 + K(x_1 - x_2)) \\ &\quad + \dot{x}_2(m\ddot{x}_2 + kx_2 - K(x_1 - x_2)) \end{aligned} \tag{14}$$

Substituting (13) into (14) we find $\frac{dE}{dt} = 0$. Therefore energy is constant (conserved) during the motion.

(c) Expressing the positions in terms of the normal modes coordinates

$$x_1 = \tfrac{1}{\sqrt{2}}(y_1 + y_2) \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2}}(y_1 - y_2)$$

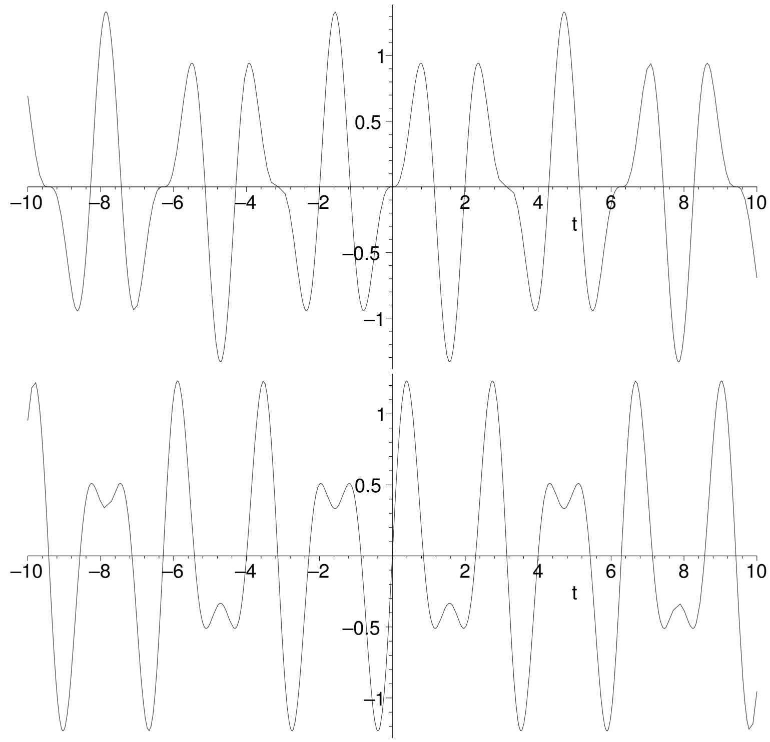

Figure 2: The displacements x_1 and x_2 for the solution to 1(d)

and substituting into the expression for E we find:

$$\begin{aligned}
E &= \frac{1}{2}m\left(\frac{1}{2}\dot{y}_1^2 + \frac{1}{2}\dot{y}_2^2 + \dot{y}_1\dot{y}_2\right) + \frac{1}{2}m\left(\frac{1}{2}\dot{y}_1^2 + \frac{1}{2}\dot{y}_2^2 - \dot{y}_1\dot{y}_2\right) \\
&\quad + \frac{1}{2}k\left(\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + y_1y_2\right) + \frac{1}{2}k\left(\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 - y_1y_2\right) \\
&\quad + Ky_2^2 \\
&= \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}m\dot{y}_2^2 + \frac{1}{2}ky_1^2 + \frac{1}{2}ky_2^2 + Ky_2^2 \\
&= \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}ky_1^2 + \frac{1}{2}m\dot{y}_2^2 + \frac{1}{2}(k+2K)y_2^2
\end{aligned}$$

Thus, the energy is the sum of

$$\frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}ky_1^2 : \text{ energy of mass } m \text{ on a spring with spring constant } k,$$

and

$$\frac{1}{2}m\dot{y}_2^2 + \frac{1}{2}(k+2K)y_2^2 : \text{ energy of mass } m \text{ on a spring with spring constant } k+2K$$

Thus each normal mode contributes to the energy as if it was a free oscillator. The total energy is the sum of these contributions.

3 (a) Denote the tension by τ. Then the equations of motion for the transverse displacements

z_1, z_2 and z_3 are

$$\begin{aligned}
\ddot{z}_1 &= -\frac{\tau}{lm}(2z_1 - z_2) \\
\ddot{z}_2 &= -\frac{\tau}{lm}(-z_1 + 2z_2 - z_3) \\
\ddot{z}_3 &= -\frac{\tau}{lm}(-z_2 + 2z_3)
\end{aligned}$$

Inserting numerical values:

$$\frac{\tau}{lm} = \frac{1}{0.5 \times 0.02} = 100s^{-2}$$

$$\begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{z}_3 \end{pmatrix} = -100 \underbrace{\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}}_{\text{M}} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

(b) In the eigenvector method we start by finding eigenvalues and eigenvectors of the matrix M:

$$\begin{aligned}
\begin{vmatrix} \lambda - 2 & 1 & 0 \\ 1 & \lambda - 2 & 1 \\ 0 & 1 & \lambda - 2 \end{vmatrix} &= 0 \\
\Leftrightarrow (\lambda - 2)^3 - 2(\lambda - 2) &= 0 \\
\Leftrightarrow (\lambda - 2)((\lambda - 2)^2 - 2) &= 0 \\
\Leftrightarrow \lambda = 2 \quad \text{or} \quad \lambda = 2 - \sqrt{2} \quad \text{or} \quad \lambda = 2 + \sqrt{2} &
\end{aligned}$$

Eigenvector for $\lambda = 2 - \sqrt{2}$ is $\vec{v}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$, corresponding angular frequency is $\omega_1 = \sqrt{100(2 - \sqrt{2})}s^{-1} \approx 7.7s^{-1}$.

Eigenvector for $\lambda = 2$ is $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, corresponding angular frequency is $\omega_2 = \sqrt{100 \times 2}s^{-1} \approx 14s^{-1}$.

Eigenvector for $\lambda = 2 + \sqrt{2}$ is $\vec{v}_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$, corresponding angular frequency is $\omega_3 = \sqrt{100(2 + \sqrt{2})}s^{-1} \approx 18.5s^{-1}$.

Hence the normal modes are

$$\begin{aligned}
\vec{z}_1 &= (A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t))\, \vec{v}_1 \\
\vec{z}_2 &= (A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t))\, \vec{v}_2 \\
\vec{z}_3 &= (A_3 \cos(\omega_3 t) + B_3 \sin(\omega_3 t))\, \vec{v}_3
\end{aligned}$$

Qualitative description:
Mode 1: All free beads oscillate in tandem.
Mode 2: Central bead at rest, outer beads oscillate in opposition.
Mode 3: Outer beads oscillate in tandem, inner bead in opposition.
The modes are sketched in Fig. 3

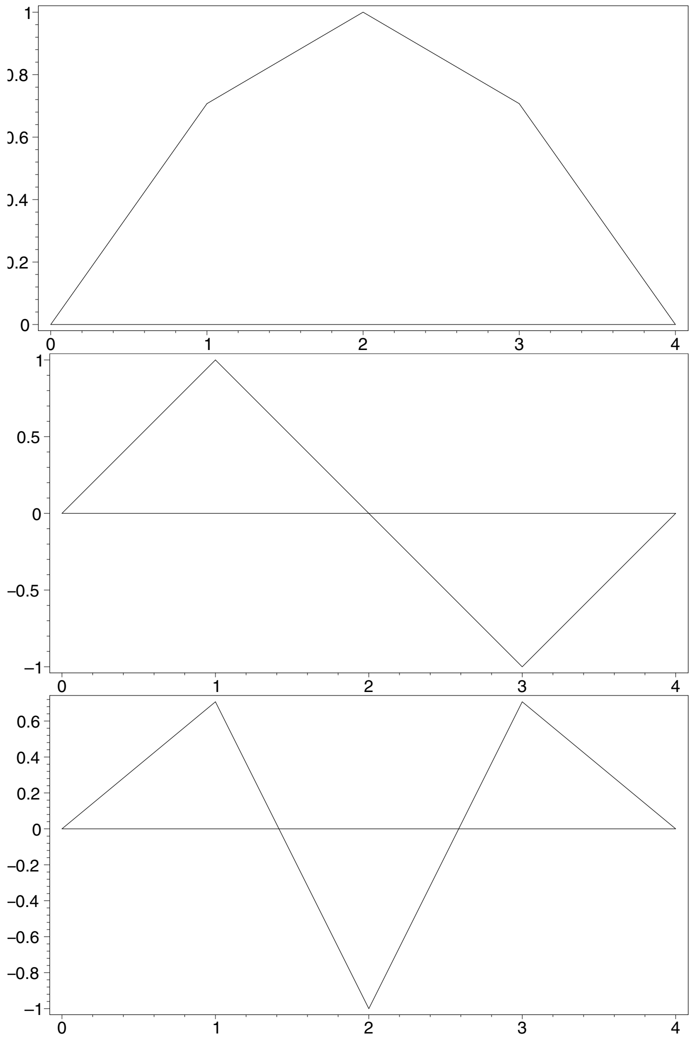

Figure 3: Normal modes of three beads on a string

4 Differentiating $f(x,t) = \sin(kx)\cos(\omega t)$ we find

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= -k^2 \sin(kx)\cos(\omega t) \\ \frac{\partial^2 f}{\partial t^2} &= -\omega^2 \sin(kx)\cos(\omega t) \end{aligned} \tag{15}$$

Hence

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

if $\omega^2 = c^2(k^2)$ or $c = \frac{\omega}{k}$ (since k, ω, and c are all positive).