Problems for Quantum Computing: week 2

Module F14ZD1 **2007-08**

1. **Adjoint operators**

Consider a finite-dimensional vector space V equipped with an inner product $(\cdot,\cdot)$. Suppose A and B are operators in V. Show the following identities.

(a) $\left(A^\dagger\right)^\dagger = A$.

(b) $(AB)^\dagger = B^\dagger A^\dagger$.

(c) If A and B are both Hermitian operators then the commutator $C = [A, B]$ is anti-Hermitian i.e. $C^\dagger = -C$

Hint: the equality of operators P and Q follows from the equality $(P|\psi\rangle, \varphi\rangle) = (Q|\psi\rangle, |\varphi\rangle)$ for all $|\psi\rangle, |\varphi\rangle \in V$

2. **Projection operators**

Let $V = \mathbb{R}^3$, equipped with the canonical inner product, i.e.

$$\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right) = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Let

$$|u\,\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad |v\rangle = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad |w\rangle = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

and write W for the two-dimensional subspace spanned by $|v\,\rangle$ and $|w\,\rangle$. Define $P_U = |u\,\rangle\langle\,u|$ and let P_W be the orthogonal projection operator onto W.

(a) Give the matrix representation of P_U and P_W relative to the canonical basis

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

of $\mathbb{R}^3$.

(b) Check that the matrix representations of P_U and P_W satisfy $P^2 = P$. Also check that both P_U and P_W are Hermitian, and that $P_U + P_W = I$.

(c) Without further calculations give eigenvectors and eigenvalues of P_U and P_W.

continued overleaf

4. **Unitary operators**

Let V be an inner product space and $U : V \to V$ a unitary operator. Show that eigenvectors $|v_1\rangle$ and $|v_2\rangle$ of U with distinct eigenvalues λ_1 and λ_2 are necessarily orthogonal.

5. **Commutators and eigenvectors**

Consider the 3×3 matrices

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Compute the commutator $[A, B] = AB - BA$.

(b) Find the eigenvalues and eigenvectors of A and B.

(c) * Show that the eigenvectors of A are also eigenvectors of B. Are the eigenvectors of B automatically eigenvectors of A?