Varieties of elliptic solitons

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Abstract. We set out a method for giving explicit algebraic coordinates on varieties of elliptic solitons, which consists of: finding the spectral curve by elimination; and solving the Jacobi inversion problem by the use of Kleinian functions and their identities. As an example we solve a 5-particle Elliptic Calogero-Moser system whose spectral curve turns out to be completely reducible over 3 equianharmonic elliptic curves.

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1. Introduction

Dubrovin and Novikov observed in 1974 [15] that the 2-gap Lamé potential $6\wp(x)$ allows an isospectral deformation

$$6\wp(x) \longrightarrow \mathcal{U}(x,t) = 2\wp(x - x_1(t)) + 2\wp(x - x_2(t)) + 2\wp(x - x_3(t)), \tag{1}$$

under the action of the KdV flow. This potential exemplifies the link between the pole dynamics of elliptic solutions of integrable hierarchies and the Elliptic Calogero-Moser system. This connection was discovered by Airault, McKean and Moser [1]. In 1980, Krichever gave the algebro-geometric description of such pole dynamics [26]: he found the curve which is a generator for the Calogero-Moser integrals of motion and whose θ-divisor gives the poles of the system. Further geometric understanding was provided by Verdier [32] and his student and collaborator Treibich [29] and the whole area was named *elliptic solitons* after them. Nevertheless, only special examples of an explicit description of elliptic solitons are known at present. [17, 18, 20].

The origin of our work lies in the following observation presented in the remarkable paper [1], p. 139: the Jacobi variety of a genus two curve, which is associated with the two-gap Lamé potential $6\wp(x)$, is the fibration whose base and fibres are the Weierstrass cubics with moduli

$$g_2, g_3 \quad \text{and} \quad \widetilde{g_2} = \frac{3^3}{2^2}(g_2^3 + 9g_3^2), \quad \widetilde{g_3} = \frac{3^5}{2^3} g_3(g_2^3 - 3g_3^2)$$

respectively. The dynamics of the Calogero-Moser system is then described by the elliptic surface whose coordinates are: the Weierstrass elliptic functions with moduli belonging to the first torus and depending on differences of the Calogero-Moser particles; and time-dependent Weierstrass functions whose moduli belong to the second torus. The evaluation of this second elliptic curve given in [1] involved some serendipity which was elucidated by one of the authors [17] from the viewpoint of the Weierstrass-Poincaré reduction theory of Abelian integrals and θ-functions to lower genera. Our paper is intended to extend the above observation to a wider class of curves to obtain a family of explicit solutions of the Elliptic Calogero-Moser system.

Our approach includes several ingredients. One is the Burchnall-Chaundy theory as applied by one of the authors [27]. Another is the Weierstrass-Poincaré reduction theory of Abelian functions to lower genera, which was applied to completely integrable equations in [4]. We also use the recent development by Buchstaber et al. [6, 7, 8] of the Weierstrass-Klein formulation of the theory of Abelian functions. In what follows we consider in such a context a reasonably wide class of (n, s)-curves, i.e. curves of the form

$$w^n - z^s + \text{ lower order terms } = 0,$$

where n, s are coprime positive integers.

The outline of our work is the following. We consider a set of (n, s)- curves, which cover elliptic curves in such a way that the associated σ-function can be factored

$$\sigma(t_1, \ldots, t_{g-1}, x) = \prod \sigma_W(x - x_i(t_1, \ldots, t_{g-1})), \quad t_g = x,$$

where σ_W is the Weierstrass elliptic function and M is the multiplicity of the σ-divisor. The variables $t_1, t_2, \ldots$ are the "times" of the integrable hierarchy or the coordinates of the Jacobi variety of the curve. Note that our indexing of the time variable is in the reverse order to the one conventionally used for KP. The functions x_i are evaluated on the flow variables of the Elliptic Calogero-Moser system.

We suppose further that the curve covers additional tori. In the most successful case, all the independent holomorphic differentials of such special curves are reduced to elliptic differentials by some rational substitution. Such class of curves include all the genus two coverings and some curves of higher genera, in particular those which admit sufficiently many automorphisms. As a result, the Jacobi inversion problem is reduced to the inversion of elliptic integrals.

We solve the Jacobi inversion problem for the Calogero-Moser particles x_i by using the solution of the problem in terms of Kleinian functions. The inversion of the elliptic integrals leads to the algebraic varieties whose coordinates are elliptic functions of the "times" $t_1, t_2 \ldots$ and elliptic functions associated with the torus parallel to the x flow.

Our approach is exemplified by the case of (3,4) curves, which describes the Calogero-Moser dynamics associated with the Boussinesq flow. The explicit description of the Calogero-Moser dynamics also serves to describe the σ-divisor of the covering in closed form.

The paper is organized as follows. In Section 2 we give a short exposition of the necessary recent results in the theory of Abelian functions of (n,s)-curves. In Section 3 we describe the methods that yield suitable curves, based on the Burchnall-Chaundy theory for the case of spectral curves that cover elliptic curves. Section 4 exemplifies our approach by treating the 5 particle dynamics of the Calogero-Moser system under the action of the Boussinesq flow.

2. (n,s)-curves and their Abelian functions

In this section we shall give a short introduction to the theory of Abelian functions for a class of algebraic (n,s)-curves, developed in [7].

Definition 2.1 *The algebraic curve $V_{n,s} = (z,w)$ is called an (n,s)-curve if n,s are coprime, $2 \le n < s$ and the curve can be realized in the form*

$$V_{n,s} := w^n - z^s + \sum_{\alpha,\beta} \lambda_{\alpha n+\beta s} z^\alpha w^\beta = 0, \tag{2}$$

where $0 \le \alpha < s-1, 0 \le \beta < n-1$, $\alpha n + \beta s < ns$ and g is the integer

$$g = \frac{(n-1)(s-1)}{2}. \tag{3}$$

The (n,s)-curve is non-degenerate if its discriminant with respect to the variable w has no multiple roots; in this case g is the genus.

Definition 2.2 *The Weierstrass gap sequence generated by the coprime numbers (n,s) is the set of the positive integers $w_1, \ldots,$ which are* not *representable in the form $an + bs$, $a, b \in \mathbb{N} \cup \{0\}$. The number of these integers is called the length.*

Recall that the Schur function $s_{\boldsymbol{\pi}}$ associated with the partition $\boldsymbol{\pi}$ of the length g, i.e. with the set of g non-increasing positive integers $(\pi_1, \ldots, \pi_g) = \boldsymbol{\pi}$ is given as

$$s_{\boldsymbol{\pi}} = \det(e_{\pi_i - i + j})_{1 \le i,j \le g}, \tag{4}$$

where e_k are the elementary symmetric functions. The functions e_k can be expressed in terms of elementary Newton polynomials p_k of weight k.

Theorem 2.1 (see [7].) *Let $\boldsymbol{W}_{n,s}$ be the Weierstrass gap sequence of an (n,s)-curve, and the set $\pi_k = w_{g-k+1} + k - g, \quad w_{g-k+1} \in \boldsymbol{W}_{n,s}, \quad k = 1, \ldots, g$, be the partition.*

The Schur function (4) associated with the partition $\boldsymbol{\pi}$ is represented as the polynomial $\sigma_{n,s}(u_1, \ldots, u_g) = \sigma_{n,s}(\boldsymbol{u})$ of g variables $u_i = p_{w_i}$ where the weights $w_i \in \boldsymbol{W}_{n,s}$. We shall call this the Schur-Weierstrass polynomial.

The degree of the Schur-Weierstrass polynomial $\sigma_{n,s}(\boldsymbol{u})$ in the variable u_1 is said to be the multiplicity $M_{n,s}$ of the σ-divisor and is given by the formula

$$M_{n,s} = \frac{(n^2-1)(s^2-1)}{24}. \tag{5}$$

It is well known that for the curve $V_{n,s}$ of genus g there exists a $2g \times 2g$-matrix $\mathfrak{M} = \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}$, where $\omega, \omega', \eta, \eta'$ are $g \times g$-matrices, such that $\det \omega \neq 0$, $\omega^{-1}\omega'$ is symmetric, $\mathrm{Re}(\omega^{-1}\omega')$ is positive definite and

$$\mathfrak{M} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \mathfrak{M}^T = \frac{\sqrt{-1}\pi}{2} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}. \tag{6}$$

The constructive definition of the matrix $\mathfrak{M}$ is the following. Introduce the Riemann surface of the curve $V_{n,s}$ and its canonical homology basis of $2g$ $\mathfrak{a}$ and $\mathfrak{b}$-cycles. Canonical holomorphic differentials $\mathrm{d}\mathbf{u} = (\mathrm{d}u_1, \ldots \mathrm{d}u_g)^T$ are defined by the formula

$$\mathrm{d}u_i(z,w) = z^{\alpha_i} w^{\beta_i} \frac{\mathrm{d}z}{f_w}, \quad i = 1, \ldots, g, \tag{7}$$

where the pair of positive integers α_i, β_i represents the i-th element from the set of the first g non-gaps, $\alpha_i n + \beta_i s \notin \boldsymbol{W}_{n,s}$. The matrices ω and ω' are

$$2\omega = \left(\oint_{\mathfrak{a}_i} \mathrm{d}u_j \right)_{i,j=1,\ldots,g}, \quad 2\omega' = \left(\oint_{\mathfrak{b}_i} \mathrm{d}u_j \right)_{i,j=1,\ldots,g}. \tag{8}$$

The matrices η and η' are are defined as the periods,

$$2\eta = \left(-\oint_{\mathfrak{a}_i} \mathrm{d}r_j \right)_{i,j=1,\ldots,g}, \quad 2\eta' = \left(-\oint_{\mathfrak{b}_i} \mathrm{d}r_j \right)_{i,j=1,\ldots,g} \tag{9}$$

of the associated meromorphic differentials $\mathrm{d}\mathbf{r} = (\mathrm{d}r_1, \ldots \mathrm{d}r_g)^T$, defined by solving the relations

$$\begin{aligned} &\frac{\mathrm{d}\Omega((z,w),(x,y))}{\mathrm{d}x} - \frac{\mathrm{d}\Omega((z,w),(x,y))}{\mathrm{d}z} \\ &\quad = \sum_{k=1}^{g} \left\{ \frac{\mathrm{d}u_i(x,y)}{\mathrm{d}x} \frac{\mathrm{d}r_i(z,w)}{\mathrm{d}z} - \frac{\mathrm{d}u_i(z,w)}{\mathrm{d}z} \frac{\mathrm{d}r_i(x,y)}{\mathrm{d}x} \right\}, \end{aligned} \tag{10}$$

where

$$\Omega((z,w),(x,y)) = \frac{1}{(x-z)f_y} \sum_{k=1}^{n} y^{n-k} \left(\frac{f(z,w)}{w^{n-k+1}} \right)_+,$$

and $(\cdot)_+$ means that we are taking only non-negative powers into account.

Now we are in a position to introduce the σ-function. Let us define the Abel map $\mathfrak{A} : (V_{n,s})^k \longrightarrow \mathbb{C}^g$ with the aid of the holomorphic integrals

$$u_j = \sum_{i=1}^{k} \int_{(x_k,y_k)}^{(\infty,\infty)} \mathrm{d}u_j(x,y), \quad j = 1, \ldots, g, \quad k \geq g. \tag{11}$$

Although we have given the both components of the limits of integration here, for typographical convenience we shall give only the first co-ordinate in these limits in the remaining part of the paper. The Jacobi variety $\mathrm{Jac}(V_{n,s}) = \mathbb{C}^g/2\omega \oplus 2\omega'$ is the natural domain of the σ-function, which is characterised as follows

Definition 2.3 *The fundamental σ-function is the entire function in* $\mathrm{Jac}(V_{n,s})$*, which satisfies the two sets of functional equations*

$$\begin{aligned}\sigma(\boldsymbol{u} + 2\omega\boldsymbol{k} + 2\omega'\boldsymbol{k}; \mathfrak{M}) &= \exp\{2(\eta\boldsymbol{k} + \eta'\boldsymbol{k}')(\boldsymbol{u} + \omega\boldsymbol{k} + \omega'\boldsymbol{k}')\}\sigma(\boldsymbol{u}; \mathfrak{M})\\ \sigma(\boldsymbol{u}; \gamma\mathfrak{M}) &= \sigma(\boldsymbol{u}; \mathfrak{M}), \gamma \in \mathrm{Sp}(2g; \mathbb{Z})\end{aligned}$$

the first of these equations display the periodicity property, while the second one the modular property. In addition, the first term of the of the σ-series is the Schur-Weierstrass polynomial $\sigma_{n,s}(\boldsymbol{u})$.

For our purposes, the fundamental σ-function is defined as an automorphic element of the ring of θ-functions. But in what follows we shall not need this definition, which explicitly generalises the known expression for the Weierstrass σ-function in terms of θ-functions. The Abelian functions are then introduced as the second logarithmic derivatives

$$\wp_{i,j}(\boldsymbol{u}) = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(\boldsymbol{u}; \mathfrak{M}), \quad i, j = 1, \ldots, g. \tag{12}$$

The following formula, due to Klein, is of great importance for our exposition.

Theorem 2.2 (Klein [24]) *Let $(y(x_0), x_0)$, (y, x) be arbitrary distinct points on $V_{n,s}$ and let $\{(y_1, x_1), \ldots, (y_g, x_g)\}$ be any set of distinct points $\in (V_{n,s})^g$. Then the following relation is valid for every $r = 1, \ldots, g$*

$$\begin{aligned}&\sum_{i,j=1}^{g} \wp_{ij}\left(\int_{x_0}^{x} \mathrm{d}\mathbf{u} - \sum_{k=1}^{g}\int_{x_0}^{x_k} \mathrm{d}\mathbf{u}\right)\mathcal{U}_i(x, y)\mathcal{U}_j(x_r, y_r)\\ &\quad = \frac{F((x, y); (x_r, y_r))}{(x - x_r)^2},\end{aligned} \tag{13}$$

where the monomials $\mathcal{U}_i(x, y)$ are numerators of the corresponding holomorphic differentials, and $F((x, y); (z, w))$ is given in terms of $\Omega((z, w), (x, y))$ and the canonical differentials $\mathrm{d}\mathbf{u}$ *and* $\mathrm{d}\mathbf{r}$*,*

$$F((x, y); (z, w)) = \frac{\mathrm{d}\Omega((z, w), (x, y))}{\mathrm{d}z} + \sum_{k=1}^{g} \frac{\mathrm{d}u_i(x, y)}{\mathrm{d}x}\frac{\mathrm{d}r_i(z, w)}{\mathrm{d}z}.$$

From this formula of Klein's one derives various relations between Abelian functions. Such relations are used to solve the Jacobi inversion problem in terms of Kleinian functions as well as to construct the meromorphic embedding of the Jacobi and Kummer varieties into projective space.

3. Burchnall-Chaundy curves

It was already mentioned in the introduction that the (n, s) curve must be a covering of an elliptic curve. A wide class of such curves can be described by the Burchnall-Chaundy theory [9, 10].

Proposition 3.1 *Let L_n and P_s be commuting differential operators of coprime orders n and s,*

$$L_n = \frac{\mathrm{d}^n}{\mathrm{d}x^n} + \mathcal{L}_{n-2}(x)\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}} + \ldots + \mathcal{L}_0(x), \tag{14}$$

$$P_s = \frac{\mathrm{d}^s}{\mathrm{d}x^s} + \mathcal{P}_{s-2}(x)\frac{\mathrm{d}^{s-2}}{\mathrm{d}x^{s-2}} + \ldots + \mathcal{P}_0(x), \tag{15}$$

where $\mathcal{L}_0, \ldots, \mathcal{L}_{n-2}$ and $\mathcal{P}_0, \ldots, \mathcal{P}_{n-2}$ are meromorphic functions, elliptic (i.e., doubly periodic) in x. Then L_n and P_s satisfy an algebraic equation of type (2), and the (n, s) curve $V_{n,s}$ covers an elliptic curve.

This statement is equivalent to the observation that the ring of functions on the spectral curve, isomorphic to $\mathbb{C}[L_n, P_s]$ as Burchnall-Chaundy proved, consists of functions that descend to an elliptic curve. We shall apply this statement, in the case when the functions $\mathcal{L}_0, \ldots, \mathcal{L}_{n-2}$ and $\mathcal{P}_0, \ldots, \mathcal{P}_{n-2}$ are elliptic functions of the curve realised as the Weierstrass cubic

$$V_{2,3} = (\mu, \nu) : \nu^2 = 4\mu^3 - g_2\mu - g_3. \tag{16}$$

The commutativity of the operators L_n and P_s guarantees a solution to the eigenvalue problem

$$L_n\Psi(x; \alpha) = z\Psi(x; \alpha), \quad P_s\Psi(x; \alpha) = w\Psi(x; \alpha) \tag{17}$$

exactly when (w, z) is a point of the curve $V_{n,s}$. In our case, in view of the periodicity in x of all the operators in $\mathbb{C}[L_n, P_s]$, the algebraic curve $V_{n,s}$ is a covering of the elliptic curve $V_{2,3}$. Moreover, the attendant solution of the KP equation is also doubly-periodic, being a constant multiple of $\mathcal{L}_0$ as a funtion of x; thus, Appendix A3 of [30] guarantees that this is a very special type of covering, namely a "tangential cover" in the sense of [29]. Even though the tangential position is very special, the curve $V_{2,3}$ is by no means unique; in fact, any isogenous quotient of $V_{2,3}$ will have the same property. Following [30], we call the tangential cover $V_{n,s} \to V_{2,3}$ "minimal" when it cannot be factored into a (non-equivalent) tangential cover and an isogeny. In what follows we tacitly assume that all tangential covers are minimal in this sense; this affects the statements on their degree.

Definition 3.1 *We shall call the operators from Proposition 3.1 a Burchnall-Chaundy pair and the algebraic curve $V_{n,s}$ the associated Burchnall-Chaundy curve. When the coefficients of the Burchnall-Chaundy operators are elliptic functions, we shall call the Burchnall-Chaundy curve a Burchnall-Chaundy tangential cover.*

Following Halphen [21] (cf. Hermite [22] p. 372), we introduce the following ansatz for the eigenfunction

$$\Psi(x; \alpha) = \mathrm{e}^{kx} \sum_{j=0}^{p} a_j(z, \mu, k)\frac{\partial^j}{\partial x^j}\Phi(x; \alpha), \tag{18}$$

where p is an appropriate positive integer, and ‡

$$\Phi(x; \alpha) = \frac{\sigma(\alpha - x)}{\sigma(\alpha)\sigma(x)} \exp\{\zeta(\alpha)x\},$$

is a solution of

$$\Phi_{xx}(x; \alpha) - (2\wp(x) + \wp(\alpha))\Phi(x; \alpha) = 0.$$

‡ Here and below we use the standard notations of the Weierstrass theory of elliptic functions [3].

The Weierstrass elliptic function $\wp(x)$ and the function $\Phi(x;\alpha)$ have the following expansion in a vicinity of $x=0$

$$\begin{aligned}\wp(x) \quad &= \frac{1}{x^2}+\frac{1}{20}g_2x^2+\frac{g_3}{28}x^4+\ldots \\ \Phi(x;\alpha) &= \frac{1}{x}-\frac{1}{2}\wp(\alpha)x+\frac{1}{6}\wp'(\alpha)x^2-\frac{1}{8}\wp(\alpha)^2x^3+\frac{1}{40}g_2x^3+\frac{1}{60}\wp(\alpha)\wp'(\alpha)x^4+\ldots\end{aligned} \tag{19}$$

Substituting the expansions (19) into the eigenvalue problems (17), we then derive two groups of equations, from the conditions of vanishing of the principal parts of the poles at $x=0$. The equation of the Burchnall-Chaundy curve as well as an explicit expression for the cover will follow from their compatibility.

The following proposition can be proved by this approach

Proposition 3.2 *Let $V_{n,s}$ be a non-degenerate (n,s)-curve which is a Burchnall-Chaundy tangential cover of the torus $\{(\wp,\wp')\}$ and whose eigenfunction satisfies the Halphen ansatz (18). Then*

(i) *The degree of the cover is the multiplicity $M_{n,s}$ of the σ-divisor.*

(ii) *The principal parts of the associated Burchnall-Chaundy operators are of the form (14,15) respectively with*

$$\mathcal{L}_{n-2}=nM_{n,s}\wp(x),\quad \ldots$$
$$\mathcal{P}_{s-2}=sM_{n,s}\wp(x),\quad \ldots$$

(iii) *The positive integer p in the Halphen ansatz is equal to $g-1$.*

Part (ii) of this proposition follows from a formal manipulation in differential algebra. Recall that the eigenfunction (18) can be written as

$$\Psi(x;k)=Se^{kx},$$

with S a formal pseudo-differential operator, where $S\frac{\mathrm{d}}{\mathrm{d}x}S^{-1}$ plays the role of the inverse of a local parameter at the point at infinity of the spectral curve $V_{n,s}$, hence L_n, P_s are analytic functions of it (cf. [28] for an exposition). Part (i) is a statement on the intersection multiplicity of the elliptic curve of a minimal tangential cover with the theta divisor of the spectral curve, and was proved in [30] (Proposition A 2.2). Lastly, (iii) follows from the Krichever theory of Baker-Akhiezer functions and the fact that g is the genus of the spectral curve.

The Weierstrass reduction theorem states (see e.g. [4, 25, 5])

Theorem 3.3 *The genus g algebraic curve $V_{n,s}$ covers $M_{n,s}$-sheetedly the elliptic curve $V_{2,3}$, $\pi: V_{n,s}\longrightarrow V_{2,3}$ if and only if there exists an element γ from $\mathrm{Sp}(2g;\mathbb{Z})$ such that the period matrix $\tau=\omega^{-1}\omega'$ can be transformed to the form*

$$\gamma\tau=\begin{pmatrix}\tau_{11} & \frac{k}{M_{n,s}} & 0 & \ldots & 0\\ \frac{k}{M_{n,s}} & & & & \\ 0 & & & & \\ \vdots & & & \widetilde{\tau} & \\ 0 & & & & \end{pmatrix}, \tag{20}$$

where $k\in\mathbb{N}$, $1\le k<M_{n,s}$ and $\widetilde{\tau}$ is a $(g-1)\times(g-1)$ matrix.

The cover π induces the reduction of one of the holomorphic differentials of the curve to the holomorphic differential of the elliptic curve. We are interested in the case when all the holomorphic differentials reduce to holomorphic differentials of the elliptic curves. In this case the (symmetric) τ-matrix is reducible to the form

$$\begin{pmatrix} \tau_1 & & q_{ij} & \\ & \tau_2 & & \\ & & \ddots & \\ & \cdots & & \tau_g \end{pmatrix}, \tag{21}$$

where q_{ij} are rational numbers.

4. (3,4)-curve: Boussinesq flow

4.1. Trigonal functions

Trigonal functions serve as a principal example in [16]. The relations for trigonal functions were also treated in [13, 14]. Recently the uniformisation of Jacobi varieties of trigonal curves by means of σ-functions was developed in [8]. Here we consider the simplest trigonal curve of genus 3, that is the $(3,4)$ curve, which we write in canonical form as

$$f(z,w) \equiv w^3 - p(z)w - q(z) = 0, \tag{22}$$

where $p(z) = p_2z^2 + p_1z + p_0$, $q(z) = z^4 + q_2z^2 + q_1z + q_0$. The associated Weierstrass gap sequence is

$$\overline{0},\ 1,\ 2,\ \overline{3,\ 4},\ 5,\ \overline{6,\ 7,\ \ldots}, \tag{23}$$

and the corresponding Schur-Weierstrass polynomial is of the form

$$\sigma(t,y,x) = x^5 - 20xy^2 + 20t + \text{higher order terms.} \tag{24}$$

We write the equations of the Jacobi inversion problem in the form

$$\begin{aligned}
&\int_\infty^{z_1}\frac{\mathrm{d}z}{f_w} + \int_\infty^{z_2}\frac{\mathrm{d}z}{f_w} + \int_\infty^{z_3}\frac{\mathrm{d}z}{f_w} = t,\\
&\int_\infty^{z_1}\frac{z\mathrm{d}z}{f_w} + \int_\infty^{z_2}\frac{z\mathrm{d}z}{f_w} + \int_\infty^{z_3}\frac{z\mathrm{d}z}{f_w} = y,\\
&\int_\infty^{z_1}\frac{w\mathrm{d}z}{f_w} + \int_\infty^{z_2}\frac{w\mathrm{d}z}{f_w} + \int_\infty^{z_3}\frac{w\mathrm{d}z}{f_w} = x.
\end{aligned}$$

These are solved in terms of Kleinian functions as follows

$$\begin{aligned}
&z^2 - \wp_{2,3}z - \wp_{3,3}w - \wp_{1,3} = 0,\\
&2wz + (-\wp_{2,2} + \wp_{2,3,3})z - \wp_{1,2} + (\wp_{3,3,3} - \wp_{2,3})w + \wp_{1,3,3} = 0.
\end{aligned}$$

The elimination of z or w from these equations leads to an equation of the third degree whose coefficients are symmetric functions of the divisor $(z_1,w_1) + (z_2,w_2) + (z_3,w_3)$.

We shall give some of the relations [16, 8] between the trigonal functions, which we shall use in our derivation

$$\begin{aligned}
\wp_{3,3,3,3} &= 4\,\wp_{3,3}p_2 - 3\,\wp_{2,2} + 6\wp_{3,3}^2,\\
\wp_{2,3,3,3} &= 6\,\wp_{2,3}\wp_{3,3} + \wp_{2,3}p_2 + p_1,\\
\wp_{2,3,3}\wp_{3,3,3} &= 4\wp_{2,3}\wp_{3,3}^2 + 2p_2\wp_{3,3}\wp_{2,3} + 2p_1\wp_{3,3} - \wp_{2,2}\wp_{2,3} - 2\wp_{1,2}.
\end{aligned}$$

The first of these relations becomes the Boussinesq equation with respect to the function $\mathcal{U}(t,y,x) = 2\wp_{33}(\boldsymbol{u})$ after double differentiation in x

$$\mathcal{U}_{xxxx} + 3\mathcal{U}_{tt} = \left(6\mathcal{U}^2 + p_2\mathcal{U}\right)_{xx}. \tag{25}$$

4.2. Burchnall-Chaundy pair

Consider the elliptic curve

$$w^2 = 4z^3 - g_3 \tag{26}$$

and its Weierstrass elliptic function (equianharmonic elliptic functions) and a related Burchnall-Chaundy pair

$$\begin{aligned} L_3 &= \frac{\mathrm{d}^3}{\mathrm{d}x^3} - 15\wp(x)\frac{\mathrm{d}}{\mathrm{d}x} - \frac{15}{2}\wp'(x), \\ P_4 &= \frac{\mathrm{d}^4}{\mathrm{d}x^4} - 20\wp(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 20\wp'(x)\frac{\mathrm{d}}{\mathrm{d}x}. \end{aligned}$$

Introduce the Halphen ansatz according to point **(iii)** of Proposition 3.2

$$\Psi(x;\alpha) = \exp(kx)\left(\Phi(x;\alpha) + a_1\Phi'(x;\alpha) + a_2\Phi''(x;\alpha)\right).$$

The eigenvalue problem $L_3\Psi(x) = z\Psi(x)$, expanded into power series in x, leads to the conditions

$$\begin{aligned} a_1 &= 2ka_2, \\ 0 &= -6k^2 a_2 + 8 - ka_1, \\ 0 &= (-15\wp + 12k^2)\, a_1 + (4k^3 + 10\wp' - 4z)\, a_2 - 18k, \\ 0 &= (15k\wp - 2k^3 + 2z)\, a_1 - 10k\wp'\, a_2 - 6k^2, \\ 0 &= \left(\frac{45}{4}\wp^2 - 10k\wp'\right) a_1 + \left(\frac{45}{2}k\wp^2 - 6\wp\wp'\right) a_2 - 2z - \\ &\quad - 5\wp' + 2k^3 + 15k, \end{aligned}$$

where $\wp = \wp(\alpha)$, etc. Eliminating a_1 and a_2 we get

$$-30k\wp + 10k^3 + 10\wp' - 4z = 0, \tag{27}$$

$$30k\wp - 10k^3 + 4z - 10\wp' = 0, \tag{28}$$

$$90k\wp^2 - 50k^2\wp' - 12\wp\wp' - 4zk^2 + 4k^5 + 30k^3\wp = 0, \tag{29}$$

to which we add the Weierstrass equation satisfied by $\wp$, $(\wp')^2 = 4\wp^3 - g_3$. By constructing a Groebner basis for (27), (28) and (26) using lexicographic ordering in variables $\wp, \wp', k, z$, [12], we can eliminate $\wp, \wp'$, and find the curve with coordinates k and z

$$2000k^3z^4 - 135000g_3z^2k^3 + 2278125g_3^2k^3 - 1024z^5 - 6400g_3z^3 = 0. \tag{30}$$

We can check this curve is consistent with (29) also.

Proceeding in the same way we can analyse the P_4 eigenvalue equation $P_4\Psi(x) = w\Psi(x)$ to get eventually the curve with coordinate w and k

$$625k^4w^4 - 43000g_3w^3k^2 - 21600000g_3^3k^2 - 54000w^2g_3^2 - 256w^5 = 0. \tag{31}$$

Taking the resultant of (30) and (31) we get the curve with coordinates z and w

$$w^3 = \left(z^2 + \frac{25}{4}g_3\right)\left(z^2 - \frac{135}{4}g_3\right). \tag{32}$$

In fact the problem $L_3\Psi(x) = z\Psi(x)$ represents the classically known Halphen equation; the solution was found as well in the original memoir [21], and also in the handbook [23]; the derivation in the framework of reduction theory was given in [19], and in [31] a generalization of the Halphen equation was studied.

4.3. The three covers

The curve (32) by construction is a tangential cover of the the elliptic curve (26), namely, both curves are tangent to the direction of the differential $\mathrm{d}u_3$. But the curve (32) also covers other elliptic curves, in fact precisely along the direction of the differentials $\mathrm{d}u_2$ and $\mathrm{d}u_1$:

$$\begin{tikzcd}
V_{3,4} = (z,w) \arrow[d, "\pi"'] \arrow[dr, "\widetilde{\pi}"] \arrow[drr, "\widetilde{\widetilde{\pi}}"] & & \\
(\wp, \wp') & (\widetilde{\wp}, \widetilde{\wp}') & (\widetilde{\widetilde{\wp}}, \widetilde{\widetilde{\wp}}')
\end{tikzcd}$$

The cover over the curve T_x is given by the birational map

$$\begin{aligned}
T_x &\equiv (\wp, \wp'), \quad {\wp'}^2 = 4\wp^3 - g_3, \\
\wp &= \frac{w^2\left(16\,z^2 + 8100\,g_3\right)}{\left(4\,z^2 - 135\,g_3\right)^2}, \\
\wp &= 2\,\frac{z\left(16\,z^4 - 19000\,z^2 g_3 - 759375\,{g_3}^2\right)}{\left(4\,z^2 - 135\,g_3\right)^2}.
\end{aligned} \tag{33}$$

This map induces the following reduction of the holomorphic differential to the elliptic differential

$$\frac{1}{3}\frac{\mathrm{d}z}{w} = \frac{\mathrm{d}\wp}{\wp'}. \tag{34}$$

The second cover induces the reduction of the holomorphic differential associated with the variable y. The corresponding curve is also equianharmonic and is given by the equation

$$T_y \equiv (\widetilde{\wp}, \widetilde{\wp}'), \quad \widetilde{\wp}'^2 = 4\widetilde{\wp}^3 + (40 g_3)^2.$$

This cover is given by the formulae

$$\widetilde{\wp} = w, \quad \widetilde{\wp}' = \frac{1}{2}(4z^2 - 55 g_3) \tag{35}$$

and the induced reduction of the holomorphic differential is

$$\frac{2}{3}\frac{z\mathrm{d}z}{w^2} = \frac{\mathrm{d}\widetilde{\wp}}{\widetilde{\wp}'}. \tag{36}$$

The third cover induces the reduction of the holomorphic differential associated with the variable t. The corresponding curve is also equianharmonic and is given by the equation

$$T_t \equiv (\widetilde{\widetilde{\wp}}, \widetilde{\widetilde{\wp}}'), \quad \widetilde{\widetilde{\wp}}'^2 = 4\,\widetilde{\widetilde{\wp}}^3 + 54\,(10 g_3)^5.$$

The cover is given by the formulae

$$\begin{aligned}\widetilde{\widetilde{\wp}} &= \frac{1}{16}\,\frac{w\left(64\,z^6-80\,g_3z^4-5300\,g_3{}^2z^2-30375\,g_3{}^3\right)}{z^2\,(25\,g_3+4\,z^2)},\\ \widetilde{\widetilde{\wp}}' &= \frac{1}{128}\,\frac{R(z)}{z^3\,(25\,g_3+4\,z^2)},\end{aligned} \tag{37}$$

where

$$\begin{aligned}R(z) = {}& 1024\,z^{10}-61509375\,g_3{}^5-15187500\,g_3{}^4z^2-700000\,g_3{}^3z^4\\ &-240000\,g_3{}^2z^6-19200\,z^8g_3,\end{aligned}$$

and the induced reduction of the holomorphic differential is

$$\frac{5}{3}\frac{\mathrm{d}z}{w^2} = \frac{\mathrm{d}\widetilde{\widetilde{\wp}}}{\widetilde{\widetilde{\wp}}'}. \tag{38}$$

Note that the second cover is of degree 2, as the curve T_y is obtained from (32) by quotienting the involution $(z,w)\mapsto(-z,w)$.

4.4. Calogero-Moser dynamics

The multiplicity of the divisor shows in that in this case the locus of poles of the Boussinesq solution consists of 5 particles, which suggests the following ansatz for the σ-function

$$\sigma(t,y,x) = \prod_{i=1}^{5}\sigma_W(x-x_i(t,y)), \tag{39}$$

and, therefore

$$\wp_{33}(t,y,x) = \sum_{i=1}^{5}\wp_W(x-x_i(t,y)), \tag{40}$$

where $\wp_W$ is the Weierstrass elliptic function with moduli $g_2=0$, and g_3 arbitrary.

The expansion of the first three Kleinian functions at $x=x_j+\varepsilon$ is

$$\begin{aligned}\wp_{33} &= \frac{1}{\varepsilon^2}+F_j+F_j'\varepsilon+O(\varepsilon^2),\\ \wp_{23} &= -\frac{1}{\varepsilon^2}\frac{\partial x_j}{\partial y}-G_j-G_j'\varepsilon+O(\varepsilon^2),\\ \wp_{22} &= \frac{1}{\varepsilon^2}\left(\frac{\partial x_j}{\partial y}\right)^2+\frac{1}{\varepsilon}\frac{\partial^2 x_j}{\partial y^2}+O(1),\end{aligned}$$

where we set

$$\begin{aligned}F_j &= \sum_{i\neq j}^{5}\wp(x_j-x_i),\quad F_j' = \sum_{i\neq j}^{5}\wp'(x_j-x_i),\\ G_j &= \sum_{i\neq j}^{5}\wp(x_j-x_i)\frac{\partial x_i}{\partial y},\quad G_j' = \sum_{i\neq j}^{5}\wp'(x_j-x_i)\frac{\partial x_i}{\partial y}.\end{aligned}$$

The substitution of these expansions into the trigonal relations leads to the dynamical equations

$$\left(\frac{\partial x_i(y,t,\ldots)}{\partial y}\right)^2 = 4\sum_{j\neq i}\wp(x_i - x_j), \quad i = 1,\ldots,5, \tag{41}$$

$$\frac{\partial^2 x_i(y,t,\ldots)}{\partial y^2} = 4\sum_{j\neq i}\wp'(x_i - x_j), \quad i = 1,\ldots,5, \tag{42}$$

which have the compatibility condition

$$\sum_{j\neq k}\wp'(x_k - x_j)\left(\frac{\partial x_k}{\partial y} + \frac{\partial x_j}{\partial y}\right) = 0, \quad k = 1,\ldots,5, \tag{43}$$

and also to the geometrical constraint (*locus*)

$$G_j^2 = 4F_j^3, \quad j = 1,\ldots,5.$$

Proposition 4.1 *Consider the trigonal curve (32) and the divisor*

$$(z_1,w_1) + (z_2,w_2) + (z_3,w_3),$$

where the coordinates depends on x and the coordinates of the particles $x_i(t,y)$, $i = 1,\ldots,5$. Then as $x \to x_j$ with $x = x_j + \varepsilon$ the limiting divisor is given by the formulae

$$\left(\frac{1}{\varepsilon^3},\frac{1}{\varepsilon^4}\right) + (Z_1^{(j)},W_1^{(j)}) + (Z_2^{(j)},W_2^{(j)}), \tag{44}$$

where

$$Z_{1,2}^{(j)} = X_j \pm \sqrt{X_j^2 + Y_j}, \quad W_{1,2}^{(j)} = \frac{\partial x_j(t,y)}{\partial y}Z_{1,2}^{(j)} + \frac{\partial x_j(t,y)}{\partial t} \tag{45}$$

and

$$X_j = -\frac{1}{4}\left(\frac{\partial^2 x_j(t,y)}{\partial y^2} - \left(\frac{\partial x_j(t,y)}{\partial y}\right)^3\right), \tag{46}$$

$$Y_j = -\frac{1}{2}\frac{\partial^2 x_j(t,y)}{\partial y\partial t} + \frac{1}{8}\left(\frac{\partial x_j(t,y)}{\partial y}\right)^2\frac{\partial x_j(t,y)}{\partial t}. \tag{47}$$

4.5. Elliptic varieties

We are now in position to display the elliptic variety associated with the 5-particle dynamics under the Boussinesq flow. With this aim we shall write the first two equations of the Jacobi inversion problem at $x = x_j$ and reduce the Abelian integrals to elliptic integrals with the aid of the reduction formulae (35,37,36,38)

$$\int_\infty^{\phi(Z_1^{(j)},W_1^{(j)})}\frac{\mathrm{d}\nu}{\sqrt{4\nu^3 + 54(10g_3)^5}} + \int_\infty^{\phi(Z_2^{(j)},W_2^{(j)})}\frac{\mathrm{d}\nu}{\sqrt{4\nu^3 + 54(10g_3)^5}} = \frac{3}{5}t, \tag{48}$$

$$\int_\infty^{W_1^{(j)}}\frac{\mathrm{d}\nu}{\sqrt{4\nu^3 + (40g_3)^2}} + \int_\infty^{W_2^{(j)}}\frac{\mathrm{d}\nu}{\sqrt{4\nu^3 + (40g_3)^2}} = \frac{3}{2}y, \tag{49}$$

where $j = 1, \ldots, 5$ and $\phi(z, w)$ is the coordinate of the cover

$$\phi(z,w) = \frac{1}{16}\,\frac{w\left(64\,z^6 - 80\,g_3 z^4 - 5300\,{g_3}^2 z^2 - 30375\,{g_3}^3\right)}{z^2\,(25\,g_3 + 4\,z^2)}$$

and $(Z_1, W_1), (Z_2, W_2)$ are given in (45). The application of the addition theorem for the Weierstrass elliptic function leads to rather complicated expressions, which involve the Weierstrass elliptic functions $\widetilde{\wp}(3y/2)$, $\widetilde{\widetilde{\wp}}(3t/5)$ and the quantities X and Y.

We shall write these equations in the particular case of fixed $t = 0$. To do that we shall use the addition theorem for the Weierstrass elliptic function and also expressions for the second cover (35) to transform (49) to the form

$$\widetilde{\wp}\left(\frac{2}{3}y\right) = -W_1^{(j)} - W_2^{(j)} + \left(\frac{{Z_1^{(j)}}^2 - {Z_2^{(j)}}^2}{W_1^{(j)} - W_2^{(j)}}\right)^2 .$$

This equation can be rewritten with the help of (45-47) as follows

$$\left(\frac{\mathrm{d}f}{\mathrm{d}y}\right)^2 - f^6 - 4\left(\widetilde{\wp}\left(\frac{2}{3}y\right) + 2g(y)\right) f^2 = 0, \tag{50}$$

where we denote $f = \mathrm{d}x_i(y)/\mathrm{d}y,\ g(y) = \partial x_i(t,y)/\partial t|_{t=0}$. The function $g(y)$ can be determined from the condition that the points $(Z_{1,2}, W_{1,2})$ given in (45) belong to the curve $V_{n,s}$.

The problem of integration of the associated Calogero-Moser particle system is then reduced to solving the ordinary differential equation (50). In particular, at $y \approx 0$ the equation (50) has 5 solutions. The first one is trivial, $f = 0$; the remaining four are found by setting $f = Ay^{\alpha}$. One finds that $\alpha = -1/2$, while $g(y) \approx y^{-2}$ in this limit. The result is in agreement with the expansion (24), which shows, that five particles are evolving on the locus under the action of Boussinesq flow.

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