Problems for Quantum Computing: week 9

Module F14ZD1 **2007-08**

1. **Reduced density operators**

The following question concerns a two qubit system with Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$. Give the density operator corresponding to the pure, entangled state

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle + |10\rangle\right).$$

Find the reduced density operator for first and second qubit and show that both reduced density operators describe mixed states.

2. **Schmidt decomposition**

Find the Schmidt decomposition of the following states of a two qubit system. In each case give the Schmidt coefficients and the Schmidt number of the state.

(a) $|\psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

(b) $|\varphi\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle\right)$

3. **Tsirelson's inequality**

(a) Deduce from the Cauchy-Schwarz inequality that

$$\langle\psi|A|\psi\rangle \leq \sqrt{\langle\psi|A^2|\psi\rangle} \tag{1}$$

for an arbitrary observable A and an arbitrary normalised state $|\psi\rangle$. Show that the equality holds if and only if $|\psi\rangle$ is an eigenstate of A.

(b) Consider now the following observable in a two-qubit system:

$$A = QS + RS + RT - QT, \tag{2}$$

where $Q = \boldsymbol{q}\cdot\boldsymbol{\sigma} \otimes I, R = \boldsymbol{r}\cdot\boldsymbol{\sigma} \otimes I, S = I \otimes \boldsymbol{s}\cdot\boldsymbol{\sigma}$ and $T = I \otimes \boldsymbol{t}\cdot\boldsymbol{\sigma}$ for unit vectors $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ and $\boldsymbol{t}$ in $\mathbb{R}^3$. Show that

$$A^2 = 4I \otimes I + [\boldsymbol{q}\cdot\boldsymbol{\sigma}, \boldsymbol{r}\cdot\boldsymbol{\sigma}] \otimes [\boldsymbol{s}\cdot\boldsymbol{\sigma}, \boldsymbol{t}\cdot\boldsymbol{\sigma}] \tag{3}$$

(c) Use (a) and (b) to show **Tsirelson's inequality**:

$$\langle\psi|QS + RS + RT - QT|\psi\rangle \leq 2\sqrt{2}$$

for any state $|\psi\rangle$ of the two qubit system. The violation of Bell's inequality found in the lectures is thus the maximal possible violation in a quantum mechanical system.