Problems for Quantum Computing: week 1

Module F14ZD1 **2007-08**

1. **Matrix representation of operators**

(a) Let A be an operator on $\mathbb{C}^2$ which has the following action on the canonical basis vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$A|0\rangle = 2|0\rangle + 3|1\rangle \qquad A|1\rangle = 1|0\rangle - 4|1\rangle.$$

(i) Find the matrix representation of A relative to the basis $\{|0\rangle, |1\rangle\}$.

(ii) Find the matrix representation of A relative to the basis $\{|v_1\rangle, |v_2\rangle\}$, where

$$|v_1\rangle = \frac{1}{\sqrt{2}}\left(i|0\rangle - |1\rangle\right) \quad \text{and} \quad |v_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right).$$

(b) Let V be the vector space of polynomials in the variable x of degree at most two. The coefficients in the polynomial are allowed to be complex, so a typical element of V is $3+6x+ix^2$. Let $D : V \to V$ be the differentiation map $\dfrac{d}{dx}$.

(i) Show that D is a linear map.

(ii) Find the matrix representation of D with respect to the basis $\{1, x, x^2\}$ of V.

(iii) Show that D is nilpotent, i.e. that $D^N = 0$ for some power N which you should determine

2. **Inner product spaces**

(a) Check that $|v\rangle = \begin{pmatrix} 2 \\ -3i \end{pmatrix}$ and $|w\rangle = \begin{pmatrix} 3 \\ 2i \end{pmatrix}$ are orthogonal in $\mathbb{C}^2$ equipped with the canonical inner product

$$\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2.$$

Compute the norm of both vectors.

(b) Consider the following three vectors in $\mathbb{C}^3$, equipped with the canonical inner product:

$$|v_1\rangle = \begin{pmatrix} 1+i \\ 1 \\ 1-i \end{pmatrix}, \quad |v_2\rangle = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad |v_3\rangle = \begin{pmatrix} 2 \\ 1-i \\ -2i \end{pmatrix}.$$

Check the validity of the Cauchy-Schwarz inequality

$$\langle v_i | v_j \rangle \langle v_j | v_i \rangle \leq \langle v_i | v_i \rangle \langle v_j | v_j \rangle$$

for all pairs $i \neq j$, $i, j = 1, 2, 3$. For which pair does the equality hold? Why does it hold for that pair but not the others?

continued overleaf

(c) Let $B = \{|b_1\rangle, |b_2\rangle, |b_3\rangle\}$ be an orthonormal basis of $\mathbb{C}^3$ with the canonical inner product. Give the matrix representation of the operators $|b_i\rangle\langle b_j|$, $i, j = 1, 2, 3$, with respect to the basis B.

3. Subspaces of a vector space

Let V be a vector space with inner product, W a subspace of V and $A : V \to V$ a linear operator. Recall the definition of the orthogonal complement of W

$$W^\perp = \{|v\rangle \in V | \langle v|w\rangle = 0 \quad \text{for all} \quad |w\rangle \in W\},$$

and the definition of the eigenspace of A with eigenvalue λ:

$$\mathrm{Eig}_\lambda = \{|v\rangle \in V | A|v\rangle = \lambda|v\rangle\}.$$

Show that both $W^\perp$ and Eig_λ are vector spaces.