Problems for Quantum Computing: week 4

Module F14ZD1 **2007-08**

1. Exponentials of operators

Consider the 3×3-matrix

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and compute

$$R(\phi) = \exp(\phi J).$$

Give a geometrical interpretation of $R(\phi)$ by working out its action on the basis vectors

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

2. Schrödinger equation

Consider the Hilbert space $V = \mathbb{C}^2$ with the Hamiltonian

$$H = b \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

where b is some positive real number.

(a) Find the time evolution operator

$$U(t) = e^{-\frac{i}{\hbar} tH}.$$

Hence solve the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

for a time dependent state $|\psi(t)\rangle$ in $\mathbb{C}^2$ which satisfies the initial condition $|\psi(0)\rangle = |1\rangle$.

(b) The observable

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is measured at time t. Compute the probablity of measuring the outcome 1 at time t.

continued overleaf

3. **Nuclear magnetic resonance**

We set $\hbar = 1$ in this question. Consider the time-dependent Hamiltonian

$$H(t) = \frac{\omega_0}{2}\sigma_3 + g(\sigma_1 \cos(\omega t) + \sigma_2 \sin(\omega t))$$

acting on a single qubit. Here ω_0, ω and g are real constants and σ_1, σ_2 and σ_3 are Pauli matrices.

(a) Show that the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{1}$$

is equivalent to the equation

$$i\frac{d}{dt}|\varphi(t)\rangle = \left(\frac{\omega_0 - \omega}{2}\sigma_3 + g\sigma_1\right)|\varphi(t)\rangle \tag{2}$$

for the "rotating " state

$$|\varphi(t)\rangle = e^{\frac{i}{2}\omega t \sigma_3}|\psi(t)\rangle.$$

(b) Find the general solution of (2) and hence the general solution of (1).

(c) Describe the solution $|\varphi(t)\rangle$ qualitatively for small g and (i) $|\omega - \omega_0|$ much bigger than g and (ii) $\omega = \omega_0$.