Abelian integrals, Picard-Vessiot groups and the Schanuel conjecture.

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Schanuel's conjecture

$x_1, ..., x_n \in \mathbf{C}$, linearly independent over $\mathbf{Q}$

$$\Rightarrow tr.deg_{\mathbf{Q}} \mathbf{Q}(x_1, ..., x_n, e^{x_1}, ..., e^{x_n}) \geq n \text{ (?)}$$

Equivalently,

$x_1, ..., x_n \in \mathbf{C}, y_1, ..., y_n \in \mathbf{C}^*, y_i = e^{x_i}$. Then,

$$\boxed{tr.deg_{\mathbf{Q}} \mathbf{Q}(x_i\text{'s}, y_i\text{'s}) \;\geq\; rk_{\mathbf{Z}}\Big(\mathbf{Z}x_1 + ... + \mathbf{Z}x_n) \quad \text{(?)}}$$

Exponential case (Lindemann-Weierstrass thm) :
. $\forall i, x_i \in \overline{\mathbf{Q}} \Rightarrow$ true (with equality).

Logarithmic case (Schneider's problem) : .
. $\forall i, y_i \in \overline{\mathbf{Q}} \Rightarrow$ (?) (with equality)

$$G = (\mathbf{G}_m)^n, \text{ } n\text{-dim'l split torus over } \mathbf{Q} \subset \mathbf{C}$$

$$TG := T_0G = Lie(G)$$

$$exp_G : TG(\mathbf{C}) \simeq \mathbf{C}^n \to G(\mathbf{C}) \simeq (\mathbf{C}^*)^n,$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \quad y = \begin{pmatrix} e^{x_1} \\ \vdots \\ e^{x_n} \end{pmatrix}.$$

Lie hull $\mathcal{G}_x$ of x := smallest algebraic subgroup H of G such that $x \in TH(\mathbf{C})$.

(NB : contains, often strictly, the *hull* G_y of y = smallest alg. subgroup H of G such that $y \in H(\mathbf{C})$).

The conjecture then reads :

$$x \in TG(\mathbf{C}), y = exp_G(x) \in G(\mathbf{C})$$

$$\Rightarrow tr.deg_{\mathbf{Q}}\mathbf{Q}(x, y) \geq dim\mathcal{G}_x \text{ (?)}$$

Abelian integrals

$k = \overline{k} \subset \mathbf{C}$, $P \in k[X,Y], f \in k(X,Y), p_0, p_1 \in k$

$$\int_{p_0}^{p_1} f(X,Y)dX \quad , \quad P(X,Y) = 0.$$

More intrinsically, X/k smooth projective algebraic curve, $\omega \in H^0(X, \Omega^1_{X/k}(D))$ for some $D \in Div^+(X)$. By Weil-Rosenlicht, there is :

- a generalized Jacobian $G = Jac(X,D)$:

$$0 \to L \to G \to A \to 0$$

where $L = \mathbf{G}_m^r \times \mathbf{G}_a^s, A = Jac(X)$;

- a canonical (Abel-Jacobi) map

$$\phi : (X, \text{ point }) \to G,$$

- an invariant differential form $\omega_G \in T^*G(k)$ on G with $\phi^*\omega_G = \omega$ mod. exact forms.

Set $y = \phi(P_1) - \phi(P_0) \in G(k)$. Up to addition of an element of k, we get

$$\int_{P_0}^{P_1} \omega = \int_0^y \omega_G.$$

More precisely, there exists $x \in TG(\mathbf{C})$ (depending on the path of integration) such that

$$y = exp_G(x), \text{ and } \int_0^y \omega_G = < \omega_G | x > .$$

$$(x, y) \in (TG \times G)(\mathbf{C}), y = exp_G(x)$$

$$\Rightarrow tr.deg_{\mathbf{Q}}\mathbf{Q}(x, y) \geq \mathcal{G}_x \text{ (??)}$$

$X = \mathbf{P}_1, D = (0) + (\infty) \rightsquigarrow G = \mathbf{G}_m$, and $x = \ell n(y)$: standard Schanuel problem.

Otherwise, (??) must be modified. One attaches to $\mathcal{M} = (X, D, (P_1) - (P_0))$ a “motivic Galois group" $\mathbf{G}_{\mathcal{M}}$, acting on TG.

André’s conjecture : $tr.deg_{\mathbf{Q}}\mathbf{Q}(x, y) \geq dim\mathbf{G}_{\mathcal{M}}.x$

(inspired by, and implying, the Grotendieck conjecture : if $k = \overline{\mathbf{Q}}$, then

$$tr.deg_{\mathbf{Q}}\mathbf{Q}(x, y) = dim\mathbf{G}_{\mathcal{M}}.x;$$

see also Kontsevich’s conjecture on periods.)

Elliptic integrals

Ref. : Whittaker-Watson.

$$g_2, g_3 \in k,\ g_2^3 - 27g_3^2 \neq 0,$$

$$j(E) = \frac{g_2^3}{g_2^3 - 27g_3^2} = j(\tau)\ ;\ \Omega \subset \mathbf{C}$$

$$Y^2 = 4X^3 - g_2X - g_3\ (E)$$

$$\omega = \frac{dX}{Y}, \eta = X\frac{dX}{Y};$$

$$Q \in E(k), \xi_Q = \tfrac{1}{2}\tfrac{Y - Y(Q)}{X - X(Q)}\tfrac{dX}{Y}, Res(\xi_Q) = -(0) + (-Q)$$

$$f(X,Y)dY = \alpha\omega + \beta\eta + dg + \sum_{i=1}^{r} \gamma_i \xi_{Q_i} + \sum_{j=1}^{r'} \gamma_i' \frac{dh_j}{h_j}$$

with $\mathbf{Z}$-linearly independent Q_i's in $E(k)$.

$$\mathcal{G} \in Ext(E, \mathbf{G}_m^r \times \mathbf{G}_a \times \mathbf{G}_m^{r'}).$$

$\mathcal{G} = \tilde{G} \times \mathbf{G}_m^{r'}$, with

- $G \in Ext(E, \mathbf{G}_m^r)$: an *essential* extension
- $\tilde{G} =$ the universal vectorial extension of G.

For $P \in E(k)$, set $u = \int_0^P \omega$, hence

$$P = (\wp(u), \wp'(u)) = exp_E(u),$$

$$\sigma(z) = z \prod_{\omega \in \Omega'} (1 - \tfrac{z}{\omega}) e^{\frac{z}{\omega} + \frac{1}{2}(\frac{z}{\omega})^2} \ , \ \zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$$

$$\boxed{f_v(z) = \frac{\sigma(v+z)}{\sigma(v)\sigma(z)} e^{-\zeta(v)z} \quad (v \notin \Omega)}$$

$$\zeta(z+\omega) = \zeta(z) + \eta(\omega), \ \eta_2\omega_1 - \eta_1\omega_2 = 2\pi i,$$

$$f_v(z+\omega) = f_v(z) e^{\lambda_v(\omega)} \ , \ \lambda_v(\omega) = \eta(\omega)v - \zeta(v)\omega.$$

$$(r=1) \ Q = exp_E(v), \ \tilde{G} \simeq_{\text{birat}} \ E \times \mathbf{G}_m \times \mathbf{G}_a.$$

$$exp_{\tilde{G}} : \mathbf{C}^3 \to \tilde{G}(\mathbf{C}) : \begin{pmatrix} \ell \\ t \\ u \end{pmatrix} \mapsto \begin{pmatrix} f_v(u)e^{-\ell} \\ \zeta(u) - t \\ \wp(u) \end{pmatrix}$$

$$Ker(exp_{\tilde{G}}) = \mathbf{Z} \begin{pmatrix} 2\pi i \\ 0 \\ 0 \end{pmatrix} \oplus \mathbf{Z} \begin{pmatrix} \lambda_v(\omega_1) \\ \eta(\omega_1) \\ \omega_1 \end{pmatrix} \oplus \mathbf{Z} \begin{pmatrix} \lambda_v(\omega_2) \\ \eta(\omega_2) \\ \omega_2 \end{pmatrix}$$

In conclusion, if $\tilde{y} = \{y_3, y_2, y_1\} \in \tilde{G}(k)$ is above $y_1 = (\wp(u), \wp'(u)) \in E(k)$,

$$\boxed{\tilde{x} = \left\{ \ell n \frac{\sigma(u+v)}{\sigma(u)\sigma(v)} - \zeta(v)u - \ell n(y_3) \ , \ \zeta(u) - y_2 \ , \ u \right\}}$$

Mumford-Tate groups

To $y \in G(k)$, we attach a one-motive M/k, a k-vector space $H^1_{DR}(M)$, a $\mathbf{Q}$-vector space $H_B(M)$, and a period matrix $\boxed{\Pi(M)}$ $\swarrow$

.

$$\begin{pmatrix} 2\pi i & \lambda_v(\omega_1) & \lambda_v(\omega_2) & \ell n f_v(u) + \ell_0 \\ 0 & \eta(\omega_1) & \eta(\omega_2) & \zeta(u) + t_0 \\ 0 & \omega_1 & \omega_2 & u \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} \xi_Q \\ \tilde{\eta} \\ \omega \\ \{df\} \end{matrix}$$

$$\gamma_0 \qquad \gamma_1 \qquad \gamma_2 \qquad \gamma_x$$

$H_B(M)$ is endowed with a mixed Hodge structure. In particular, an increasing weight filtration $W_\bullet$ with

$$W_{-2} = H_B(\mathbf{G}_m), W_{-1} = H_B(G), W_0 = H_B(M)$$

$$Gr_{-1} = H_B(E), Gr_0 = \mathbf{Z}$$

and a Hodge filtration $F^\bullet$ on $H_B(M) \otimes \mathbf{C}$. Similarly with $H^1_{DR}(M)$.

The canonical pairing

$$< \omega|\gamma >= \int_\gamma \omega$$

induces an isomorphism

$$H^1_{DR}(M) \otimes_k \mathbf{C} \to H_B(M)^* \otimes_{\mathbf{Z}} \mathbf{C}$$

(represented by the period matrix $\Pi(M)$ above), which respects both filtrations.

Mixed Hodge structures form a $\mathbf{Q}$-linear tannakian category, with fiber functor H_B. The Mumford-Tate group of M is

$$\mathbf{MT}_M = Aut^{\otimes}(H_B(M)).$$

$Isom^{\otimes}(H^1_{DR}(M), H_B(M)^* \otimes k)$ is represented by a scheme Z/k, which is a $\mathbf{MT}_M \otimes k$-torsor.

$\rightsquigarrow$ Alternative rephrasings of the conjectures :

$$tr.deg._{\mathbf{Q}}\mathbf{Q}(\tilde{x}, \tilde{y}) \geq dim(\mathbf{MT}_M.\gamma_x) \quad (?)$$

(if $k = \overline{\mathbf{Q}}$) : $\Pi(M)$ is a generic point of Z/k (?)

Function field analogue I

Let (F,∂), with $F^{\partial} = \mathbf{C}$, be a sufficiently large differential field extension of $(K = \mathbf{C}(t), d/dt)$. For $x \in F$, define $y = e^x \in F^*/\mathbf{C}^*$ as a solution of the diff'l equation

$$\frac{\partial y}{y} = \partial x.$$

$K(x,y)$ is well-defined (and depends only on the classes of x in $F/\mathbf{C}$).

Ax (1970) : $x_i \in F, y_i = e^{x_i} (i = 1, ..., n)$. Then

$$\boxed{tr.deg_K K(x_i'\text{s}, y_i'\text{s}) \geq rk_{\mathbf{Z}}\Big(\mathbf{Z}x_1 + ... + \mathbf{Z}x_n \text{ mod } \mathbf{C}\Big).}$$

$\swarrow$

NB : $$rk_{\mathbf{Z}}\Big(....\Big) = dim\mathcal{G}_x$$

where $\mathcal{G}_x$ is the smallest algebraic group H of $G = \mathbf{G}_m^n$ such that $x \in TH(F) + TG(\mathbf{C})$.

$$\Updownarrow (now)$$

$\mathcal{G}_x = G_y :=$ smallest algebraic group H of G such that $y \in H(F) + G(\mathbf{C})$.

$G =$ an algebraic group **defined over** $\mathbf{C}$. By Kolchin, there is a canonical logarithmic derivative map

$$\partial \ell n_G : G(F) \to \underline{TG}(F) = TG \times_{\mathsf{Ad}} G;$$

e.g. if $G \subset GL_n : \partial \ell n_G(U) = \partial U.U^{-1}$.

When G is commutative, "ℓn_G inverts exp_G modulo the constants".

For $y \in G(F)$, define the **relative hull** G_y of y as the smallest algebraic group $H/\mathbf{C}$ such that $y \in H(F)$ mod. $G(\mathbf{C})$.

Theorem 1.a (Ax, Kirby) : *assume that* G *is a semi-abelian variety (no additive subgroup),* $(x, y) \in (TG \times G)(F), y = exp_G(x)$. *Then,*

$$tr.deg._K K(x, y) \geq dim G_y.$$

This *cannot* hold true in general if additive subgroups occur. However

Brownawell-Kubota : $E/\mathbf{C}$ ell. curve, $u_1, ..., u_n$ $\in F$, linearly independent over $End(E)$ mod $\mathbf{C}$. Then

$$tr.deg_K K(u_i, \wp(u_i), \zeta(u_i); i = 1, ..., n) \geq 2n$$

Theorem 1.b : *let further $\tilde{G}$ (resp. $\tilde{G}_y$) be the universal vectorial extension of G (resp. G_y). For any $\tilde{x} \in T\tilde{G}(F)$ s.t. $exp_{\tilde{G}}(\tilde{x}) = \tilde{y}$ projects to $y \in G(F)$,*

$$tr.deg._K K(\tilde{x}, \tilde{y}) \geq dim \tilde{G}_y.$$

E.g., for $v_i \in \mathbf{C}, Q_i = exp_E(v_i) \in E(\mathbf{C})$, l.i. $/\mathbf{Z}$

$$tr.deg_K K(u_i, \wp(u_i), \zeta(u_i), \frac{\sigma(v_i+u_i)}{\sigma(u_i)}; i = 1, ..., n) \geq 3n$$

as well as $\quad ..., \ \ell n \frac{\sigma(v_i+u_i)}{\sigma(u_i)}; ... \quad \geq 3n$

NB : B-K also got : $\quad ..., \ \sigma(u_i); .... \quad \geq 3n$

Proof : a kind of intersection theory + rigidity of alg. groups.

i) wlog, assume that $G_y = G$. Amost by definition, $\tilde{G}$ is an **essential** extension of G ; hence $\tilde{G}_y = \tilde{G}$. Must now prove that

$$tr.deg.(\mathbf{C}(\tilde{x}, \tilde{y})/\mathbf{C}) \geq dim(\tilde{G}) + 1.$$

ii) reduce by Seidenberg (cf. J. Kirby) to the analytic case $\rightsquigarrow$

- $\mathbf{X} = T\tilde{G} \times \tilde{G}$ (alg. group over $\mathbf{C}$),
- $\mathbf{A} =$ graph of $exp_{\tilde{G}}$ (anal. subgroup of $\mathbf{X}$),
- $\mathbf{K} =$ the analytic curve defined by the image of $\{\tilde{x}, \tilde{y}\} : \mathbf{C} \supset U \to \mathbf{X}(\mathbf{C})$.Wlog, assume that $0 \in \mathbf{K}$ and let $\mathbf{V}$ be its Zariski closure in $\mathbf{X}/\mathbf{C}$, so that $tr.deg.(\mathbf{C}(\tilde{x}, \tilde{y})/\mathbf{C}) = dim\mathbf{V}$.

iii) Ax's theorem (1972) : there exists an analytic subgroup $\mathbf{B}$ of $\mathbf{X}$ containing both $\mathbf{A}$ and $\mathbf{V}$ such that $dim\mathbf{K} \leq dim\mathbf{V} + dim\mathbf{A} - dim\mathbf{B}$.

We shall prove that $\mathbf{B} = \mathbf{X}$. Consequently :

$$\begin{array}{ccc} dim\mathbf{V} & \geq & dim\mathbf{X} - dim\mathbf{A} + dim\mathbf{K}, \\ \| & & \| \\ tr.deg.(\mathbf{C}(\tilde{x}, \tilde{y})/\mathbf{C}) & & dim\tilde{G} + 1 \end{array}$$

Since $\mathbf{V}$ is a connected algebraic variety $\ni 0$, the abstract group it generates in $\mathbf{X}$ is an algebraic subgroup $g(\mathbf{V})$ of $\mathbf{X} = T\tilde{G}\times\tilde{G}$. Since $\mathbf{K} \subset \mathbf{V}$, and since $G_y = G$, the image $G' \subset \tilde{G}$ of $g(\mathbf{V})$ under the 2nd projection projects onto G, and therefore coincides with $\tilde{G}$. Let $T' \subset T\tilde{G}$ be the image of $g(\mathbf{V})$ under the 1st projection.

Now, $g(\mathbf{V})$ is an algebraic subgroup of $T' \times \tilde{G}$ with surjective images under the two projections. But any such subgroup induces an isomorphism from a quotient of $\tilde{G}$ to a quotient of T' : setting $H = g(\mathbf{V}) \cap (0 \times \tilde{G})$, and $H' = g(\mathbf{V}) \cap (T' \times 0)$, we have $\tilde{G}/H \simeq T'/H'$. If these quotients were not trivial, the 2nd one would admit $\mathbf{G}_a$ among its quotients, and ditto for the 1st one, hence for $\tilde{G}$; contradiction. Consequently, $\tilde{G}/H = 0$, and $g(\mathbf{V})$, hence $\mathbf{B}$, contains $0 \times \tilde{G}$.

Finally, $\mathbf{B} \supset \mathbf{A}$ projects onto $T\tilde{G}$ by the 1st projection. Hence, $\mathbf{B} = \mathbf{T\tilde{G}} \times \mathbf{\tilde{G}} = \mathbf{X}$.

Where are the Picard-Vessiot groups?

[French : remboursez !]

(= [Scots.] Gie'e ma' bawbies back.)

This seems to have little to do with differential Galois theory : relatively to ∂, $K(x, y)/K$ need not even be a differential extension !

However, it *is* a differential extension, and in fact a strongly normal one, in each of the "unmixed" cases $\tilde{x} \in TG(K)$, resp. $\tilde{y} \in G(K)$, where on recalling that $\mathcal{G}_x = G_y$, Theorem 1 amounts to

• (exponential case) : set $\tilde{b} = \partial\tilde{x} \in T\tilde{\mathcal{G}}_x(K)$. Then the (Kolchin) differential Galois group of $\partial\ell n_G(\tilde{y}) = \tilde{b}$ is

$$Aut_\partial(K(\tilde{y}))/K) = \tilde{\mathcal{G}}_x.$$

• (logarithmic case) : set $\tilde{a} = \partial\ell n_{\tilde{G}}\tilde{y} \in T\tilde{G}_y(K)$. Then the (Picard-Vessiot) differential Galois group of $\partial\tilde{x} = \tilde{a}$ is

$$Aut_\partial(K(\tilde{x}))/K) = T\tilde{G}_y.$$

At least in the split case, the latter result could be deduced from

• the purely differential fact [cf. Bible, I.33] that if connected, the Picard-Vessiot group of any system $\partial Y = AY, A \in gl_n(K)$ is the $\mathbf{C}$-Lie hull $\mathcal{G}_A \subset gl_n(\mathbf{C})$ of (a convenient gauge transform of) A,

combined with

• a more geometric observation of the type : logarithmically exact differentials on a curve S which are linearly independent over $\mathbf{Z}$ remain so over $\mathbf{C}$ (and even so when taken modulo exact forms on S).

Function field analogue II

[in the logarithmic case]

Until now, we considered

$$x(t) = \int_1^{y(t)} \frac{dy}{y}\ ,\ x(t) = \int_0^{y(t)} f(x,y)dx,$$

i.e. integrals between non-constant points of a constant diff. form on a curve $X/\mathbf{C}$.

In a more natural frame-work, X **and** ω **vary with** t as well, bringing back the symmetry between objects such that u and v, and, more deeply, allowing for notions of duals in the space of generalized periods.

$$S = \text{ curve}/\mathbf{C}, \pi : \mathcal{X} \to S, K = \mathbf{C}(S), X/K$$

Fix a non constant $t \in \mathbf{C}(S), \partial = d/dt$ and K-**rational** sections p_0, p_1 of π.

$$\int_{p_0(t)}^{p_1(t)} f(t,X,Y)dX \quad , \quad P(t,X,Y) = 0.$$

All the previous notions from the theory of one-motives admit relative versions over S (variation of mixed Hodge structures). Moreover, the O_S-module $H^1_{DR}(\mathcal{M}/S)$ carries a Gauss-Manin (= generalized Picard-Fuchs) connection ∇, whose space of horizontal sections is generated over $\mathbf{C}$ by the local system $R^1\pi_*\mathbf{Q} = H_B(\mathcal{M}/S)^*$.

$$\mathcal{H}(M) := H^1_{DR}(M/K)^* \ , \ D = \nabla^*_{d/dt}$$

is a $K[d/dt]$-module, again filtered (in the elliptic case and with $r = 1$ as above) by the sub-equations

$$W_{-2} = \mathcal{H}(\mathbf{G}_m) \simeq \mathbf{1}; W_{-1} = \mathcal{H}(G), W_0 = \mathcal{H}(M)$$

$$Gr_{-1} = \mathcal{H}(E), Gr_0 = \mathcal{H}(\mathbf{Z}) \simeq \mathbf{1}.$$

Over a sufficiently small domain $U \subset S(\mathbf{C})$,

$$\boxed{\Pi(M)(t)} : U \to GL(H^1_{DR}(\mathcal{M}/U) \otimes O_U^{an})$$

represents a fundamental matrix of analytic solutions of $\mathcal{H}(M)$, and its last vector $\hat{x} = (\tilde{x}(t), 1)$ satisfies $exp_{\tilde{G}_t}(\tilde{x}(t)) = \tilde{y}(t) \in \tilde{G}(K)$.

The field $K(\tilde{x}) = K(\tilde{x}, \tilde{y})$ depends only on the projection x of $\tilde{x}$ on TG.

Let $\mathbf{PV}_M$ be the Picard-Vessiot group of the D-module $\mathcal{H}(M)$: $\forall g \in \mathbf{PV}_M$, $g\widehat{x} - \widehat{x} \in W_{-1}$ also depends only on $x \in TG$. Write $\mathbf{PV}_M . x \subset H_B(\mathcal{G}/U) \otimes \mathbf{C}$ for the corresponding orbit.

Exercise : $tr.deg_K K(\tilde{x}, \tilde{y}) = dim \mathbf{PV}_M . x$.

i.e. the last columns of the elements of $\mathbf{PV}_M$ govern Schanuel's problem *in the logarithmic case*. Here is an elliptic illustration.

Theorem 2 : $g_2(t), g_3(t) \in K, j(t) \notin \mathbf{C}; E/K$ *the corresponding elliptic curve ;* $\{u_i(t); i = 1, ..., n\}$ *holomorphic functions on* $U \subset \mathbf{C}$*, such that* $P_i = exp_E(u_i), i = 1, ..., n$ *are* $\mathbf{Z}$*-linearly independent points in* $E(K)$*. Then,*

$$tr.deg_K K(u_i, \zeta(u_i), \ell n \sigma(u_i); i = 1, ..., n) = 3n.$$

$$[exp_E = exp_{E(t)}, \zeta = \zeta_t, \sigma = \sigma_t \,;\ j \notin \mathbf{C} \Rightarrow \textbf{no}\ \text{CM}]$$

The proof combines three ingredients :

• (A) an essentially geometric fact (Manin)

$$G \in Ext_{gr.sch./K}(E, \mathbf{G}_m) \simeq \hat{E} \simeq E(K) \ni Q$$

$$\rightsquigarrow \mathcal{H}(G) = \mathcal{H}^*(Q) \in Ext_{D-mod.}(\mathcal{H}(E), \mathbf{1})$$

and dually

$$P \in E(K) = Ext_{gr.sch./K}(\mathbf{Z}, E)$$

$$\rightsquigarrow \mathcal{H}(P) := W_0/W_{-2} \in Ext_{D-mod.}(\mathbf{1}, \mathcal{H}(E)).$$

Manin's **kernel theorem** is that the kernel of these maps is generated by the points of height 0, i.e. the constant part of E (here 0) and the torsion points of E.

• (B) pure PV theory (cf. C. Hardouin's talk, in the general framework of a neutral tannakian category), viz. :

Let $\mathcal{V}$ be an irreducible D-module, $V = \mathcal{V}^{sol}$, and let $\mathcal{E}_1, ..., \mathcal{E}_n$ be $\mathbf{C}$-lin. ind. extensions in $Ext_{D-mod.}(\mathbf{1}, \mathcal{V})$. Then, *the unipotent radical of* $\mathbf{PV}(\mathcal{E}_1 \oplus ... \oplus \mathcal{E}_n)$ *fills up* V^n.

• (C) Rigidity of algebraic groups.

$\mathcal{V} = \mathcal{H}(E)$, with $V = H_B(\mathcal{E}/U) \otimes \mathbf{C}$, has an (antisymmetric) polarization $< \mid >$. Let $\mathbf{H} \in Ext_{gr}(V, \mathbf{C})$ be the **Heisenberg group** on V,

$$\mathbf{H} = \left\{ \begin{pmatrix} 1 & v^\flat & c \\ 0 & \mathbf{I}_2 & v \\ 0 & 0 & 1 \end{pmatrix} ; v \in V, c \in \mathbf{C} \right\}$$

For $n = 1$, and $P = Q$ non-torsion, A + B + rigidity force an isomorphism

$$\psi_P : R_u(\mathcal{H}(M)) \simeq \mathbf{H}$$

For $i = 1, ..., n$ and the $P_i = Q_i$'s lin. indep. over $\mathbf{Z}$, let R_u be the unipotent radical of $\mathbf{PV}(\mathcal{H}(M_1) \oplus ... \oplus \mathcal{H}(M_n))$.

$$\Psi = (\psi_{P_1}, ..., \psi_{P_n}) : R_u \hookrightarrow \mathbf{H}^n,$$

and by A + B, $\Psi(R_u)$ projects onto V^n. But since $< \mid >$ is non degenerate, the derived group of any subgroup of $\mathbf{H}^n$ projecting onto V^n fills up $\mathbf{C}^n$, so that $\mathbf{H}^n$ is again an essential extension ! Hence, $R_u = \mathbf{H}^n$, and

$$tr.deg._K K(u_i, \zeta(u_i), \ell n\sigma(u_i)) = dim\mathbf{H}^n = 3n.$$