Solutions 6 for Oscillations and Waves

Module F12MS3 **2007-08**

1 (a) Wave equation is

$$\frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2},$$

where t is time, x is the coordinate along the string and $z(x,t)$ is the transverse displacement from the equilibrium position.
$v = \sqrt{\tau/\mu}$, τ: tension and μ: mass density.
With $\tau = 40\text{N}$, $\mu = 1\text{kg}/10\text{m} = 0.1\text{kgm}^{-1}$

$$v = \sqrt{\frac{40}{0.1}} = 20\text{ms}^{-1}$$

(b) Normal modes

$$z_n(x,t) = (A_n \cos(\omega_n t) + B_n \sin(\omega_n t)) \sin(k_n x)$$

where $k_n = \frac{n\pi}{10}$ m^{-1}, $\omega_n = v k_n = 2n\pi$ s^{-1}.
The lowest three modes at time $t = 0$ are sketched in Fig. 1.

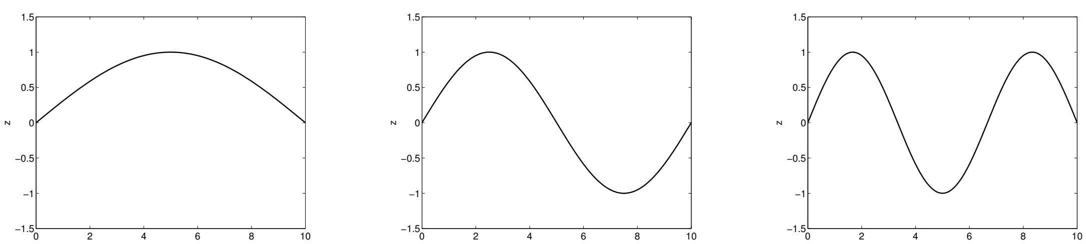

Figure 1: Lowest three normal modes of vibrating string

(c) Wavelength: $\lambda_n = 2\pi k_n = \frac{20}{n}$ m
Angular frequency: $\omega_n = 2n\pi$ s^{-1}
Period: $T_n = \frac{2\pi}{\omega_n} = \frac{1}{n}$ s
Frequency: $\nu_n = \frac{1}{T_n} = n$ Hertz

2 We use two basic facts about integration:

- Substitution:

$$\int_a^b f(y(x))\frac{dy}{dx}\,dx = \int_{y(a)}^{y(b)} f(y)\,dy$$

- Exchange of limits:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Apply this to the odd function f, and $y = -x$. Then, by the substitution rule

$$\int_0^L f(-x)(-1)\,dx = \int_0^{-L} f(y)\,dy = \int_0^{-L} f(x)\,dx$$

(y is a dummy variable). So, since $-f(-x) = f(x)$, we have

$$\int_0^L f(x)\,dx = \int_0^{-L} f(x)\,dx.$$

Now using the exchange of limits rule

$$\int_0^{-L} f(x)\,dx = -\int_{-L}^0 f(x)\,dx,$$

hence

$$\begin{aligned}
\int_{-L}^L f(x)\,dx &= \int_{-L}^0 f(x)\,dx + \int_0^L f(x)\,dx \\
&= \int_{-L}^0 f(x)\,dx + \left(-\int_{-L}^0 f(x)\,dx\right) \\
&= 0.
\end{aligned}$$

3 (a) $g(x) = |x|$, $-\pi < x < \pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx \\
&= \frac{2}{\pi}\left(\left[\frac{x}{n}\sin(nx)\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\right) \\
&= \frac{2}{\pi n^2}[\cos(nx)]_0^{\pi} \\
&= \frac{2}{\pi n^2}((-1)^n - 1) \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin(nx)\,dx = 0.
\end{aligned}$$

So

$$\begin{aligned}
|x| &= \sum_{n=1}^{\infty} \frac{2}{\pi n^2}((-1)^n - 1)\cos(nx) \\
&\approx -\frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x) - \frac{4}{25\pi}\cos(5x)\ldots
\end{aligned}$$

The function to which the Fourier series converges is sketched in Fig. 2.

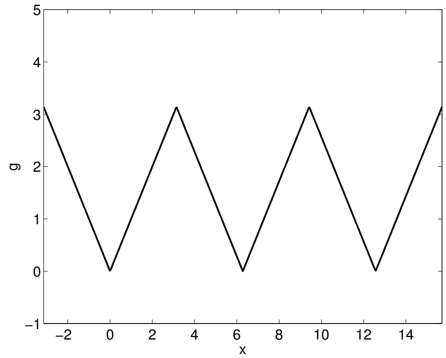

Figure 2: The function to which the Fourier series in 3(a) converges

(b) $h(x) = 1 - x^2$, $-1 < x < 1$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-1}^{1} (1-x^2)\,dx = \frac{1}{2}\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \\
a_n &= \int_{-1}^{1} (1-x^2)\cos(n\pi x)\,dx \\
&= \left[(1-x^2)\frac{1}{n\pi}\sin(n\pi x)\right]_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1} 2x\sin(n\pi x) \\
&= \left[-\frac{1}{(n\pi)^2}2x\cos(n\pi x)\right]_{-1}^{1} + \underbrace{\frac{2}{(n\pi)^2}\int_{-1}^{1}\cos(n\pi x)\,dx}_{=0} \\
&= \frac{2}{n^2\pi^2}\left((-1)^n + (-1)^n\right) \\
&= \frac{4}{n^2\pi^2}(-1)^{n+1} \\
b_n &= \int_{-1}^{1} (1-x^2)\sin(n\pi x)\,dx = 0.
\end{aligned}$$

So

$$\begin{aligned}
1 - x^2 &= \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^{n+1}\cos(n\pi x) \\
&\approx \frac{2}{3} + \frac{4}{\pi^2}\cos(\pi x) - \frac{1}{\pi^2}\cos(2\pi x) + \frac{4}{9\pi^2}\cos(3\pi x)\ldots
\end{aligned}$$

The function to which the Fourier series converges is sketched in Fig. 3.

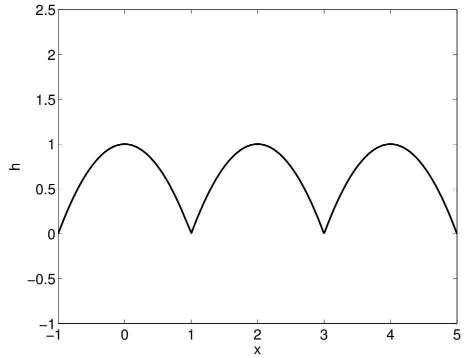

Figure 3: The function to which the Fourier series in 3(b) converges

4 (a)

$$z(x,t) = f(x - vt) + f(x + vt)$$

Then

$$\begin{aligned}
\frac{\partial z}{\partial t}(x,t) &= f'(x - vt)(-v) + f'(x + vt)v \\
&= vf'(x + vt) - vf'(x - vt) \\
\frac{\partial^2 z}{\partial t^2}(x,t) &= v^2 f''(x + vt) + v^2 f''(x - vt)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial z}{\partial x}(x,t) &= f'(x - vt) + f'(x + vt) \\
\frac{\partial^2 z}{\partial x^2}(x,t) &= f''(x - vt) + f''(x + vt).
\end{aligned}$$

So

$$\frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} = f''(x - vt) + f''(x + vt) = \frac{\partial^2 z}{\partial x^2}.$$

(b)

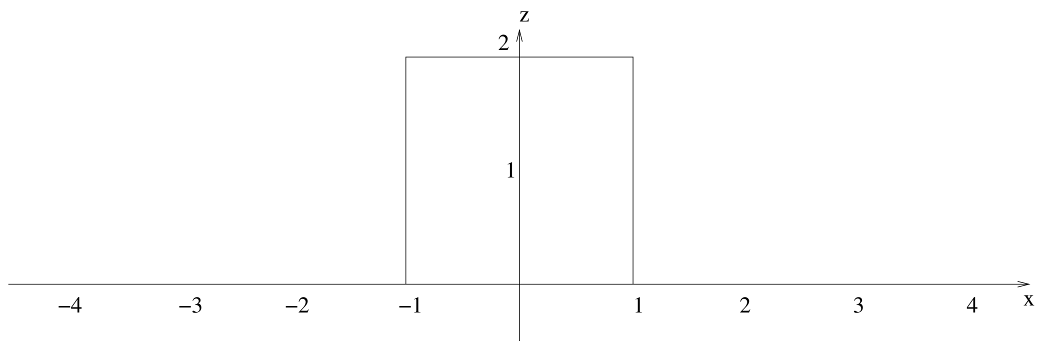

Figure 4: The wave at $t = 0$: $z(x, t = 0) = 2f(x)$

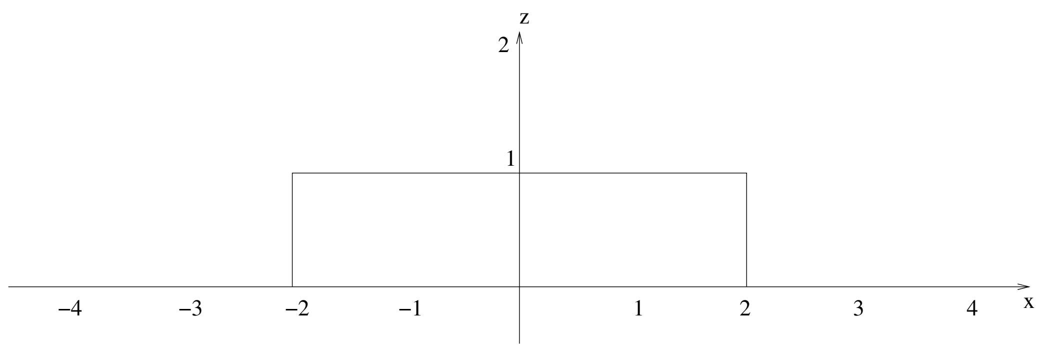

Figure 5: The wave at $t = 1$: $z(x, t = 1) = f(x - 1) + f(x + 1)$

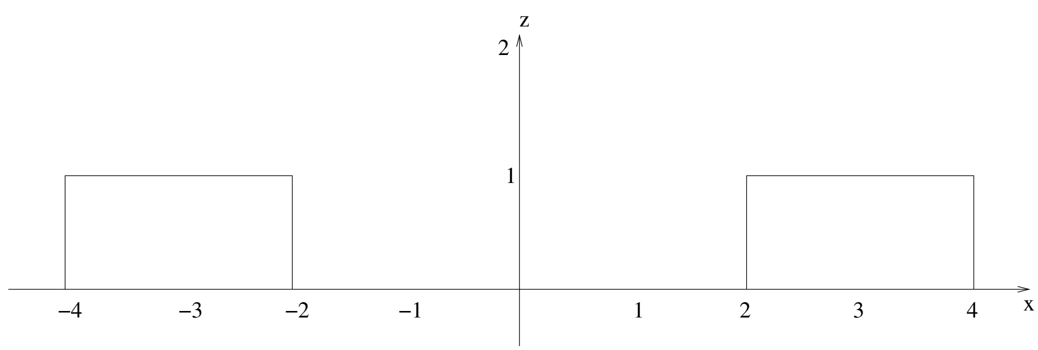

Figure 6: The wave at $t = 3$: $z(x, t = 3) = f(x - 3) + f(x + 3)$