Problems for Quantum Computing: week 10

Module F14ZD1 **2007-08**

1. Controlled gates

(a) Let H be the unitary operation implemented by the Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Show that $H\sigma_1 H = \sigma_3$.

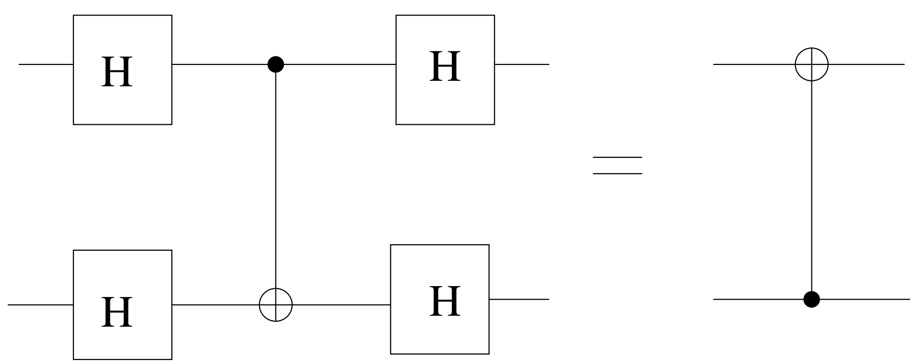
(b) A controlled σ_3-gate has the following matrix representation relative to the canonical basis

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Construct a CNOT gate from controlled σ_3 gates and controlled Hadamard gates.

2. Swapping control and target

Show that



3. (Most of a) specimen exam question

(a) Show that an operator $U : V \to V$ which is both unitary and Hermitian satisfies $U^2 = I$. Deduce that for an arbitrary state $|\psi\rangle \in V$ the states

$$|+\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle + U|\psi\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle - U|\psi\rangle) \tag{1}$$

are eigenvectors of U. Find the corresponding eigenvalues.

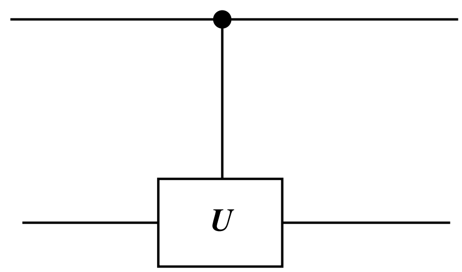


Figure 1: The gate for the controlled U operation

(b) The gate for the controlled U operation is shown in Fig. 1. With the convention that the upper strand refers to the first qubit and the lower strand to the second qubit, it does nothing if the first qubit is in the state $|0\rangle$ and performs the operation U on the second qubit when the first qubit is in the state $|1\rangle$ i.e. it maps

$$|0\rangle \otimes |\psi\rangle \mapsto |0\rangle \otimes |\psi\rangle, \qquad |1\rangle \otimes |\psi\rangle \mapsto |1\rangle \otimes U|\psi\rangle.$$

Consider now the quantum circuit shown in Fig. 2, where H stands for Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

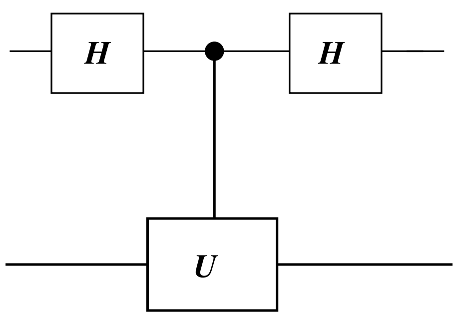


Figure 2: Quantum circuit

(i) Find the output state produced by the circuit from the input state $|0\rangle \otimes |\psi\rangle$.

(ii) Suppose the observable $P = |1\rangle\langle 1| \otimes I$ is measured on the output state. Show that the possible outcomes of the measurement are 0 and 1.

(iii) Also show that the state after the measurement is $|0\rangle \otimes |+\rangle$ if the outcome is 0 and $|1\rangle \otimes |-\rangle$ if the outcome is 1. Here $|+\rangle$ and $|-\rangle$ are the states defined in (1).