Integrable equations

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Introduction.

1. Integrable ODEs

$$\frac{d}{dt}\mathbf{U} = \mathbf{F}(\mathbf{U}), \quad \mathbf{U} = (U_1, \ldots, U_N)$$

- First Integrals $I = I(\mathbf{U})$

$$\frac{d}{dt}I = \sum_{k=0}^{N} \frac{\partial I}{\partial U_k} F_k(\mathbf{U}) = 0$$

- Symmetries $\mathbf{G}(\mathbf{U})$

$$\frac{d}{d\tau}\mathbf{U} = \mathbf{G}(\mathbf{U}), \quad \frac{d}{d\tau}\mathbf{F}(\mathbf{U}) = \frac{d}{dt}\mathbf{G}(\mathbf{U})$$

2. 1+1 dimensional systems of PDEs (evolutionary)

$$u_t = f(u, u_1, \ldots, u_n), \quad u_1 = u_x, u_2 = u_{xx}, u_3 = u_{xxx}, \ldots$$

- No first Integrals
- Infinite hierarchy of local conservation laws
- Infinite hierarchy of local symmetries
- Multi-Hamiltonian structure
- Recursion operators

- Master symmetry
- Bäclund transformations
- the Lax representation
  - Inverse spectral transform and solution of IVP
  - Multi-soliton and algebra-geometric solutions
  - Darboux transformations
- Bi-linear representations and the τ function
- Connection with the Painlevé theory

3. Non-evolutionary equations, multi-dimensional equations, integro-differential, differential-difference, discrete, ...

Examples of Integrable Equations

Gardner Green Kruskal and Miura 1967, the KdV equation

$$u_t = u_{xxx} + 6uu_x$$

and the discovery of the inverse scattering method.

Zakharov and Shabat 1971, the NLS equation

$$iu_t = u_{xx} \pm 2|u|^2 u$$

1972, the mKdV equation

$$u_t = u_{xxx} \pm 6u^2 u_x$$

1973, N-wave equations. For $N = 3$

$$\begin{aligned} u_{1t} + v_1 u_{1x} &= iu_2^* u_3 \\ u_{2t} + v_2 u_{2x} &= iu_1^* u_3 \\ u_{3t} + v_3 u_{3x} &= iu_1 u_2 \end{aligned}$$

1973, the Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0$$

1974, the Boussinesq equation

$$u_{tt} = u_{xx} \pm u_{xxxx} + (u^2)_{xx}$$

1976, the massive Thirring model

$$iu_t + v + u|v|^2 = 0$$
$$iv_x + u + v|u|^2 = 0$$

1979, the Landau and Lifshitz equation $\mathbf{S} = \{S_1, S_2, S_3\}$, $\mathbf{S}\cdot\mathbf{S}$
1.

$$\mathbf{S}_t = \mathbf{S} \bigwedge \mathbf{S}_{xx} + \mathbf{S} \bigwedge \mathbf{J}\mathbf{S}$$

1979, the 2-d Toda lattice

$$u_{n\,tt} - u_{n\,xx} = \exp(u_{n+1} - u_n) - \exp(u_n - u_{n-1})$$

and the Tzetzeika equation

$$u_{tt} - u_{xx} + \exp(u) - \exp(-2u) = 0$$

2+1 dimensional equations

1973 ,the Kadomtsev-Petviashvili equation

$$(u_t - u_{xxx} - 6uu_x)_x = \pm u_{yy}$$

Nizhnik 1980, Veselov-Novikov 1984

$$u_t + u_{zzz} + u_{\bar{z}\bar{z}\bar{z}} = 3(uv_z)_z + 3(uw_{\bar{z}})_{\bar{z}},\ u = v_{\bar{z}} = w_z$$

4-d equations (self-dual Yang Mills) 1973.

$$(g_z g^{-1})_{\bar{z}} + (g_y g^{-1})_{\bar{y}} = 0$$

Differential-difference (Volterra, Toda), discrete, ODEs (N-dim. Euler Top), integro-differential (Benjamin-Ono), ...

Examples of the Lax representations.

KdV (P.Lax 1968)

$$u_t = u_{xxx} + 6uu_x \iff L_t = [L, A]$$

where

$$L = D_x^2 + u, \quad A = 4D_x^3 + 6uD_x + 3u_x$$

Two linear problems

$$\phi_{xx} + u\phi - \lambda\phi = 0 \quad \text{and} \quad \phi_t = A\phi$$

are compatible if and only if $u(x,t)$ solves the KdV equation. In the basis ϕ, ϕ_x we can represent

$$\widehat{L} = D_x + \begin{pmatrix} 0 & -1 \\ u - \lambda & 0 \end{pmatrix},$$

$$\widehat{A} = \begin{pmatrix} u_x & -2u - 4\lambda \\ u_{xx} + 2u^2 + 2\lambda u - 4\lambda^2 & -u_x \end{pmatrix}$$

The condition $[\widehat{L}, D_t - \widehat{A}] = 0$ is equivalent to the KdV equation.

We always can consider two linear problems

$$D_x\phi = U\phi \quad D_t\phi = V\phi$$

where U, V are two $n \times n$ matrices which depend on a spectral parameter λ and our dynamical variables (dependent variables and their derivatives).

Example (NLS):

$$L = D_x + \begin{pmatrix} i\lambda & -q \\ \pm\bar{q} & -i\lambda \end{pmatrix} = D_x + i\lambda\sigma_3 + W$$

$$A = D_t - \begin{pmatrix} i\lambda^2 \pm i|q|^2 & 2i\lambda q + iq_x \\ \mp 2i\lambda\bar{q} \pm i\bar{q}_x & -i\lambda^2 \mp i|q|^2 \end{pmatrix}$$

The compatibility condition gives the Nonlinear Schrödinger equation

$$iq_t = q_{xx} \pm |q|^2 q\,.$$

Example: For the Tzetzeika equation

$$u_{xy} + \exp(u) - \exp(-2u) = 0$$

the corresponding operator L is of the form

$$L = D_x - i\frac{\sqrt{3}}{3}u_x \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} q & 0 & 0 \\ 0 & \bar{q} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $q = \exp(2\pi i/3)$.

Example: The Landau and Lifshitz equation

$$\mathbf{S}_t = \mathbf{S} \bigwedge \mathbf{S}_{xx} + \mathbf{S} \bigwedge \mathbf{J}\mathbf{S}$$

$$L = D_x - i\sum_{k=1}^{3} W_k(\lambda) S_k \sigma_k$$

where $W_n(\lambda)^2 - W_m(\lambda)^2 = J_n - J_m$ and σ_k are Pauli matrices.

1. Structure of Lax pairs.

We consider two differential operators

$$L = D_x - U, \qquad A = D_t - V,$$

where $U = U(x,t), V = V(x,t)$ are two $n \times n$ matrices. The compatibility condition

$$[L, A] = D_t(U) - D_x(V) + [U, V] = 0 \tag{1}$$

provides the existence of a fundamental solution to the over-determined linear systems

$$L\Psi = \Psi_x - U\Psi = 0, \quad A\Psi = \Psi_t - V\Psi = 0$$

Equation (1) is a nonlinear PDE, but trivial. Its general solution is given by

$$U = \Psi_x \Psi^{-1}, \quad V = \Psi_t \Psi^{-1},$$

where $\Psi = \Psi(x,t)$ is any nonsingular matrix function.

Equation (1) becomes non-trivial if we assume that matrices U, V also depend on an auxiliary (spectral) parameter λ and are rational functions of λ. We also require that equation (1) is satisfied for all values of λ.

Example: $U = U_0 + \lambda U_1, V = V_0 + \lambda^{-1} V_1$, then (1) yields

at	λ	$D_t(U_1) - [V_0, U_1] = 0$
at	λ^0	$D_t(U_0) - D_x(V_0) + [U_0, V_0] + [U_1, V_1] = 0$
at	λ^{-1}	$D_x[V_1] - [U_0, V_1] = 0$

Solution of a matrix Riemann-Hilbert problem $\Psi(x,t,\lambda)$

$$\Psi_x \Psi^{-1} = U_0 + \lambda U_1, \quad \Psi_t \Psi^{-1} = V_0 + \lambda^{-1} V_1 \, .$$

Gauge freedom, gauge transformations

$$L \to \widehat{L} = g^{-1} L g, \quad A \to \widehat{A} = g^{-1} A g\,.$$

$$\widehat{L} = D_x - \widehat{U}_0 - \lambda \widehat{U}_1, \quad \widehat{U}_0 = g^{-1} U_0 g - g^{-1} g_x,\ \widehat{U}_1 = g^{-1} U_1 g$$

$$\widehat{A} = D_t - \widehat{V}_0 - \lambda^{-1} \widehat{V}_1, \quad \widehat{V}_0 = g^{-1} V_0 g - g^{-1} g_t,\ \widehat{V}_1 = g^{-1} V_1 g$$

For example

$$\mathbf{S}_t = \mathbf{S} \bigwedge \mathbf{S}_{xx} \text{ and } i q_t = q_{xx} + 2|q|^2 q\,,$$

are gauge equivalent.

We can extend the gauge group by external automorphisms

$$L \to -h^{-1} L^{\mathsf{A}} h, \quad A \to -h^{-1} A^{\mathsf{A}} h\,.$$

Matrices g, h may also depend on λ, be differential operators, ...

Miura transformations are examples of gauge transformations.

Change of the spectral parameter $\lambda \to \mu = \sigma(\lambda)$

Example: $\lambda = \frac{\mu+1}{\mu-1}$

$$L \to D_x - \widetilde{U}_0 + \frac{\widetilde{U}_1}{\mu - 1}, \quad A \to D_t - \widetilde{V}_0 + \frac{\widetilde{V}_1}{\mu + 1}\,,$$

where $\tilde{U}_0 = U_0 + U_1, \tilde{U}_1 = 2U_1, \tilde{V}_0 = V_0 + V_1, \tilde{V}_1 = -2V_1$. By a gauge transformation one can set $\tilde{U}_0 = \tilde{V}_0 = 0$. Result is a Lax pair for the Principal Chiral field model.

Algebraic structure

$+,\ [\cdot,\cdot],\ D_x,\ D_t$ - Lie algebra $U, V \in \mathcal{A}$.

Nonlinear coupled equations $\Rightarrow$ the Lie algebra $\mathcal{A}$ is simple.

Solvable $\mathcal{A} \Rightarrow$ linear triangular system of equations.

Reductions, the reduction group

Example: The Tzetzeika equation

$$L = D_x - i\frac{\sqrt{3}}{3}u_x \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} q & 0 & 0 \\ 0 & \bar{q} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $q = \exp(2\pi i/3)$.

We start with a general operator: $L = D_x - iU_0 - \lambda U_1$

$$g^{-1}Lg \to \hat{U}_1 = g^{-1}U_1 g = \text{diag}(a_1, a_2, a_3), \quad \text{diag}\hat{U}_0 = 0$$

Thus

$$L = D_x - i \begin{pmatrix} 0 & u_{12} & u_{13} \\ u_{21} & 0 & u_{23} \\ u_{31} & u_{32} & 0 \end{pmatrix} - \lambda \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

We impose a symmetry Q, s.t. $Q^3 = id$:

$$Q : L(\lambda) \to J^{-1} L(\bar{q}\lambda) J = L(\lambda),\ J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $a_n = aq^n$ and

$$L = D_x - i \begin{pmatrix} 0 & w & v \\ v & 0 & w \\ w & v & 0 \end{pmatrix} - \lambda a \begin{pmatrix} q & 0 & 0 \\ 0 & \bar{q} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Imposing another symmetry P, ($P^2 = id$):

$$P : L(\lambda) \to -L^{\mathsf{A}}(-\lambda) = L(\lambda)$$

we find $w = -v$. Transformations P, Q form the S_3 group.

Symmetry H, ($H^2 = id$):

$$H : L(\lambda) \to h^{-1} \bar{L}(\bar{\lambda}) h = L(\lambda),\ h = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

implies that w and a are real.

These symmetries act on solutions $L\Psi = 0$

$$\begin{aligned} Q: &\quad \Psi(\lambda) \to J\Psi(q\lambda) \\ P: &\quad \Psi(\lambda) \to (\Psi^{\mathrm{tr}}(-\lambda))^{-1} \\ H: &\quad \Psi(\lambda) \to h\bar{\Psi}(\bar{\lambda}) \end{aligned}$$

Local Symmetries, conservation laws and the Lax pairs

How to find symmetries and local conservation laws for equations having the Lax representations (such as KdV $L = D_x^2 + u$, Nonlinear Schrödinger equation, ...)?

A few general definitions:

1. We define a differential ring $\mathcal{R}[u]$ of polynomials of infinite number of variables $u, u_1, u_2, \ldots$ over $\mathbb{C}$ with a derivation D defined by

$$D(u_n) = u_{n+1}, \quad D(\alpha) = 0, \alpha \in \mathbb{C}\,.$$

We assume that $1 \notin \mathcal{R}[u]$. Derivation D represents D_x, and u_n represents $\partial_x^n u$.

An evolutionary equation, such as the KdV

$$u_t = u_3 + 6uu_1 = f[u] \in \mathcal{R}[u],$$

defines another derivation D_t of the $\mathcal{R}[u]$ by

$$D_t(u) = f[u],\ D_t(u_n) = D^n(f[u]),\ D_t(\alpha) = 0, \alpha \in \mathbb{C}$$

which commutes with D. Derivations of $\mathcal{R}[u]$ commuting with D we call evolutionary derivations.

2. A symmetry can be defined as an evolutionary derivation D_τ commuting with D_t. It is sufficient to define the action of D_τ on u, i.e. an element $D_\tau(u) = g[u] \in \mathcal{R}[u]$. Element $g[u]$ is usually called a symmetry generator.

For KdV:

$$u_{\tau_1} = u_1$$
$$u_{\tau_3} = u_3 + 6uu_1$$
$$u_{\tau_5} = u_5 + 10uu_3 + 20u_1u_2 + 30u^2u_1$$

are symmetries, and there are infinitely many symmetries. All corresponding derivations commute $[D_{\tau_n}, D_{\tau_m}] = 0$.

3. Local conservation laws. Element $\rho \in \mathcal{R}[u]$ is said to be a density of a local conservation law if

$$D_t(\rho) = D(\sigma), \quad \sigma \in \mathcal{R}[u]\,,$$

i.e. $D_t : \rho \to D(\mathcal{R}[u])$.

$\rho = D(h), h \in \mathcal{R}[u]$ is a trivial density.

$\rho \in \mathcal{R}[u]/D(\mathcal{R}[u])$. Densities ρ_1, ρ_2 are equivalent, if $\rho_1 - \rho_2 \in D(\mathcal{R}[u])$

$$h \in D(\mathcal{R}[u]) \Longleftrightarrow \frac{\delta h}{\delta u} = 0$$

$$\frac{\delta h}{\delta u} = \sum_{k=0} (-D)^k (\frac{\partial h}{\partial u_k})$$

For KdV $u, \rho_0 = u^2, \rho_2 = u_1^2 - 2u^3, \ldots$ are densities of local conservation laws.