5 Boundary value problems and Green's functions

Many of the lectures so far have been concerned with the initial value problem

$$L[y] = f(x), \quad y(x_0) = \alpha, \ y'(x_0) = \beta, \tag{5.1}$$

where L is the differential operator

$$L[y] = \frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y. \tag{5.2}$$

From Picards' theorem we know that, if a_1 and a_0 are smooth everywhere, then a unique solution of (5.1) exists everywhere. We have also developed an arsenal of methods for finding that solution.

In this last section of the course we look at **boundary value problems**, where we solve a differential equation subject to conditions imposed at two different points $x = a$ and $x = b$. The most general boundary value problem we will consider is

$$L[y] = f(x), \quad B_a(y) = 0, \quad B_b[y] = 0, \tag{5.3}$$

where we have used the abbreviation

$$B_a[y] = \alpha_1 y(a) + \beta_1 y'(a) \quad \text{and} \quad B_b[y] = \alpha_2 y(b) + \beta_2 y'(b). \tag{5.4}$$

Choosing, for example, $\beta_1 = \beta_2 = 0$ and $\alpha_1 = \alpha_2 = 1$ we obtain the condition that y vanishes at a and b. This boundary condition arises physically for example if we study the shape of a rope which is fixed at two points a and b. Choosing $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1$ we obtain $y'(a) = y'(b) = 0$. The general conditions we impose at a and b involve both y and y'.

Unlike initial value problems, **boundary value problems do not always have solutions**, as the following example illustrates. Suppose we try to solve

$$y'' + y = f(x), \quad y(0) = y(\pi) = 0. \tag{5.5}$$

Multiplying the equation by $\sin x$ and integrating yields

$$\begin{aligned}
\int_0^\pi f(x)\sin x\,dx &= \int_0^\pi y''(x)\sin x\,dx + \int_0^\pi y(x)\sin x\,dx \\
&= y'(x)\sin x|_0^\pi - \int_0^\pi y'(x)\cos x\,dx + \int_0^\pi y(x)\sin x\,dx \\
&= -y(x)\cos x|_0^\pi - \int_0^\pi y(x)\sin x\,dx + \int_0^\pi y(x)\sin x\,dx \qquad (5.6) \\
&= 0. \qquad (5.7)
\end{aligned}$$

Thus a necessary condition for (5.5) to have a solution is

$$\int_0^\pi f(x)\sin x\,dx = 0 \tag{5.8}$$

This is not satisfied, for example, if $f(x) = x$.

We shall now explain how to find solutions to boundary value problems in the cases where they exist. Our main tool will be Green's functions, named after the English mathematician George Green (1793-1841).

A Green's function is constructed out of two independent solutions y_1 and y_2 of the homogeneous equation

$$L[y] = 0. \tag{5.9}$$

More precisely, let y_1 be the unique solution of the initial value problem

$$L[y] = 0, \quad y(a) = \beta_1, \quad y'(a) = -\alpha_1 \tag{5.10}$$

and y_2 be the unique solution of

$$L[y] = 0, \quad y(b) = \beta_2, \quad y'(b) = -\alpha_2. \tag{5.11}$$

These solutions thus satisfy

$$B_a[y_1] = 0 \quad \text{and} \quad B_b[y_2] = 0, \tag{5.12}$$

where we use the notation (5.4). In fact y_1 and y_2 are essentially the only solutions satisfying the boundary conditions at, respectively, a and b:

Lemma 5.1 *A function u satisfies*

$$L[u] = 0 \quad \textit{and} \quad B_a[u] = 0 \tag{5.13}$$

if and only if $u = \lambda y_1$ for some real number λ

Proof: If $u = \lambda y_1$ then it is straightforward to check that u satisfies (5.13). Suppose that u satisfies (5.13). Then

$$\begin{aligned} & \alpha_1 u(a) + \beta_1 u'(a) = 0 \\ \Leftrightarrow\ & -y_1'(a)u(a) + y_1(a)u'(a) = 0 \\ \Leftrightarrow\ & W(y_1, u)(a) = 0, \end{aligned} \tag{5.14}$$

where $W(y_1, u)$ is the Wronskian of y_1 and u. Hence, by corollary (2.6), u is a multiple of y_1. □

Clearly one can similarly prove that any solution u of $L[u] = 0$ and $B_b[u] = 0$ must be a multiple of y_2. It might of course happen that y_1 and y_2 are dependent. The following simple check follows directly from the above lemma

Corollary 5.2 *The solutions y_1 and y_2 are independent if and only if $B_a(y_2) \neq 0$.*

For our construction of the Green's function we require y_1 and y_2 to be independent, which we assume in following. The next ingredient we require is a particular solution of the homogeneous equation

$$L[y] = f. \tag{5.15}$$

This is a problem we solved in section 2.5.2 using the method of variation of parameters. The particular solution constructed there is of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) \tag{5.16}$$

with c_1 and c_2 satisfying the first order differential equation (2.88). Writing that equation out in components

$$\begin{aligned} c_1'(x) &= -\frac{y_2(x)f(x)}{W(y_1, y_2)(x)} \\ c_2'(x) &= \frac{y_1(x)f(x)}{W(y_1, y_2)(x)} \end{aligned} \tag{5.17}$$

we give solutions in the following form

$$\begin{aligned} c_1(x) &= -\int_a^x \frac{y_2(s)f(s)}{W(y_1, y_2)(s)} ds \\ c_2(x) &= \int_a^x \frac{y_1(s)f(s)}{W(y_1, y_2)(s)} ds. \end{aligned} \tag{5.18}$$

Hence we have the particular solution

$$\begin{aligned} y_p(x) &= -\int_a^x \frac{y_2(s)f(s)}{W(y_1, y_2)(s)} ds\ y_1(x) + \int_a^x \frac{y_1(s)f(s)}{W(y_1, y_2)(s)} ds\ y_2(x) \\ &= \int_a^x \frac{(y_1(s)y_2(x) - y_1(x)y_2(s))\, f(s)}{W(y_1, y_2)(s)} ds \end{aligned} \tag{5.19}$$

Differentiating we find

$$
\begin{aligned}
y_p'(x) &= \frac{y_1(x)f(x)}{W(y_1,y_2)(x)}\; y_2(x) + \int_a^x \frac{y_1(s)f(s)}{W(y_1,y_2)(s)} ds\; y_2'(x) \\
&\quad - \frac{y_2(x)f(x)}{W(y_1,y_2)(x)}\; y_1(x) - \int_a^x \frac{y_2(s)f(s)}{W(y_1,y_2)(s)} ds\; y_1'(x) \\
&= \int_a^x \frac{(y_1(s)y_2'(x) - y_1'(x)y_2(s))f(s)}{W(y_1,y_2)(s)} ds \qquad (5.20)
\end{aligned}
$$

It follows that $y_p(a) = y_p'(a) = 0$ and hence

$$B_a[y_p] = 0. \qquad (5.21)$$

On the other hand

$$
\begin{aligned}
B_b[y_p] &= \int_a^b \frac{(y_1(s)B_b[y_2] - B_b[y_1]y_2(s))f(s)}{W(y_1,y_2)(s)}\, ds \\
&= -B_b[y_1] \int_a^b \frac{y_2(s)f(s)}{W(y_1,y_2)(s)} ds \\
&\neq 0. \qquad (5.22)
\end{aligned}
$$

Thus y_p satisfies the boundary condition at a but not at b. In order to satisfy the boundary condition at b we thus turn to the most general solution of $L[y] = f(x)$. According to the theory of inhomogeneous differential equations this is

$$y(x) = Ay_1(x) + By_2(x) + y_p(x). \qquad (5.23)$$

It thus remains to determine the constants A and B so that the boundary conditions are satisfied. Since $B_a[y_1] = B_a[y_p] = 0$ but $B_a[y_2] \neq 0$ we have

$$B_a[y] = 0 \Rightarrow B = 0. \qquad (5.24)$$

Similarly using $B_b[y_2] = 0$, $B_b[y_1] \neq 0$ and equation (5.22) we deduce

$$B_b[y] = 0 \Rightarrow A = \int_a^b \frac{y_2(s)f(s)}{W(y_1,y_2)(s)} ds. \qquad (5.25)$$

Inserting the values for A and B into (5.23) and using the form (5.19) for y_p we obtain the solution

$$
\begin{aligned}
y(x) &= \int_a^b \frac{y_1(x)y_2(s)f(s)}{W(y_1,y_2)(s)} ds + \int_a^x \frac{(y_1(s)y_2(x) - y_1(x)y_2(s))f(s)}{W(y_1,y_2)(s)} ds \\
&= \int_a^x \frac{y_1(s)y_2(x)f(s)}{W(y_1,y_2)(s)} ds + \int_x^b \frac{y_1(x)y_2(s)f(s)}{W(y_1,y_2)(s)} ds. \qquad (5.26)
\end{aligned}
$$

To write this solution in a convenient form, define the Green's function

$$
G(x,s) = \begin{cases} \dfrac{y_1(s)y_2(x)}{W(y_1,y_2)(s)} & \text{if} \quad a \le s \le x \le b \\ \dfrac{y_1(x)y_2(s)}{W(y_1,y_2)(s)} & \text{if} \quad a \le x \le s \le b \end{cases} \qquad (5.27)
$$

so that (5.26) is

$$y(x) = \int_a^b G(x,s) f(s)\, ds. \tag{5.28}$$

In our derivation, the Green's function only appeared as a particularly convenient way of writing a complicated formula. The importance of the Green's function stems from the fact that it is very easy to write down. All we need is fundamental system of the homogeneous equation. Thus the quickest way of solving boundary problems like (5.3)is to proceed in the following **four steps**:

1. Find a fundamental system $\{u_1, u_2\}$ of $L[y] = 0$.
2. By taking suitable linear combinations of u_1 and u_2 find solutions y_1 and y_2 of $L[y] = 0$ satisfying $B_a[y_1] = 0$ and $B_b[y_2] = 0$ (often possible by inspection).
3. Define the Green's function G according to (5.27).
4. Compute the solution according to (5.28).

To illustrate the properties and use of the Green's function consider the following **examples**.

Example 1. Find the Green's function for the following boundary value problem

$$y''(x) = f(x), \quad y(0) = 0,\ y(1) = 0. \tag{5.29}$$

Hence solve $y''(x) = x^2$ subject to the same boundary conditions.

The homogeneous equation $y'' = 0$ has the fundamental solutions $u_1(x) = 1$ and $u_2(x) = x$. Take $y_1(x) = x$ and $y_2(x) = 1 - x$ to satisfy the boundary conditions $B_0[y] = y(0) = 0$ and $B_1[y] = y(1) = 0$ respectively. Then $W(y_1, y_2)(x) = -1$ and therefore

$$G(x,s) = \begin{cases} s(x-1) & \text{if} \quad 0 \le s \le x \\ x(s-1) & \text{if} \quad x \le s \le 1 \end{cases} \tag{5.30}$$

Thus solve (5.29) with

$$y(x) = \int_0^x s f(s)\, ds\ (x-1) + \int_x^1 (s-1) f(s)\, ds\ x. \tag{5.31}$$

Inserting $f(s) = s^2$ and carrying out the integration yields

$$y(x) = \frac{1}{12}(x^4 - x). \tag{5.32}$$

Example 2. Find the Green's function for the boundary value problem

$$y''(x) + y(x) = f(x), \quad y(0) = 0,\ y'(1) = 0. \tag{5.33}$$

The equation $y'' + y = 0$ has the fundamental system $u_1(x) = \sin x$ and $u_2(x) = \cos x$. To satisfy $B_0[y] = y(0) = 0$ take $y_1(x) = \sin x$ and to satisfy $B_1[y] = y'(1) = 0$ take $y_2(x) = \cos(x-1)$. Then check that $W(y_1, y_2)(x) = -\cos 1$ and find

$$G(x,s) = \begin{cases} -\dfrac{\sin s \cos(x-1)}{\cos 1} & \text{if} \quad 0 \le s \le x \\ -\dfrac{\sin x \cos(s-1)}{\cos 1} & \text{if} \quad x \le s \le 1. \end{cases} \tag{5.34}$$

Example 3. Consider the Green's function found in example 1.

(a) Show that G is symmetric in the sense that $G(x,s) = G(s,x)$.

(b) Show that

$$\frac{\partial^2 G}{\partial s^2}(x,s) = \delta(s-x). \tag{5.35}$$

To show (a) we check

$$\begin{aligned} G(s,x) &= \begin{cases} x(s-1) & \text{if} \quad 0 \le x \le s \\ s(x-1) & \text{if} \quad s \le x \le 1 \end{cases} \\ &= G(x,s) \end{aligned} \tag{5.36}$$

To prove (b) we differentiate (5.30) to obtain

$$\frac{\partial G}{\partial s}(x,s) = \begin{cases} x-1 & \text{if} \quad 0 \le s \le x \\ x & \text{if} \quad x \le s \le 1 \end{cases} \tag{5.37}$$

which we can write in terms of the Heaviside function as

$$\frac{\partial G}{\partial s}(x,s) = x - 1 + u_x(s). \tag{5.38}$$

Then using the definition of the Dirac delta function as the derivative of the Heaviside function we obtain (5.35).

It follows from the symmetry of the Green's function (and also by direct computation) that

$$\frac{\partial^2 G}{\partial x^2}(x,s) = \delta(x-s) = \delta(s-x). \tag{5.39}$$

This result suggests a new way of understanding the fundamental formula (5.28). According to that formula, the solution of (5.29) in terms of the Green's function (5.30) is

$$y(x) = \int_0^1 G(x,s) f(s)\, ds \tag{5.40}$$

Differentiating twice with respect to x and using (5.39) we find immediately

$$\frac{d^2 y}{dx^2} = \int_0^1 \frac{\partial^2 G}{\partial x^2}(x,s) f(s) ds = \int_0^1 \delta(s-x) f(s) ds = f(x), \tag{5.41}$$

where we used theorem 4.6 about integrals involving the Dirac delta function. The equation (5.35) suggests that we can think of the Green's function as the response function to a unit impulse at $s = x$. As we have seen, it then follows immediately that (5.40) solves the inhomogeneous equation (5.29). This point of view provides useful intuition when dealing with Green's functions and is important in the further development of the theory.