Problems for Quantum Computing: week 8

Module F14ZD1 **2007-08**

1. **Hadamard operator**

The Hadamard operator on $\mathbb{C}^2$ is defined via

$$H = \frac{1}{\sqrt{2}} \left[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1| \right].$$

Show that

$$H \otimes H = \frac{1}{2} \sum_{x_1, x_2, y_1, y_2 \in \{0,1\}} (-1)^{x_1 y_1 + x_2 y_2} |x_1, x_2\rangle\langle y_1, y_2|.$$

2. **Measurements in composite systems**

Consider a two qubit system with Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ and suppose the system is in the state $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.

(a) What is the probability of finding the system in the state $|00\rangle$? Give the observable which "measures" if the system is in the state $|00\rangle$, and compute its expectation value and standard deviation in the state $|\psi\rangle$.

(b) What is the probability of finding the first qubit in the state $|0\rangle$? Give the observable which "measures" if the first qubit is in the state $|0\rangle$ and compute its expecation value. Also give the state of the systems immediately after the measurement.

3. **Time evolution and measurements in a 3-qubit system**

Consider the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and the operator T with matrix respresentation $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the canonical basis.

(a) Find the matrix representation of $T^\dagger$ in the canonical basis, and check if T and $T^\dagger$ commute. Show that $T^2 = (T^\dagger)^2 = 0$.

(b) The time evolution is determined by the Hamiltonian $H = (T + T^\dagger)$. Show that $H^{2m} = K$, where

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes I + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I$$

and that $H^{2m+1} = H$, for any integer $m > 0$. Deduce that the time evolution operator is given by

$$e^{-\frac{i}{\hbar}Ht} = (I - K) + \cos(\frac{t}{\hbar})\, K - i \sin(\frac{t}{\hbar})\, H$$

(c) Use the time evolution operator found in (b) to find the state $|\psi(t)\rangle$ of the three qubit system at time $t > 0$ if the state at time $t = 0$ is $|\psi(0)\rangle = |010\rangle$. What is the probability of finding the left qubit in the state $|0\rangle$ in a measurement on the left qubit (partial measurement) at time t? What is the state of the system immediately after such a measurement?