4.1 History and Motivation

One of Maxwell’s equations \( \nabla \cdot \mathbf{b} = 0 \) says that there are no magnetic monopole. There is overwhelming experimental evidence that this statement is correct. Nonetheless, magnetic monopoles exert an inexorable fascination on physicists and mathematicians. The main reason for this is that the mathematics of magnetic monopoles is very rich - and in many ways deeper and more interesting than the mathematics of electric charges in Maxwell’s theory. The study of magnetic monopoles in \( U(1) \) gauge theories like Maxwell’s and their generalisation in other gauge theories has proved to be very fertile for mathematics and has sharpened our understanding of the structure of gauge theories. A simple example is the observation of Dirac’s original paper that the existence of a magnetic monopole would explain the observed quantisation of all observed electric charges as integer multiples of the electron charge.

In this lecture we will concentrate on the basics of monopole theory. We interpret the Dirac monopole in terms of language of gauge theory, and discuss its simplest generalisation, the ’t Hooft-Polyakov monopole. There is a vast literature on magnetic monopoles, which itself is a subset of the even larger literature on ‘topological defects’ or ‘topological solitons’. A good starting point is the recent textbook [3].

In the paper [1], Dirac considers the Schrödinger equation coupled to the electromagnetic field, as discussed in our second lecture. He discusses the implications of the local phase-ambiguity of the wavefunction - and notices the remarkable relation to magnetic poles. In modern language one might say that Dirac discovers the implications of topological non-trivial fibre bundles for physics. We will describe Dirac’s discovery in the language of fibre bundles below.

Dirac’s paper is remarkable in other respects, too. In the introduction, he describes a new way of doing physics, with mathematics guiding the search for new physical laws. One could argue that this approach is dominant in much of contemporary mathematical physics, particularly in string theory. Dirac puts it as follows:

There are at present fundamental problems in theoretical physics awaiting solution, e.g., the relativistic formulation of quantum mechanics and the nature of atomic nuclei (to be followed by more difficult ones such as the problem of life), the solution of which problems will presumably require a more drastic revision of our fundamental concepts than any that have gone before. Quite likely these changes will be so great that it will be beyond the power of human intelligence to get the necessary new ideas by direct attempts to formulate the experimental data in mathematical terms. The theoretical worker in the future will therefore have to proceed in a more indirect
way. The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities.

NOTATION: We write \( x = (x_1, x_2, x_3) \) for an element of Euclidean space \( \mathbb{R}^3 \) when we interpret it as an event. We often write \( \partial_t \) for \( \frac{\partial}{\partial t} \) and \( \partial_i \) for \( \frac{\partial}{\partial x_i} \) and use \( \nabla = (\partial_1, \partial_2, \partial_3) \) for the gradient operator in \( \mathbb{R}^3 \).

4.2 The Dirac Monopole

4.2.1 The Dirac Quantisation Condition (First Chern Class Revisited)

In this section we make use of the standard polar coordinates \( r, \theta, \varphi \) for \( \mathbb{R}^3 \), so

\[
x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = \cos \theta.
\]

(4.1)

We use the notation

\[
n = \frac{x}{r} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),
\]

(4.2)

for the unit vector on \( S^2 \). In analogy with the electric field of an electric point-charge, the magnetic field of a magnetic monopole is

\[
B = \frac{m}{2} \frac{x}{r^3}
\]

(4.3)

It satisfies

\[
\nabla \cdot B = 2\pi m \delta^3(x).
\]

(4.4)

In differential form notation

\[
f = \frac{m}{2} \frac{1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} (x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2)
\]

\[
= \frac{m}{2} \sin \theta d\theta \wedge d\varphi,
\]

(4.5)

so that the flux through a 2-sphere enclosing the origin (see Exercises) is

\[
\int_{S^2_R} f = 2\pi m.
\]

(4.6)

We would like to explore the physical implications of the magnetic monopole for quantum theory. Recall from the second lecture, that Schrödinger-Maxwell - the coupling of Maxwell electrodynamics to non-relativistic quantum theory - requires gauge potentials. For the magnetic field of a monopole, we seem to have a problem, since \( f \) cannot be exact on all of \( S^2 \). Even though \( f \) is not exact on \( S^2 \), it is exact in suitable open sets. Writing \( N \) for the ‘North pole’ \((0, 0, 1) \in S^2 \) and \( S \) for the ‘South pole’ \((0, 0, -1) \in S^2 \), we define northern and southern open sets

\[
U_N = S^2 \setminus \{S\}, \quad U_S = S^2 \setminus \{N\},
\]

(4.7)

and give gauge potentials \( a_N \) and \( a_S \) defined on the respective patches

\[
a_N = \frac{m}{2} (1 - \cos \theta) d\varphi, \quad a_S = \frac{m}{2} (-1 - \cos \theta) d\varphi.
\]

(4.8)
One checks that
\[ f = da_N = da_S, \quad a_N = a_S + m d\varphi. \] (4.9)
With \( A = i a, \ F = i f \) and
\[ g_{NS}(\theta, \varphi) = e^{-im\varphi} \] (4.10)
we have
\[ A_N = A_S + g_{NS}dg_{NS}^{-1} \] (4.11)
The transition function \( g_{NS} \) is well-defined on \( U_N \cap U_S = S^2 \setminus \{N, S\} \) if and only if \( m \in \mathbb{Z} \). This is the Dirac quantisation condition. It is equivalent to the statement that the first Chern class of any \( U(1) \) bundle over the 2-sphere is an integer:
\[ \frac{i}{2\pi} \int_{S^2} F = -m \in \mathbb{Z}. \] (4.12)

### 4.2.2 The Hopf bundle

In the previous lectures we saw that the Lie group \( SU(2) \) is diffeomorphic to the 3-sphere \( S^3 \) and that it can be viewed as a principal \( U(1) \) fibre bundle. This is the famous Hopf fibration of the 3-sphere as a \( S^1 \)-fibration over the 2-sphere
\[ S^1 \to S^3 \]
\[ \downarrow \pi \]
\[ S^2 \] (4.13)
This bundle can be described in different ways, and in this section we will make use of the group structure of \( SU(2) \), so we have
\[ U(1) \to SU(2) \]
\[ \downarrow \pi \]
\[ S^2. \] (4.14)
In order to write down the projection \( \pi \) and the right action of \( U(1) \) on \( SU(2) \) we need to pick a \( U(1) \) subgroup of \( SU(2) \). We use the following generators for \( \mathfrak{su}(2) \)
\[ t_1 = -\frac{1}{2}i\tau_1 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad t_2 = -\frac{1}{2}i\tau_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_3 = -\frac{1}{2}i\tau_3 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \] (4.15)
where \( \tau_a, a = 1, 2, 3 \), are the Pauli matrices; the commutators are \( [t_a, t_b] = \epsilon_{abc} t_c \). As a vector space, \( \mathfrak{su}(2) \) can naturally be identified with Euclidean 3-space, with the inner product given by
\[ \langle X, Y \rangle = -2\text{tr}(XY). \] (4.16)
The normalisation is chosen so that
\[ \langle t_a, t_b \rangle = \delta_{ab}. \] (4.17)
We then use Euler angles \( \theta \in [0, \pi), \ \varphi \in [0, 2\pi) \) and \( \psi \in [0, 4\pi) \) for parametrising \( SU(2) \) as follows
\[ h(\varphi, \theta, \psi) = e^{et_3}e^{i\theta t_2}e^{i\psi t_3} = \begin{pmatrix} e^{-\frac{1}{2}(\psi+\varphi)}\cos \frac{1}{2}\theta & -e^{\frac{1}{2}(\psi-\varphi)}\sin \frac{1}{2}\theta \\ e^{\frac{1}{2}(\psi-\varphi)}\sin \frac{1}{2}\theta & e^{\frac{1}{2}(\psi+\varphi)}\cos \frac{1}{2}\theta \end{pmatrix}. \] (4.18)
The angles turn out to be convenient when working with the Hopf fibration. Picking the diagonal generator $t_3$ for selecting a $U(1)$ subgroup of $SU(2)$, we define the projection map $\pi$ via

$$\pi : SU(2) \to S^2 \subset su(2), \quad h \mapsto ht_3h^{-1}. \tag{4.19}$$

Since $\langle ht_3h^{-1}ht_3h^{-1} \rangle = 1$, the image of $\pi$ is indeed the 2-sphere. We expand

$$ht_3h^{-1} = n_1 t_1 + n_2 t_2 + n_3 t_3, \tag{4.20}$$

and identify this Lie algebra element with the coordinate vector $n \in \mathbb{R}^3$. In terms of Euler angles, we then find

$$\pi(h(\varphi, \theta, \psi)) = n(\theta, \varphi), \tag{4.21}$$

with the unit vector $n$ given in terms of polar coordinates in (4.2). The right action of $U(1)$ on $SU(2)$ can now be written as

$$R(\alpha) : h \mapsto he^{-\alpha t_3}, \quad \alpha \in [0, 4\pi). \tag{4.22}$$

The right action shifts $\psi$:

$$R(\alpha) : \psi \mapsto \psi - \alpha. \tag{4.23}$$

With our normalisation $\alpha \in [0, 4\pi)$, the natural basis of the $U(1)$ Lie algebra is $i/2$. The vector field on $SU(2)$ generated by the $U(1)$ right-action is therefore

$$R_* \left( i \frac{i}{2} \right) = -\partial_\psi. \tag{4.24}$$

Before we relate this data to magnetic monopoles we turn to a closely related bundle, namely the Lens space $S^3/\mathbb{Z}_m$, obtained from $S^3$ by the right action of the cyclic group $\mathbb{Z}_m$, $m \neq 0$, whose generator acts via

$$h \mapsto he^{4\pi t_3/m}. \tag{4.25}$$

This action commutes with the right-action $R(\alpha)$ so we obtain a family of $U(1)$ bundles

$$S^4 \to S^3/\mathbb{Z}_m \quad \downarrow \pi \quad S^2 \tag{4.26}$$

The right-action is now

$$R(\alpha) : h \mapsto he^{-\alpha t_3}, \quad \alpha \in [0, \frac{4\pi}{m}), \tag{4.27}$$

i.e., as in (4.22) but with a different range for rotation angle. With $\alpha \in [0, 4\pi/m)$, the associated basis of the $U(1)$ Lie algebra is $mi/2$. The vector field on $SU(2)$ generated by the $U(1)$ right-action is still $\partial_\psi$, but now we have

$$R_* \left( m \frac{i}{2} \right) = -\partial_\psi. \tag{4.28}$$

We would like to study the Dirac monopole in terms of the language of fibre bundles. From the previous lecture, we recall

**Definition 4.1** A connection on a principal $G$-bundle with total space $P$ can be described by a $\mathfrak{g}$-valued 1-form $A$ on $P$ which satisfies two properties. Writing $R(g)$ for the right-action of by an element $g$ of $G$ on $P$, every Lie algebra element $\chi \in \mathfrak{g}$, defines a vector field on $P$, which we denote by $R_*(\chi)$. Then the required properties are
\[(1)\] \(R(g)A = Ad_{g^{-1}}(A)\)
\[(2)\] \(A(R_e(\chi)) = \chi\).

We now try to identify such a form on the Hopf bundle or the quotients \(S^3/\mathbb{Z}_m\). The magnetic monopole field is spherically symmetric, and we therefore begin by looking at left-invariant forms on the total space \(SU(2)\). Using the maps \(h\) \([4.18]\), we define left-invariant 1-forms \(\eta_a\) on \(SU(2)\) via
\[
h^{-1}dh = \eta_1 t_1 + \eta_2 t_2 + \eta_3 t_3.
\]

We compute to find
\[
\eta_1 = \sin \psi d\theta - \cos \psi \sin \theta d\varphi,
\eta_2 = \cos \psi d\theta + \sin \psi \sin \theta d\varphi,
\eta_3 = d\psi + \cos \theta d\varphi,
\]

satisfying \(d\eta_1 = -\eta_2 \wedge \eta_3\) and similar equations obtained by cyclic permutation.

The forms \(\eta_1, \eta_2, \eta_3\) span the space of left-invariant 1-forms. In order to obtain a connection 1-form for the Hopf bundle, we need to pick out the form which is invariant under the right-action \(R(\alpha)\) \([4.22]\). Under this action (which shifts \(\psi\)), \(\eta_1\) and \(\eta_2\) get rotated into each other, but \(\eta_3\), or any constant multiple of it, is invariant. In order to fix the constant, we use the second condition in the Definition 4.1 with the action \([4.28]\), i.e., we demand
\[
A_m(-\partial_\psi) = \frac{im}{2}
\]

Then the form
\[
A_m = -\frac{im}{2} \eta_3 = -\frac{im}{2} (d\psi + \cos \theta d\varphi).
\]

satisfies all the requirement for a left-invariant connection 1-form on \(S^3/\mathbb{Z}_m\). The curvature of this connection is
\[
F = dA = \frac{im}{2} \sin \theta d\theta \wedge d\varphi
\]

We deduce that the magnetic monopole of charge \(m \neq 0\) is the curvature of the rotationally invariant \(U(1)\) connection on the Lens space \(S^3/\mathbb{Z}_m\).

With the local sections
\[
e_N(\theta, \varphi) = e^{\varphi t_3}e^{\theta t_2}e^{-\varphi t_1}, \quad e_S(\theta, \varphi) = e^{\varphi t_3}e^{\theta t_2}e^{\varphi t_3}
\]
defined on the northern and southern patch, respectively, we obtain the local gauge potentials of the Dirac monopole via pull-back, as expected:
\[
e^*_N A_m = A_N = \frac{im}{2} (1 - \cos \theta)d\varphi, \quad e^*_S A_m = A_S = \frac{im}{2} (-1 - \cos \theta)d\varphi
\]

### 4.2.3 Complex Bundles

It is instructive to repeat the discussion of the previous section in different coordinates which bring out the complex geometry. We can view the 3-sphere also as a subset of the complex plane:
\[
S^3 = \{(z_1, z_2) \in \mathbb{C}^2| |z_1|^2 + |z_2|^2 = 1\}.
\]
Recall the definition of the complex projective line
\[ \mathbb{C}P^1 = \{(z_1, z_2) \in \mathbb{C}^2\}/\sim, \quad (z_1, z_2) \sim \lambda(z_1, z_2), \quad \lambda \in \mathbb{C}^*, \] (4.37)
or the equivalent definition
\[ \mathbb{C}P^1 = \{(z_1, z_2) \in S^3\}/\sim, \quad (z_1, z_2) \sim \lambda(z_1, z_2), \quad \lambda \in U(1). \] (4.38)
This last definition makes it obvious that we have a projection
\[ \pi : S^3 \rightarrow \mathbb{C}P^1, \quad (z_1, z_2) \mapsto [(z_1, z_2)]. \] (4.39)
In order to identify this projection with the Hopf map (as the notation implies) we need to identify \( \mathbb{C}P^1 \) with \( S^2 \). This can be done by using stereographic projection from the South pole
\[ \text{St} : U_N \subset S^2 \rightarrow \mathbb{C}, \quad (x_1, x_2, x_3) \mapsto z = \frac{1}{1 + x_3}(x_1 + ix_2) \] (4.40)
and stereographic projection from the North pole, followed by complex conjugation
\[ \overline{\text{St}} : U_S \subset S^2 \rightarrow \mathbb{C}, \quad (x_1, x_2, x_3) \mapsto \zeta = \frac{1}{1 - x_3}(x_1 - ix_2) \] (4.41)
You should be able to check that \( \zeta = 1/z \)
At the same time, we can introduce coordinates on the \( \mathbb{C}P^1 \) as follows. Define
\[ U_1 = \{[(z_1, z_2)] \in \mathbb{C}P^1 | z_1 \neq 0\}, \quad U_2 = \{[(z_1, z_2)] \in \mathbb{C}P^1 | z_2 \neq 0\}. \] (4.42)
Then we can construct charts of \( \mathbb{C}P^1 \) via
\[ P_1 : U_1 \rightarrow \mathbb{C}, [(z_1, z_2)] \mapsto z = \frac{z_2}{z_1}, \] (4.43)
and
\[ P_2 : U_2 \rightarrow \mathbb{C}, [(z_1, z_2)] \mapsto \zeta = \frac{z_1}{z_2}, \] (4.44)
On the overlap, the coordinates are related via \( \zeta = 1/z \) as for the stereographic coordinates of \( S^2 \). We have identical description of \( S^2 \) and \( \mathbb{C}P^1 \) in terms of two charts and a transition function and thus established the equivalence of the two as differentiable manifolds.
In order to compute the connection 1-form in terms of complex coordinates, we note that, with the constraint \( |z_1|^2 + |z_2|^2 = 1 \) understood, the 1-form
\[ \bar{z}_1 dz_1 + \bar{z}_2 dz_2 \] (4.45)
is invariant under the left-action under \( g \in SU(2) \):
\[ \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) \mapsto g \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) \] (4.46)
and under the right-action
\[ (z_1, z_2) \mapsto (z_1 e^{i\alpha}, z_2 e^{i\alpha}), \quad \alpha \in \left[ 0, \frac{2\pi}{m} \right) \] (4.47)
The vector field generating this action is
\[ R_*(mi) = i z_1 \frac{\partial}{\partial z_1} + i z_2 \frac{\partial}{\partial z_2}, \] (4.48)
so that requirements in the Definition 4.1 are satisfied if we set
\[ A_m = m(\bar{z}_1 dz_1 + \bar{z}_2 dz_2) \] (4.49)

A local section defined on \( U_1 \) is
\[ e_N(z) = \frac{1}{\sqrt{1 + |z|^2}}(z, 1) \] (4.50)

and on \( U_2 \)
\[ e_S(\zeta) = \frac{1}{\sqrt{1 + |\zeta|^2}}(1, \zeta). \] (4.51)

Pull-back give the local connection 1-forms
\[ e^*_N A_m = A_N = m \frac{\bar{z} dz - zd\bar{z}}{1 + |z|^2}, \] (4.52)
\[ e^*_S A_m = A_S = m \frac{\bar{\zeta} d\zeta - \zeta d\bar{\zeta}}{1 + |\zeta|^2}. \] (4.53)

For the curvature we have
\[ dA_m = m(d\bar{z}_1 \wedge dz_1 + d\bar{z}_2 \wedge dz_2) = m \frac{d\bar{z} \wedge dz}{(1 + |z|^2)^2} = m \frac{d\bar{\zeta} \wedge d\zeta}{(1 + |\zeta|^2)^2}, \] (4.54)

with the equalities holding wherever the expressions are defined.

Finally, we relate the complex and the Euler angle parametrisation of \( SU(2) \) via
\[ h(z_1, z_2) = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}. \] (4.55)

with the constraint \(|z_1|^2 + |z_2|^2 = 1\) understood, so that, comparing with (4.18), we have
\[ z_1 = e^{-\frac{i}{2}(\varphi + \psi)} \cos \frac{1}{2} \theta, \quad z_2 = e^{\frac{i}{2}(\varphi - \psi)} \sin \frac{1}{2} \theta. \] (4.56)

Then we observe that
\[ z = \frac{z_2}{z_1} = \tan \frac{\theta}{2} e^{i\varphi}, \] (4.57)

and
\[ \zeta = \frac{z_1}{z_2} = \cot \frac{\theta}{2} e^{-i\varphi}, \] (4.58)

which is consistent with the stereographic projections (4.40) and (4.41) and the use of polar coordinates on \( S^2 \).

We can now check that
\[ \bar{z}_1 dz_1 + \bar{z}_2 dz_2 = -\frac{i}{2} (\cos \theta d\varphi + d\psi) = -\frac{i}{2} \eta_2 \] (4.59)

Substituting \( z = \tan \frac{\theta}{2} e^{i\varphi} \) and \( \zeta = 1/z = \cot \frac{\theta}{2} e^{-i\varphi} \) we also find agreement between the expressions for the local gauge potentials
\[ \frac{1}{2} \frac{\bar{z} dz - zd\bar{z}}{1 + |z|^2} = \frac{i}{2} (1 - \cos \theta) d\varphi, \]
\[ \frac{1}{2} \frac{\bar{\zeta} d\zeta - \zeta d\bar{\zeta}}{1 + |\zeta|^2} = -\frac{i}{2} (1 + \cos \theta) d\varphi. \] (4.60)
4.2.4 Remarks on Complex Line Bundles

In the previous lecture, we learned that for every principal fibre bundle with a representation of the structure group on a vector space $V$ we have an associated vector bundle, with typical fibre $V$. The case where $V = \mathbb{C}$ is called a complex line bundle. Given one of the irreducible representations of $U(1)$ on $\mathbb{C}$ we can therefore naturally associated complex line bundles to the principle fibre bundles discussed in the previous section.

The complex line bundle associated to $U(1)$ fibration of the the Lens space $S^3/\mathbb{Z}_m$ (4.26) is often denoted $L^m$ or $\mathcal{O}(-m)$ in complex geometry.

The line bundle $L^1$ plays a special role. It is the tautological line bundle, and denoted simply by $L$. It can be defined without reference to the Hopf bundle by making use of the definition of $\mathbb{C}P_1$ as set set of lines in $\mathbb{C}^2$. We simple define the fibre over a line $[(z_1, z_2)]$ to be the line defined by $[(z_1, z_2)]$:

$$L = \{(l, z) \in \mathbb{C}P_1 \times \mathbb{C}^2 | z \in l\}$$ (4.61)

This is the line bundle associated to the Hopf bundle (check it!), so it follows from our earlier calculations that

$$c_1(L) = -1$$ (4.62)

The dual line bundles $L^*$ has $c_1(L^*) = 1$ and one can show that

$$TS^2 \simeq L^* \otimes L^*, \quad T^*S \simeq L \otimes L.$$ (4.63)

For details on these and related constructions, see [6] for a succinct summary, or any book on Algebraic Geometry for a careful treatment.

4.2.5 The Levi-Civita Connection: Covariant Derivatives via Projection

Instead of obtaining vector bundles ‘by association’ with principal bundles we can often construct them directly. Even though this uses non-intrinsic structures (like ambient spaces in which we embed the bundles), these often arise naturally. The best known example is perhaps the tangent bundle to the 2-sphere. This can be constructed by its embedding in the trivial bundle $S^2 \times \mathbb{R}^3$ by requiring tangent vectors to be orthogonal to the normal to the sphere at each point. The construction uses the embedding space $\mathbb{R}^3$ and its Euclidean structure $\langle \cdot, \cdot \rangle$:

$$TS^2 = \{ (n, x) \in S^2 \times \mathbb{R}^3 | \langle n, x \rangle = 0 \}. \quad \text{(4.64)}$$

The embedding also allows one to construct a natural covariant derivative. If we want derivatives of tangent vector again to be tangent vectors, we simply project onto the tangent space. Using the projection operator $P = \text{id} - nn^t$, we can write

$$D_PX = PdX = dX - d(nn^t)X$$ (4.65)

for any tangent vector field $X$. Here we used $(nn^t)X = 0$, so $d(nn^t)X = -(nn^t)dX$. In order to make contact with the discussion in the previous section we need to pick an orthonormal frame for the tangent space. Such a frame only exists locally, but

$$u_1 = \begin{pmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ \sin \theta \end{pmatrix}, \quad u_2 = \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{pmatrix}, \quad \text{(4.66)}$$
is defined everywhere except at the North and South Poles. An arbitrary section can be written locally as \( \psi_1 u_1 + \psi_2 u_2 \), or with

\[
\mathbf{u} = \begin{pmatrix} \cos \theta \cos \varphi & -\sin \varphi \\ \cos \theta \sin \varphi & \cos \varphi \\ \sin \theta & 0 \end{pmatrix} \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\] (4.67)

as \( u \psi \). The covariant derivative is

\[
D P(u \psi) = u d \psi + du \psi = u (d \psi + u^t du, \psi)
\] (4.68)

so that we can read off the local connection 1-form as

\[
A = u^t du.
\] (4.69)

Carrying out the differentiation, and introducing the generator

\[
I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\] (4.70)

of the \( SO(2) \) Lie algebra we find the connection and curvature

\[
A = \cos \theta d \varphi I, \quad F = dA = -\sin \theta d \theta \wedge d \varphi I.
\] (4.71)

Identifying \( SO(2) \) with \( U(1) \) by setting \( I = i \), and thus thinking of the tangent bundle as complex line bundle, we find the first Chern number

\[
\frac{i}{2\pi} \int_{S^2} F = 2,
\] (4.72)

in agreement with (4.63). In other words: the Levi-Civita connection on \( TS^2 \) is a magnetic monopole of charge \( m = -2 \).

A slight modification of this construction gives the line bundle \( L \) associated to the Hopf fibration and the Dirac monopoles of charge \( \pm 1 \). This time we start with the trivial bundle \( S^2 \times \mathbb{C}^2 \) and use the following \( 2 \times 2 \) matrix associated to a point \( \mathbf{n} \in S^2 \):

\[
\mathbf{n} \cdot \mathbf{\tau} = n_1 \tau_1 + n_2 \tau_2 + n_3 \tau_3,
\] (4.73)

where \( \tau_1, \tau_2, \tau_3 \) are the Pauli matrices. Since \((\mathbf{n} \cdot \mathbf{\tau})^2\), the map

\[
P = \frac{1}{2} (\text{id} + \mathbf{n} \cdot \mathbf{\tau})
\] (4.74)

is a projection operator, i.e. \( P^2 = P \). It has eigenvalues \( \pm 1 \). We claim that the complex line bundle defined by the eigenspaces for eigenvalue 1 is the tautological line bundle. Computing, we find the eigenvector

\[
u_+ = \begin{pmatrix} e^{-\frac{i}{2} \varphi} \cos \left( \frac{\theta}{2} \right) \\ e^{\frac{i}{2} \varphi} \sin \left( \frac{\theta}{2} \right) \end{pmatrix}
\] (4.75)

for eigenvalue 1 and

\[
u_- = \begin{pmatrix} -e^{-\frac{i}{2} \varphi} \sin \left( \frac{\theta}{2} \right) \\ e^{\frac{i}{2} \varphi} \cos \left( \frac{\theta}{2} \right) \end{pmatrix}.
\] (4.76)
for eigenvalue $-1$. As an aside, we note that, since $\mathbf{n} \cdot \mathbf{\tau} = h \tau_3 h^{-1}$, the eigenvectors are up to phase the column vectors in $h$, as expected, i.e, in terms of the parameterisation (4.55),

$$u_+ \propto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad u_- \propto \begin{pmatrix} -\bar{z}_2 \\ z_1 \end{pmatrix}. \quad (4.77)$$

In analogy with the calculation above, the Levi-Civita connection induced by orthogonal projection in $\mathbb{C}^2$ on the eigenspace bundle for eigenvalue $1$ is

$$A_+ = u_+^\dagger du_+ = -i \cos \theta d\varphi$$

so that $F = \frac{i}{2} \sin \theta d\theta \wedge d\varphi$ and we recover the magnetic field of the Dirac monopole (with $m = 1$ and first Chern number $-1$) as promised. Thus we can identify the eigenspace bundle for eigenvalue $+1$ with the tautological bundle $L$. If we work with the eigenspace for $-1$ we obtain

$$A_- = u_-^\dagger du_- = i \cos \theta d\varphi,$$

so $F = -\frac{i}{2} \sin \theta d\theta \wedge d\varphi$ - which is the magnetic field of the anti-monopole with $m = -1$ and first Chern number $1$. The eigenspace bundle for eigenvalue -1 is the bundle $L^*$. We have therefore shown that the trivial $\mathbb{C}^2$-bundle over the 2-sphere split into $L$ and $L^*$:

$$S^2 \times \mathbb{C}^2 \simeq L \oplus L^*.$$  

(4.80)

### 4.3 Monopoles as Solitons

Dirac monopoles are singular at the location of the magnetic charge, and the total energy, as measured by the usual electromagnetic energy integral $\int_{\mathbb{R}^3} \mathbf{B}^2 d^3x$ diverges. It turns out that one can obtain non-singular and finite-energy configuration which look like Dirac monopoles ‘from afar’ by embedding the abelian gauge group $U(1)$ in non-abelian gauge groups like $SU(2)$. An example of this is discussed in the second Exercise sheet. A detailed treatment can be found in [3]. We will return to monopoles in non-abelian theories at the end of the course, time permitting. The form part of a large research area concerned with solitons - localised and smooth solutions of partial differential equations.

The first recorded observation of a soliton in nature stems from 1834 - and took place on the union canal near Heriot-Watt University. We end this lecture with the description given by the discoverer, Victorian engineer Scott Russell:

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.
References


