Linear Functional Analysis, Second Edition

Misprints and further comments:

- **Page 50, Ex. 2.12 (b):** replace $z_n = (r - n^{-1})z$ with $z_n = (1 - n^{-1})z$.
- **Page 55, Ex. 3.8:** replace $(\cdot,\cdot) : \mathbb{F}^k \times \mathbb{F}^k \to \mathbb{F}$ with $(\cdot,\cdot) : \ell^2 \times \ell^2 \to \mathbb{C}$.
- **Page 72, Ex. 3.20:** replace 'inner product space' with 'Hilbert space'
- **Page 83, Proof of Theorem 3.56:** replace the given definition of the function f_δ with the following definition:

  $$f_\delta(x) = \begin{cases} 0, & \text{if } x \in [0,\delta] \cup [\pi - \delta, \pi], \\ f(x), & \text{if } x \in (\delta, \pi - \delta), \end{cases}$$

  with $0 < \delta < \pi/2$.
- **Page 81, Ex. 3.25:** change $x \in X$ to $x \in M$.
- **Page 85, Ex. 3.28 (c):** delete the term $(2^n n!)^2$ on the right hand side of the formula (the calculation in the solution is correct, but needs to be divided by $(2^n n!)^2$ to yield the formula in the question)
- **Page 114:** Since

  $$\|\sum_{j=1}^{\infty} 2^{-j} x_j\| \le \sum_{j=1}^{\infty} 2^{-j} = 1,$$

  the series $\sum_{j=1}^{\infty} 2^{-j} x_j$ is absolutely convergent, so it follows immediately from Theorem 2.30 that it converges to some $x \in \overline{B_{0,X}(1)}$ (the partial sums argument given on page 114 is written slightly incorrectly, and is not needed anyway since it is already in the proof of Theorem 2.30).
- **Page 118, line 8:** change $\mathrm{Im}(T_f) = C[0,1]$ to $\mathrm{Im}(T_f) \supset C[0,1]$.
- **Page 135:** add the following hypothesis to Corollary 5.22: 'Suppose that $X \neq \{0\}$'.
- **Page 144, Exercise 5.11:** in addition to being non-empty and convex, the set A must be open. In fact, if A is not open the stated result need not be true — we can construct an example as follows.

  Let $x_0 = 0$, choose U to be a dense linear subspace of X with $U \neq X$ (for example, see Exercise 5.2 (c); see also Exercise 5.7), and choose a point $a \notin U$ and let $A = \{a\}$. Since U is dense, its closure $\overline{U} = X$, so there is no closed hyperplane H containing U (the only closed subspace containing U is X, which of course intersects A and is not a hyperplane, see Definition 5.31).

  In addition, on rereading this exercise and its solution, it seems that some further explanation would clarify the solution, so a rewritten version is as follows.

Choose $a_0 \in A$ and let $w_0 = x_0 - a_0$ and $C = w_0 + A - B$, as before. We first note that $w_0 \in U \iff x_0 - a_0 = u$, for some $u \in U$, that is, $x_0 - u \in A$, which contradicts the condition $A \cap (x_0 + U) = \emptyset$, so we have $w_0 \notin U$. Hence, we can define $W = \mathrm{Sp}\{w_0\} \oplus U$ and $f_W(\alpha w_0 + u) = \alpha$ for $\alpha \in \mathbb{R}$, $u \in U$. Next, it also follows from $A \cap (x_0 + U) = \emptyset$ that, for any $u \in U$, $w_0 + u \notin C$, and so $p_C(w_0 + u) \geqslant 1$. Hence, f_W satisfies (5.3), and so f_W has an extension $f \in X'$. Now, with $\gamma = f(x_0)$, it follows that $f(a) < \gamma = f(x_0 + u)$ for $a \in A$, $u \in U$, that is, $x_0 + U \subset H = f^{-1}(\gamma)$, and $A \cap H = \emptyset$.

- **Page 187:** the proof of Theorem 6.39 (a) refers to Lemma 4.35, but the statement of Lemma 4.35 is not strong enough to give the required result. The required result is as follows.

Lemma 4.35

Let X, Y, Z be normed linear spaces, and $T_1 \in B(X,Y)$, $T_2 \in B(Y,Z)$.

(A) If T_1, T_2 are invertible then:

(a) T_1^{-1} is invertible with inverse T_1;

(b) T_2T_1 is invertible with inverse $T_1^{-1}T_2^{-1}$.

(B) If $X = Y = Z$ and T_1 and T_2 commute then:

(a) if T_1 is invertible then T_1^{-1} and T_2 commute;

(b) if T_1T_2 is invertible then T_1 and T_2 are invertible.

Proof

(A) The proof is in Exercise 4.15.

(B) (a) Multiply each side of the equation $ST = TS$ by S^{-1}.

(B) (b) Writing $S = T_1T_2$, the inverse of T_1 (for example) is $S^{-1}T_2$ (check).

- **Page 281:** the solutions to Exercises 5.2 and 5.3 are slightly mixed up, and one part is missing. For simplicity, the full statements of these exercises and their solutions are written out below.

Ex. 5.2 Show the following:

(a) ℓ^p is separable for $1 \leqslant p < \infty$;

(b) ℓ^∞ is not separable;

(c) $\mathcal{S}$ is separable and dense in ℓ^p, $1 \leqslant p < \infty$, but $\mathcal{S}$ is not dense in ℓ^∞.

Sln. 5.2 (a) A simple adaptation of the proof of Theorem 3.52 (b) shows that ℓ^p is separable for all $1 \leqslant p < \infty$.

(b) Let x^k, $k \geqslant 1$, be an arbitrary sequence in ℓ^∞, with each x^k having the form $x^k = (x_1^k, x_2^k, \dots)$. Now define $z \in \ell^\infty$ as follows: for each $n \geqslant 1$, let

$$z_n = \begin{cases} x_n^n + 1, & \text{if } |x_n^n| \leqslant 1, \\ 0, & \text{if } |x_n^n| > 1. \end{cases}$$

Clearly, $\|z - x^k\|_\infty \geqslant |z_k - x_k^k| = 1$ for all $k \geqslant 1$, so the set $\{x^k\}$ is not dense in ℓ^∞.

(c) Suppose that $1 \leqslant p < \infty$, and let $z \in \ell^p$ and $\epsilon > 0$ be arbitrary. By definition, there exists an integer $k \geqslant 1$ such that $\left(\sum_{n=k}^{\infty} |z_n|^p\right)^{1/p} < \epsilon$. Now define $x \in \mathcal{S}$ by $x_n = z_n$, $n \leqslant k$, $x_n = 0$, $n > k$. Then $\|z - x\|_p < \epsilon$, which shows that $\mathcal{S}$ is dense in ℓ^p. Since ℓ^p is separable, $\mathcal{S}$ must be separable (by Theorem 1.43).

Next, define $z = (1, 1, \dots) \in \ell^\infty$. If $x \in \mathcal{S}$ then x has only finitely many non-zero entries, so $\|z - x\|_\infty \geqslant 1$, and hence $\mathcal{S}$ cannot be dense in ℓ^∞.

Ex. 5.3 Let c_0 be the linear subspace of ℓ^∞ consisting of all sequences which converge to 0. Show the following:

(a) $\mathcal{S}$ is dense in c_0, and $\mathcal{S}$ and c_0 are separable (with respect to the ℓ^∞ norm);

(b) c_0 is closed in ℓ^∞;

(c) A linear operator $T_{c_0} : \ell^1 \to c_0'$, can be constructed as in Theorem 5.5, and T_{c_0} is an isometric isomorphism.

Sln. 5.3 (a) An adaptation of the proof of Theorem 3.52 (b) shows that the (countable) set of sequences in $\mathcal{S}$ with rational terms is dense in $\mathcal{S}$. Next, let $z \in c_0$ and $\epsilon > 0$ be arbitrary. By the definition of c_0, there exists $k \geqslant 1$ such that $|z_n| < \epsilon$ for all $n \geqslant k$. Now define $x \in \mathcal{S}$ by $x_n = z_n$, $n \leqslant k$, $x_n = 0$, $n > k$. Then $\|z - x\|_\infty < \epsilon$, so $\mathcal{S}$ is dense in c_0. Hence, $\mathcal{S}$ and c_0 are separable.

(b) Suppose that c_0 is not closed in ℓ^∞. Then there exists a sequence $x^k \in c_0$, $k = 1, 2, \dots$, and an $x \in \ell^\infty \setminus c_0$ such that $\|x^k - x\|_\infty \to 0$. Since $x \notin c_0$, there is a $\delta > 0$ such that $|x_n| > \delta$ for infinitely many $n \geqslant 1$. However, for each $k \geqslant 1$ we have $\lim_{n\to\infty} |x_n^k| = 0$, so by the definition of $\|\cdot\|_\infty$, $\|x^k - x\|_\infty \geqslant \delta$ which is a contradiction.

(c) The proof that $T_{c_0} : \ell^1 \to (c_0)'$, is a linear isometric isomorphism now follows the proof of Theorem 5.5, with some minor differences to the inequalities due to the sequences lying in ℓ^∞ and ℓ^1, rather than in ℓ^p and ℓ^q.

- **Page 232-3, Exercise 7.27, and its solution:** since we allow $r(S) = \infty$ here, statements of the form $n = 1, \dots, r(S)$ seem somewhat poorly written; these should be replaced with $1 \leqslant n \leqslant r(S)$ (with the obvious meaning of $n \leqslant \infty$).

- **Page 255, line -3:** change $\xi_0 = \xi(a)$ to $\xi_0 = h(a)$.

- **Pages 259-260:** the equations at the bottom of p. 259 and the top of p. 260 are missing an e_n throughout the eigenvector expansions, and should read:

$$u - \sum_{n=1}^{k} (u, e_n) e_n = G\left(Tu - \sum_{n=1}^{k} (Tu, e_n) e_n \right),$$

$$\left\| u - \sum_{n=1}^{k} (u, e_n) e_n \right\|_X = \left\| G\left(Tu - \sum_{n=1}^{k} (Tu, e_n) e_n \right) \right\|_X$$

$$\leqslant M(b-a)^{1/2} \left\| Tu - \sum_{n=1}^{k} (Tu, e_n) e_n \right\|_{\mathcal{H}} \to 0, \quad \text{as } k \to \infty$$

Page 262, line 6: (8.31) should be (8.27).

- **Page 292, Solution 6.19:** the given solution is incomplete. At one point the solution says 'Hence if $|\lambda| < 2$ then $\lambda \in \sigma(T)\ldots$', but the argument given to prove this in fact only shows that at least one of $\lambda, -\lambda$ must belong to $\sigma(T)$. It will complete the proof if we can show that $\lambda \in \sigma(T) \iff -\lambda \in \sigma(T)$.

  To do this we define $S : \ell^2 \to \ell^2$ by

  $$S(x_1, x_2, x_3, x_4, \ldots) = (x_1, -x_2, x_3, -x_4, \ldots).$$

  Then $S \in B(\ell^2)$ and $S^2 = I$ (so S is invertible), and we have

  $$\begin{aligned} STS(x_1, x_2, x_3, x_4, \ldots) &= ST(x_1, -x_2, x_3, -x_4, \ldots) \\ &= S(0, 4x_1, -x_2, 4x_3, -x_4, \ldots) \\ &= (0, -4x_1, -x_2, -4x_3, -x_4, \ldots) \\ &= -T(x_1, x_2, x_3, x_4, \ldots). \end{aligned}$$

  Hence,

  $$\begin{aligned} \lambda \notin \sigma(T) &\iff S(T - \lambda I)S \ \text{ is invertible} \\ &\iff STS - \lambda I \ \text{ is invertible} \\ &\iff -T - \lambda I \ \text{ is invertible} \\ &\iff T + \lambda I \ \text{ is invertible} \\ &\iff -\lambda \notin \sigma(T), \end{aligned}$$

  which completes the proof.

- **Page 300, Solution 7.11 (c):** For the definition of a Hilbert-Schmidt operator to make sense we need to assume that H is separable, and so has an orthonormal basis (the rest of the exercise does not require this). The given solution then works if the orthonormal sequence $\{e_n\}$ is a basis for H. If not, let $Y = \overline{\mathrm{Sp}}\{e_n\}$ and let $\{g_n\}$ be an orthonormal basis for $Y^\perp$. The union $\{e_n\} \cup \{g_n\}$ is then a basis for H (see Exercise 3.26) and it is clear from the definitions that $Tg_n = 0$ for all the vectors in $\{g_n\}$. Hence, we can use the basis $\{e_n\} \cup \{g_n\}$, together with the previous calculation, to show that T is Hilbert-Schmidt.

- **Page 302, Solution 7.16:** this solution should use the adjoint operator T^* as follows.
  Since $\mathcal{H}$ is not separable it follows from Theorem 7.8 that $\overline{\mathrm{Im}\, T^*} \neq \mathcal{H}$ (by Theorem 7.14, T^* is compact), so $\mathrm{Ker}\, T = \mathrm{Ker}\, T^{**} \neq \{0\}$, by Lemma 6.11 (c). Thus, there exists $e \neq 0$ such that $Te = 0$, that is, e is an eigenvector of T with eigenvalue 0.