Problems for Quantum Computing : Week 8 Solutions

1) Hadamard Operator

$$H = \frac{1}{\sqrt{2}}\Big[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\Big]$$

$$\Rightarrow H \otimes H = \frac{1}{2}\Big[(|0\rangle+|1\rangle)\langle 0| + (|0\rangle-|1\rangle)\langle 1|\Big] \otimes \Big[(|0\rangle+|1\rangle)\langle 0| + (|0\rangle-|1\rangle)\langle 1|\Big]$$

$$= \frac{1}{2}\Big[(|0\rangle+|1\rangle)\langle 0| \otimes (|0\rangle-|1\rangle)\langle 0| + (|0\rangle+|1\rangle)\langle 0| \otimes (|0\rangle-|1\rangle)\langle 1| + (|0\rangle-|1\rangle)\langle 1| \otimes (|0\rangle+|1\rangle)\langle 0| + (|0\rangle-|1\rangle)\langle 1| \otimes (|0\rangle-|1\rangle)\langle 1|\Big]$$

In general $|a\rangle\langle b| \otimes |c\rangle\langle d| = |ac\rangle\langle bd|$

$$\Rightarrow H \otimes H = \frac{1}{2}\Big[|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 0| + |0\rangle\langle 0| \otimes |0\rangle\langle 1| - |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| - |1\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| - |0\rangle\langle 1| \otimes |1\rangle\langle 1| - |1\rangle\langle 1| \otimes |0\rangle\langle 0| - |1\rangle\langle 1| \otimes |1\rangle\langle 0| - |1\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \Big]$$

$$= \frac{1}{2}\Big[|00\rangle\langle 00| + |01\rangle\langle 00| + |00\rangle\langle 01| - |01\rangle\langle 01| + |10\rangle\langle 00| + |11\rangle\langle 00| + |10\rangle\langle 01| - |11\rangle\langle 01| + |00\rangle\langle 10| + |01\rangle\langle 10| + |00\rangle\langle 11| - |01\rangle\langle 11| - |10\rangle\langle 10| - |11\rangle\langle 10| - |10\rangle\langle 11| + |11\rangle\langle 11| \Big]$$

$\therefore$ Now taking values $x_1, x_2, y_1, y_2 \in \{0,1\}$ we get

$$H \otimes H = \frac{1}{2} \sum_{x_1,x_2,y_1,y_2 \in \{0,1\}} (-1)^{x_1 y_1 + x_2 y_2} |x_1, x_2\rangle\langle y_1, y_2| \quad \text{as required.}$$

$\square$

②

2) Measurements in Composite Systems

Two qubit system with Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is in state

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

a) Want to find probability of finding system in state $|00\rangle$
$\Rightarrow$ need $P = |00\rangle\langle 00|$, the projection operator

$$\begin{aligned}
\therefore P_\psi &= \langle\psi|P|\psi\rangle \\
&= \frac{1}{4}(\langle 00| + \langle 01| + \langle 10| + \langle 11|)(|00\rangle\langle 00|)(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
&= \frac{1}{4}(\langle 00|00\rangle + \langle 01|00\rangle + \langle 10|00\rangle + \langle 11|00\rangle)(\langle 00|00\rangle + \langle 00|01\rangle + \langle 00|10\rangle + \langle 00|11\rangle) \\
&= \frac{1}{4}(1 + 0 + 0 + 0)(1 + 0 + 0 + 0) \\
&= \frac{1}{4}
\end{aligned}$$

In this case, the observable which "measures" if the system is in the state $|00\rangle$ is also the projection operator P, so $A = P$

Now Expectation value of $A = \langle\psi|A|\psi\rangle$
$= \langle\psi|P|\psi\rangle$
$= \frac{1}{4}$

$\Rightarrow$ (standard deviation)^2 is: $E(A^2) - [E(A)]^2$
$= E(P^2) - [E(A)]^2$

$P^2 = P$ since it is a projection operator.

$= E(P) - [E(P)]^2$
$= \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$
$= \cancel{\frac{1}{8}}$

Standard deviation $= \sqrt{\frac{3}{16}} = \frac{\sqrt{3}}{4}$

2b) The eiganspace for eigan value 1 in the operator $P = |0\rangle\langle 0| \otimes I$ satisfies our requirement. Hence we require the probability of 1 in state $|\psi\rangle$

$$\Rightarrow P_\psi(P=1) = \langle\psi|P|\psi\rangle$$

$$= \frac{1}{4}\left(\langle 00| + \langle 01| + \langle 10| + \langle 11|\right)\left(|0\rangle\langle 0| \otimes I\right)\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$= \frac{1}{4}\left(\langle 00| + \langle 01| + \langle 10| + \langle 11|\right)\left(|00\rangle + |01\rangle\right)$$

$$= \frac{1}{4}(1+1) = \frac{1}{2}$$

Since P is a projector, the expectation equals the probability found above.

Expectation $= \frac{1}{2}$

The state after the measurement would be given by $\frac{P_1|\psi\rangle}{\sqrt{P_\psi(1)}}$

$$\Rightarrow \frac{\frac{1}{2}\left(|0\rangle\langle 0| \otimes I\right)\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)}{\sqrt{\frac{1}{2}}}$$

$$= \frac{\sqrt{2}}{2}\left(|00\rangle + |01\rangle\right)$$

3a) If $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 4

$T^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$TT^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I$$

$$T^\dagger T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes I$$

$\therefore$ $T^\dagger T \neq TT^\dagger$, they do not commute.

$$T^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes I = 0$$

$$(T^\dagger)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes I = 0$$

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b) $H^2 = (T+T^\dagger)(T+T^\dagger) = T^2 + TT^\dagger + T^\dagger T + (T^\dagger)^2$

by the above calculations $\Rightarrow$

$$0 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes I + 0$$

$= K$.

This is the case for $m=1$ in H^{2m}

It follows that $H^{2\times 2} = H^2 H^2 = KK$

$$KK = \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} \otimes I$$

$$+ \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} \otimes I$$

$$+ \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} \otimes \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} \otimes I$$

$$+ \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} \otimes \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} \otimes I$$

$$= \begin{pmatrix}0&0\\0&1\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix} \otimes I + \begin{pmatrix}1&0\\0&0\end{pmatrix} \otimes \begin{pmatrix}0&0\\0&1\end{pmatrix} \otimes I$$

$$= K$$

and $K^n = K \ \forall\, n > 1$ thus $H^{2m} = K^m = K$

For H^{2m+1}:

$$H^{2m+1} = H^{2m} H = KH$$

$$KH = \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} \otimes I \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$+ \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} \otimes \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} \otimes I \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$+ \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} \otimes I \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$+ \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} \otimes \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} \otimes I \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= \begin{pmatrix}0&0\\1&0\end{pmatrix} \otimes \begin{pmatrix}0&1\\0&0\end{pmatrix} \otimes \begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix} \otimes \begin{pmatrix}0&0\\1&0\end{pmatrix} \otimes \begin{pmatrix}0&1\\1&0\end{pmatrix} = H$$

$$e^{-\frac{i}{\hbar}Ht} = \sum_{n=0}^{\infty} \left(\frac{-\frac{i}{\hbar}Ht}{n!} \right)^n$$

$$\Big\|$$

When $n=0 \Rightarrow I$

$n =$ even, $\neq 0 \Rightarrow \sum_{n \geq 2} \frac{-\frac{i}{\hbar} H^n t^n}{n!} = \left(\cos\left(\frac{t}{\hbar}\right) - 1\right) K$

$n =$ odd $\neq 0 \Rightarrow \sum_{n\,odd} \frac{-\frac{i}{\hbar} H^n t^n}{n!} = -i \sin\left(\frac{t}{\hbar}\right) H$

$$I + \left(\cos\left(\frac{t}{\hbar}\right) - 1\right) K - i \sin\left(\frac{t}{\hbar}\right) H$$

$$= (I - K) + \cos\left(\frac{t}{\hbar}\right) K - i \sin\left(\frac{t}{\hbar}\right) H$$

3c) At $t=0$ $|\psi(0)\rangle = |010\rangle$

$$\Rightarrow |\psi(t)\rangle = (I-K)|010\rangle + \cos\left(\frac{t}{\hbar}\right) K |010\rangle - i \sin\left(\frac{t}{\hbar}\right) H |010\rangle$$

$$K|010\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes I \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes I \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 = |010\rangle$$

$$H|010\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0 = |101\rangle$$

$$(I-K)|010\rangle = |010\rangle - |010\rangle = 0$$

$$\Rightarrow |\psi(t)\rangle = \cos\left(\frac{t}{\hbar}\right)|010\rangle - i \sin\left(\frac{t}{\hbar}\right)|101\rangle .$$

Following the same reasoning as 2b) $\Rightarrow$ $P = |0\rangle\langle 0| \otimes I$

We seek $P_\psi(P=1) = \langle\psi|P|\psi\rangle$

$$\Rightarrow \left(\cos\left(\frac{t}{\hbar}\right)\langle 010| + i \sin\left(\frac{t}{\hbar}\right)\langle 101|\right)\left(|0\rangle\langle 0| \otimes I\right)\left(\cos\left(\frac{t}{\hbar}\right)|010\rangle - i\sin\left(\frac{t}{\hbar}\right)|101\rangle\right)$$

$\rightarrow$

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$$\Rightarrow \quad (|0\rangle\langle 0| \otimes I \otimes I)\left(\cos\left(\frac{t}{\hbar}\right)|010\rangle - i\sin\left(\frac{t}{\hbar}\right)|101\rangle\right)$$

$$= \quad \cos\left(\frac{t}{\hbar}\right)|010\rangle$$

$$\Rightarrow \quad \left(\cos\left(\frac{t}{\hbar}\right)\langle 010| + i\sin\left(\frac{t}{\hbar}\right)\langle 101|\right)\left(\cos\left(\frac{t}{\hbar}\right)|010\rangle\right)$$

$$= \quad \left(\cos\left(\frac{t}{\hbar}\right)\right)^2 = P$$

The state after the measurement would be given by $\dfrac{P_0|\psi\rangle}{\sqrt{P_\psi(0)}}$

$$|0\rangle\langle 0| \otimes I \otimes I \left(\cos\left(\frac{t}{\hbar}\right)|010\rangle - i\sin\left(\frac{t}{\hbar}\right)|101\rangle\right)$$
$$\Rightarrow \quad \cos\left(\frac{t}{\hbar}\right)$$

$$= \quad \cancel{} \; |010\rangle$$

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