Problems for Quantum Computing: week 6

Module F14ZD1 **2007-08**

1. **Ensembles versus density matrices**

Consider the Hilbert space $\mathbb{C}^2$, the observable

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and the ensemble

$$\mathcal{E} = \{(\frac{1}{3}, |0\,\rangle), (\frac{2}{3}, |1\,\rangle)\}.$$

This is a special case of the ensemble considered in the lectures, with $|\alpha|^2 = \frac{1}{3}$ and $|\beta|^2 = \frac{2}{3}$. Write down the associated density operator $\rho_{\mathcal{E}}$ and use the density matrix formalism to compute

(a) the probability of measuring the eigenvalue 2 of the observable A,

(b) the expectation value of A,

(c) the density matrix after the eigenvalue 2 of A has been measured.

In each case compare with the calculations in the lecture.

2. **Bloch sphere**

Given a vector $\boldsymbol{r} \in \mathbb{R}^3$ with $|\boldsymbol{r}| \leq 1$ define the density operator

$$\rho(\boldsymbol{r}) = \frac{1}{2}(I + \boldsymbol{r}\cdot\boldsymbol{\sigma}). \tag{1}$$

(a) Show that $\rho(\boldsymbol{r})$ is a density operator of a pure state iff $\boldsymbol{r}\cdot\boldsymbol{r} = 1$.

(b) Parametrising a unit vector $\boldsymbol{k} \in \mathbb{R}^3$ in terms of angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ via

$$\boldsymbol{k} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

and using the notation $|(\theta, \phi)^+\,\rangle$ for the eigenstate of $\boldsymbol{k}\cdot\boldsymbol{\sigma}$ with eigenvalue 1, show that

$$\rho(\boldsymbol{k}) = |(\theta, \phi)^+\,\rangle\langle\,(\theta, \phi)^+|.$$

(c) Consider now the general density operator $\rho(\boldsymbol{r})$ given in (1) for some vector $\boldsymbol{r} \in \mathbb{R}^3$ of length ≤ 1. If $\boldsymbol{m} \in \mathbb{R}^3$ show that

$$\mathrm{tr}(\rho(\boldsymbol{r})\boldsymbol{m}\cdot\boldsymbol{\sigma}) = \boldsymbol{m}\cdot\boldsymbol{r}.$$

What is the quantum-mechanical interpertation of this quantity?