Problems for Quantum Computing: week 5

Module F14ZD1 **2007-08**

1. Epsilon symbol

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be two vectors in $\mathbb{R}^3$, and $\boldsymbol{r} = \boldsymbol{p} \times \boldsymbol{q}$ be their vector product. If

$$\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \qquad \boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

show that the components of $\boldsymbol{r}$ are given by

$$r_a = \sum_{b,c=1}^{3} \epsilon_{abc} p_b q_c, \quad a = 1, 2, 3.$$

2. Pauli matrices

Let σ_1, σ_2 and σ_3 be the Pauli matrices defined in the lecturers.

(a) Write out the multitplication law

$$\sigma_a \sigma_b = \delta_{ab} I + i \sum_{c=1}^{3} \epsilon_{abc} \sigma_c \tag{1}$$

for each of the values $a = 1, 2, 3$ and $b = 1, 2, 3$. Check the resulting multiplication table by an explicit matrix multiplication.

(b) Show that for any vectors $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^3$,

$$(\boldsymbol{p}\cdot\boldsymbol{\sigma})(\boldsymbol{q}\cdot\boldsymbol{\sigma}) = \boldsymbol{p}\cdot\boldsymbol{q}\, I + i(\boldsymbol{p} \times \boldsymbol{q})\cdot\boldsymbol{\sigma} \tag{2}$$

by writing out $\boldsymbol{p}\cdot\boldsymbol{\sigma} = p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3$ and $\boldsymbol{q}\cdot\boldsymbol{\sigma} = q_1\sigma_1 + q_2\sigma_2 + q_3\sigma_3$ and carrying out the multiplication term by term, using the multiplication table found in (a).

(c) * Prove the formula (2) using the expressions $\boldsymbol{p}\cdot\boldsymbol{\sigma} = \sum_{a=1}^{3} p_a \sigma_a$ and $\boldsymbol{q}\cdot\boldsymbol{\sigma} = \sum_{b=1}^{3} q_a \sigma_b$ and the general formula (1).

3. Spin eigenstates

(a) Write each of the Hermitian operators

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \qquad B = \frac{1}{2} \begin{pmatrix} \sqrt{3} & -1 \\ -1 & -\sqrt{3} \end{pmatrix} \tag{3}$$

in the form $\boldsymbol{k}(\theta, \phi)\cdot\boldsymbol{\sigma}$, where

$$\boldsymbol{k}(\theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \tag{4}$$

and $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ are angles which you should determine.

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(b) Use the recipe given in the lectures to write down eigenstates $|(\theta, \phi)^{\pm})\,\rangle$ with eigenvalues ± 1 for the operators (3).

4. **Heisenberg meets Pauli**

Let $\alpha, \beta \in \mathbb{C}$ and $|\psi\,\rangle = \alpha|0\,\rangle + \beta|1\,\rangle$ be a general state in $\mathbb{C}^2$. Setting $\hbar = 1$ the spin operators are

$$S_a = \frac{1}{2}\sigma_a, \qquad a = 1, 2, 3.$$

Consider the vector $\boldsymbol{s}$ with components

$$s_a = \langle\,\psi|S_a|\psi\,\rangle.$$

(a) Compute the components s_1, s_2 and s_3 and show that

$$\boldsymbol{s}^2 = s_1^2 + s_2^2 + s_3^2 = \frac{1}{4} \tag{5}$$

(b) Show that

$$S_1^2 + S_2^2 + S_3^2 = \frac{3}{4}I$$

and deduce that the "sum over spin variances" is 1/2:

$$\sum_{a=1}^{3} \langle\,\psi|S_a^2|\psi\,\rangle - (\langle\,\psi|S_a|\psi\,\rangle)^2 = \frac{1}{2}.$$