Non-abelian Dyons

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ABSTRACT The dyonic quantum states of magnetic monopoles in Yang-Mills-Higgs theory with a non-abelian unbroken gauge group display a subtle interplay between magnetic and electric properties. This is described in detail in the theory with the gauge group $SU(3)$ broken to $U(2)$ and shown to be captured by the representation theory of the semi-direct product $U(2) \ltimes \mathbb{R}^4$. The implications of this observation for the fusion rules and electric-magnetic duality properties of dyonic states are pointed out.

1 Outline of the problem

One of the most exciting applications of the soliton concept is the physics of elementary particles. This application is very natural in view of the particle-like properties of solitons, but it requires that one complements the classical theory of solitons by quantum concepts. A fully (3+1)-dimensional model in which both the classical and the (semi-) quantised properties of solitons are particularly well-studied is Yang-Mills-Higgs (YMH) theory in the Prasad-Sommerfield limit. The soliton solutions in this theory carry magnetic charge and are called magnetic monopoles. Naturally most is known about the monopole solutions in the theory with the simplest gauge group, namely $SU(2)$ broken to $U(1)$, see [AH88] for a review. They provide a paradigmatic example for classical and quantised soliton properties and also inspired Montonen and Olive to formulate the first precise duality conjecture in a (3+1)-dimensional field theory [MO77].

In this talk I want to discuss the qualitatively new questions that arise when one tries to understand the classical and quantum properties of monopoles in YMH theory with non-abelian unbroken gauge group. The talk is based on research carried out jointly with Sander Bais and reported in the paper [FB97]. To present the questions I want to address in as sharp a light as possible, let me briefly review the salient features of monopoles in $SU(2)$ YMH theory broken to $U(1)$.

1. The solitons carry an integer topological charge which one may interpret as the particle number. The topological charge is an element of the homotopy group $\Pi_2(SU(2)/U(1)$ and equals the monopole's magnetic charge.

2. At low energy it is possible to separate the solitonic "particle" degrees of freedom from other degrees of freedom (such as radiation) in the field theory. One may then truncate the theory and model

soliton dynamics by the time evolution of finitely many collective coordinates. For $SU(2)$ monopoles in the Prasad-Sommerfield limit the moduli spaces of static soliton solutions may be used as collective coordinates. The moduli space of magnetic charge K monopoles is denoted M_K. An elementary but crucial property is that the dimension of M_K increases linearly with K (in this case $\dim M_K = 4K$). Thus the K-particle moduli space has enough degrees of freedom to allow for independent motion of the K particles.

3. The unbroken gauge group acts smoothly on the moduli spaces. Semiclassically, quantum states are realised as wavefunctions on the moduli space and it follows that they can be organised into irreducible representations of the unbroken gauge group. In the case at hand where the unbroken gauge group is $U(1)$ the representations are labelled by a single integer N which physically corresponds to the electric charge. General dyonic quantum states are therefore labelled by two integers: K characterising the magnetic and N characterising the electric charge.

4. Although magnetic and electric charges have a very different mathematical status (the former being a topological charge, the latter being a Noether charge) one can envisage an action on the dyonic states which exchanges the integers K and N. This is what happens in the electric-magnetic duality conjecture of Montonen and Olive.

If one goes through the above list and checks which of these properties still holds for monopoles in theories with non-abelian unbroken gauge group one encounters some surprises. We consider the simplest case, where the gauge group $SU(3)$ is broken to $U(2)$, and formulate some pertinent questions.

1. Here the situation is analogous. Monopoles are topologically classified in terms of $\Pi_2(SU(3)/U(2)) = \mathbb{Z}$. Thus there is again an integer K which specifies the monopole's topological charge and which one can interpret as a particle number.

2. The moduli spaces are still labelled by the topological magnetic charge, but in addition they are stratified. For topological charge $K \geq 0$ there are $[K/2] + 1$ strata (square brackets denote the integer part), with each stratum corresponding to one of $[K/2]+1$ distinct ways in which K monopoles can be put together. Different strata have different dimensions: the largest has dimension $6K$, but all others have lower dimension. What is the physical significance of the strata?

3. The action of the unbroken gauge group $U(2)$ on the moduli space depends on the stratum. This is a manifestation of the effect first noted by Abouelsaood in [Abo83b] and [Abo83a] that dyonic excitations

of monopoles in theories with non-abelian magnetic gauge group do not generally fall into representations of the unbroken gauge group. Instead there is a subtle dependence of the "electric group" on the magnetic charge in such theories. What is the algebraic structure behind this interplay?

4. With topological magnetic charge given by a single integer, and the electric charge sometimes given by $U(2)$ representations and sometimes (as we shall see) by a $U(1) \times U(1)$ representation it is clear that electric-magnetic duality cannot be realised simply as the exchange of magnetic and electric properties. How should one formulate it instead?

The goal of the rest of the talk is to provide a conceptual framework for addressing and partly answering these questions.

2 $SU(3)$ monopoles and their moduli spaces

A monopole solution of $SU(3)$ Yang-Mills-Higgs theory with coupling constant e in the Bogomol'nyi limit is a pair (A_i, Φ) of a SU(3) connection A_i and an adjoint Higgs field Φ on $\mathbb{R}^3$ satisfying the Bogomol'nyi equations

$$D_i\Phi = B_i, \tag{1.1}$$

where $D_i = \partial_i + eA_i$ is the covariant derivative and B_i is the non-abelian magnetic field constructed from A_i. Appropriate boundary conditions have to be imposed to ensure that the energy is finite, and one also demands that the Higgs field has the following form along the positive z-axis:

$$\Phi(0,0,z) = \Phi_0 - \frac{G_0}{4\pi z} + \mathcal{O}(\frac{1}{z^2}), \tag{1.2}$$

where Φ_0 is a constant element of the Lie algebra of $SU(3)$, chosen to lie in the Cartan subalgebra of diagonal traceless matrices. Φ_0 determines the symmetry breaking pattern. If it has three distinct eigenvalues the symmetry is broken maximally to $U(1) \times U(1)$. If two eigenvalues coincide the symmetry is broken minimally to $U(2)$; this is the case of interest here.

The Bogomol'nyi equation relates the Higgs field to the magnetic field and shows that one may interpret G_0 as the vector magnetic charge of the monopole. It also follows from the Bogomol'nyi equation (1.1) that G_0 commutes with Φ_0 [Wei80] so that one may rotate G_0 into the Cartan subalgebra. According to the generalised Dirac condition [EW76],[PGO77], G_0 has to lie on the dual root lattice after that rotation, which means that it has the form

$$G_0 = \frac{2\pi}{e}\mathrm{diag}(m_1, m_2 - m_1, m_2) \tag{1.3}$$

for integers m_1 and m_2. This condition is usually derived without reference to the symmetry breaking pattern. In the case of maximal symmetry breaking one can show that both the integers appearing in (1.3) have a topological significance. In the case of minimal symmetry breaking, however, there is only one topological charge. What is the relevance of the second integer?

In the Bogomol'nyi limit the answer to this question was given in the recent mathematical literature, particularly in the work of Donaldson and Murray. Extending Donaldson's work relating magnetic monopoles to rational maps, Murray showed in [Mur89] that the integer which lacks a topological interpretation characterises holomorphic properties of the monopole. Thus one can say that in general the vector magnetic magnetic charge has topological and holomorphic components. One important difference between the two sorts of charges is revealed by the action of the unbroken gauge group on G_0. In general this generates an orbit, but more precisely it is the holomorphic components which sweep out a non-trivial orbit while the topological components remain invariant. In [FB97] we christened these non-trivial orbits 'magnetic orbits' and pointed out that one can usefully translate the holomorphic charges defined by Murray into numbers characterising these orbits.

In the theory at hand the magnetic orbits are the cosets $U(2)/(U(1) \times U(1))$ and thus have the topology of a two-sphere. As explained in [FB97] these two-spheres all have their centres on the one-dimensional lattice $\{K\Phi_0\}_{K\in \mathbf{Z}}$ in the Lie algebra of $SU(3)$. While the topological charge specifies the position of the centre of the two-sphere the holomorphic charge specifies its radius. As a consequence of the Dirac condition the allowed values for the radius are (positive) half-integers. Thus one may picture the magnetic properties of monopoles in YMH theory with gauge group $SU(3)$ broken to $U(2)$ as two-spheres in the Lie algebra of $SU(3)$ with quantised radii and and centers. Note that these orbits intersect the Cartan subalgebra on the lattice defined by the Dirac condition (1.3). Murray showed that solutions of the Bogomol'nyi equations only exist for certain holomorphic charges; translated into our language his results say that solutions exist for magnetic orbits with arbitrary centers K but only for radii k in the set $\{0, 1, 2, ..., |K|/2\}$ if K is even, and in the set $\{1/2, 3/2, ..., |K|/2\}$ if K is odd.

Monopole moduli spaces are defined as the set of all monopole solutions of a given topological charge K, divided by the group

$$\mathcal{G}_0 = \{g : \mathbb{R}^3 \to SU(3) \,|\, \lim_{z\to\infty} g(0,0,z) = \mathrm{id}\} \tag{1.4}$$

of framed gauge transformations. Murray showed, however, that in a given moduli space monopoles with the same holomorphic charge are grouped together into a stratum. In our picture the magnetic orbit's centre labels the moduli spaces and the magnetic orbit's radius labels the strata. Thus we denote the strata by $M_{K,k}$. The different strata for given K generally have different dimensions, but they form part of one connected space.

So far we have only described the moduli spaces as sets, but it is only when we induce more structure from the field theory onto these spaces that we can use them to answer physical questions. The first question we want to address is the question of how the exact symmetry group is realised in the various magnetic sectors. Seminal papers by Abeoulsaood [Abo83b] [Abo83a] and Nelson and Manohar [NM83] made it clear that "electric" excitations of a monopole with given vector magnetic charge do not, as one might naively expect, fall into representations of the exact group $U(2)$ but only form representations of the subgroup of $U(2)$ which commutes with the vector magnetic charge (the centraliser subgroup). The basic reason for this is that infinitesimal deformations which change the magnetic field at infinity do not satisfy Gauss' law and are therefore physically ruled out. In the $SU(3)$ example these results imply that electric excitations of monopoles with "purely topological" vector magnetic charges ($k = 0$) form $U(2)$ representations while for the vector magnetic charges the electric excitations only carry representations of a $U(1) \times U(1)$ subgroup of $U(2)$. Moreover, in the latter case one of the $U(1)$ groups depends on the vector magnetic charge. In other words, monopoles in this sector may be charged with respect to different $U(1)$ subgroups.

How are these physical facts reflected in the moduli spaces? For $K = 1$ there is only one allowed value of the orbit radius, so the moduli space only has one stratum, namely $M_{1,1/2}$. Murray showed that this space is six-dimensional and that it has the topology $\mathbb{R}^3 \times S^3$. The first factor parametrises the monopole's position and in [FB97] it is further explained that the S^3 part should physically be interpreted by thinking of it as Hopf-fibred over a two-sphere. The base space of this fibration is the magnetic two-sphere described earlier and the fibre is an electric circle, familiar from $SU(2)$ monopoles. Motion on this circle is physical and gives the monopole electric charge, but motion tangent to the base space is forbidden because it violates Gauss' law. The fibration thus captures and makes precise the interplay between magnetic and electric properties suggested by the work of Abeoulsaood *et. al.*. A point on the magnetic two-sphere specifies the monopole's magnetic charge, and only the $U(1)$-rotations about this magnetic direction are permissible electric excitations.

Moving on to topological charge $K = 2$ we find a moduli space with two strata. The space $M_{2,0}$, called the large stratum, contains monopoles with purely topological charge. It is a 12-dimensional smooth manifold and was studied by Dancer in a series of papers [Dan92],[Dan93],[Dan94]. The whole unbroken gauge group $U(2)$ acts smoothly on this space. The stratum $M_{2,1}$, called the small stratum, corresponds to a magnetic orbit of radius one and is 10-dimensional; like the space $M_{1,1/2}$ it is fibred over the magnetic orbit and in a fibre over a given magnetic charge only that charge's centraliser subgroup of $U(2)$ acts smoothly.

Apart from the group action of the unbroken gauge group, a further physically important property of the moduli spaces is the metric they inherit from the kinetic energy of the field theory. No metric can be defined on those parts of the moduli space whose tangent vectors violate Gauss' law, namely the magnetic orbits. In general, the metric structure and the group action of $U(2)$ can be summarised as follows. Those strata of the moduli spaces which are labelled by magnetic orbits of zero radius are smooth manifold with hyperkähler metrics and smooth $U(2)$ actions. All other strata have the structure of a fibre bundle over the magnetic orbit; the fibres of this fibration are smooth hyperkähler manifolds which only permit smooth actions of that $U(1) \times U(1)$ subgroup of $U(2)$ which leaves the magnetic charge labelling the fibre invariant.

3 Dyonic quantum states and the emergence of $U(2) \ltimes \mathbb{R}^4$

In the case where the unbroken gauge symmetry is abelian the following semi-classical bosonic quantisation scheme for monopoles in the BPS limit has been standard in the literature. The Hilbert space of states is taken to be the space of (square-integrable) wavefunctions on the monopole moduli space and the covariant Laplacian on the moduli space plays the role of the quantum Hamiltonian. This prescription can be extended to a supersymmetric situation. If one thinks of the monopoles as classical bosonic solutions in $N = 4$ supersymmetric Yang-Mills theory, the quantum mechanical model for their motion is $N = 4$ supersymmetric quantum mechanics on the moduli space: the Hilbert space is the space of all (square-integrable) forms on the moduli space and the Hamiltonian is the Laplacian acting on forms. An important consistency requirement for this quantum mechanical model is the hyperkähler property of the metric.

In theories with unbroken non-abelian symmetry the above scheme has to be modified. The fibration of the strata is crucial. In each stratum the above quantisation scheme can be applied to the wavefunctions on the fibres (as remarked, these have hyperkähler metrics). By contrast points on the magnetic orbits (the base spaces of the fibration) serve as labels of superselection sectors of the theory. Since a point on the magnetic orbit also specifies which subgroup of the unbroken gauge group can be implemented physically as the electric group one arrives at the conclusion that dyonic quantum states are labelled by a point on the magnetic orbit together with a representation of the centraliser subgroup of that point. In the YMH theory with gauge group $SU(3)$ broken to $U(2)$ we therefore have the following labelling of dyonic states. Writing K for the topological magnetic charge as before and k for the radius of the magnetic sphere, we specify the magnetic charges by giving K and a vector $\mathbf{k}$ on the magnetic sphere (and thus of length k). If $k = 0$ the electric group is the full group $U(2) = (SU(2) \times U(1))/\mathbf{Z}_2$. States in $U(2)$ representations are labelled by three integers j, m, N, with $j = 0, 1/2, ..$ and $m \in \{-j, -j+1, ..., j-1, j\}$ specifying a state in a $SU(2)$ representation and N specifying a $U(1)$ representation; the $\mathbf{Z}_2$ identification requires that $N + 2j$ be even. Thus dyonic states on the strata with trivial magnetic orbits are of the form

$$|K, 0; N, j, m\rangle. \tag{1.5}$$

Introducing an explicit parametrisation of $U(2)$ in terms of a $U(1)$-angle $\chi \in [0, 2\pi)$ and Euler angles (α, β, γ) for $SU(2)$ one can represent the above state as a function on $U(2)$, using the Wigner functions D^j_{ms} on $SU(2)$:

$$\langle \chi, \alpha, \beta, \gamma | K, 0; N, j, m\rangle = e^{N\chi} D^{j*}_{ms}(\alpha, \beta, \gamma) \tag{1.6}$$

(Different values of s lead to equally valid realisations of the state (1.5)). The angles $(\chi, \alpha, \beta, \gamma)$ explicitly appear in the parametrisation of large strata such as $M_{2,0}$, so the above formula shows how to realise dyonic states in that sector as wavefunctions on the moduli space.

In the strata with magnetic orbits of radius $k > 0$, the magnetic charge is specified by giving K and $\mathbf{k}$ as defined above, while the electric group is $U(1) \times U(1)$ (with the second factor being the centraliser group of $\mathbf{k}$ in $SU(2)$) whose representations are labelled by one integer N and one half-integer s. Thus dyonic quantum states in these strata can be written as

$$|K, \mathbf{k}; N, s\rangle. \tag{1.7}$$

In particular for a single monopole, $K = 1$ and $k = 1/2$, we have the additional constraint $N = 2s$, and can represent the above state as a function on $M_{1,1/2}$. Using again Euler angles (α, β, γ) for the S^3 part of that space, and parametrising the direction of $\mathbf{k}$ by spherical coordinates $(\hat{\beta}, \hat{\alpha})$ (so that $\mathbf{k} = (\sin\hat{\beta}\cos\hat{\alpha}, \sin\hat{\beta}\sin\hat{\alpha}, \cos\hat{\beta})$) one finds

$$\langle \alpha, \beta, \gamma | 1, \mathbf{k}, 2s, s\rangle = \delta(\cos\beta - \cos\hat{\beta})\delta(\alpha - \hat{\alpha})e^{is\gamma}. \tag{1.8}$$

A key observation of [FB97] is that all these dyonic states can profitably be interpreted as states in representations of the semi-direct product $U(2) \ltimes \mathbb{R}^4$. The non-trivial part of this group is the double cover of the three dimensional Euclidean group, in whose representation theory the interplay between orbits and centraliser representations is familiar, albeit in a different guise. In that context irreducible representations $V_{k,s}$ are labelled by the magnitude $k > 0$ of the momentum vector $\mathbf{k}$ and a half integer s specifying the helicity, which is a representation of the $U(1)$ subgroup which leaves a specified momentum vector invariant. These representations are infinite dimensional, and in it translation and helicity eigenstates are labelled by a momentum vector $\mathbf{k}$ of length k and the half-integer s. If $\mathbf{k} = 0$ the representation spaces are written $V_{0,j}$ and isomorphic to the usual $(2j+1)$-dimensional spin j representations of $SU(2)$. Here I have chosen a notation that makes the correspondence between the different physical situations evident; for explicit formulae which also include the $U(1)$ part of $U(2)$ I refer the listener to the paper [FB97].

Interpreting dyonic states as elements of $U(2) \ltimes \mathbb{R}^4$ representations answers the questions posed under points 2. and 3. in out initial list. The labels of the different strata of the moduli spaces now gain a group-theoretic interpretation, and quantum states on different strata can be combined according to the Clebsch-Gordan coefficients of $U(2) \ltimes \mathbb{R}^4$. In particular one can now understand how a dyonic state in the large stratum $M_{2,0}$ (which carries a $U(2)$ representation) can be a tensor product of two dyonic states of two single monopoles (which only carry $U(1)$ electric charges). The trick that made this possible was to interpret both magnetic and electric properties as representation labels of one algebraic object. I should also emphasise that this trick depends crucially on the inclusion of the full magnetic orbit in the discussion. Restricting attention to a particular magnetic charge, or (as many authors have done) to the Weyl orbit of a particular charge, makes a consistent fusion algebra of non-abelian dyons impossible.

4 Discussion and outlook

While much progress in solving our initial puzzles could be made by interpreting non-abelian dyons in the theory discussed here as carriers of $U(2) \ltimes \mathbb{R}^4$ representations, a number of open questions remain. The first concerns the Dirac quantisation of the magnetic orbits. While this is a fundamental fact from the point of view of monopole physics it has to be imposed artificially in the representation theory of $U(2) \ltimes \mathbb{R}^4$. Also, to respect this condition when multiplying dyonic states in a tensor product one has to require that the magnetic charge vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ of the two dyons to be multiplied are either parallel or anti-parallel. While these conditions can be imposed consistently they suggest that there is a further algebraic object, related to semi-direct products but in some way more restrictive, whose representation theory incorporates the Dirac conditions from the start.

Similarly (and perhaps relatedly) more work is required to answer the question posed under point 4. or our list, namely the proper formulation of duality in Yang-Mills theory with non-abelian unbroken gauge group. The organisation of dyonic states into representations of semi-direct product groups allows one to formulate this question in sharper language. In [FB97] a natural candidate for a electric-magnetic duality transformation is discussed which acts on the representations $V_{k,s}$ by exchanging the magnetic orbit label k with the electric centraliser label s; since k is quantised by the Dirac condition such an exchange makes sense. However, while this appears to be satisfactory as long as both k and s are non-vanishing, it is problematic when $s = 0$. For $k = 1/2$ for example this prescription would then relate the *infinite*-dimensional representation $V_{1/2,0}$ containing all single monopole states with the purely electric *two*-dimensional representation $V_{0,1/2}$. Since duality is supposed to relate degrees of freedom which are in a suitable sense equivalent, such a mismatch of dimensions is not acceptable. It seems that further conceptual progress is required before we fully understand non-abelian dyons.

Acknowledgments: The work reported here was done in collaboration with Sander Bais and financially supported by a Pionier fund of the Nederlandse Organisatie voor Wetenschappelijk Onderzoek. I thank the organisers for organising a stimulating workshop and for inviting me to speak.

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