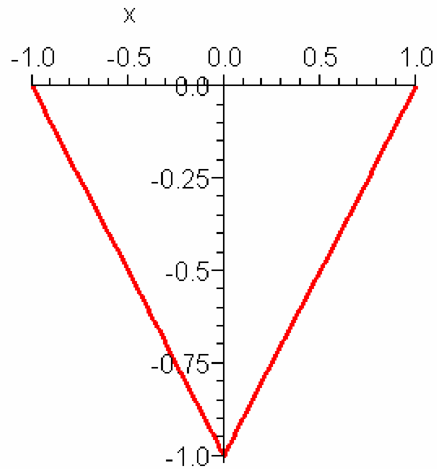
Boundary value problems for nonlinear first-order ODEs - constructing generalised solutions via the max-plus algebra

$$(y'(x))^2 = 1$$

$$y(-1) = y(1) = 0$$



$$a \oplus b = \max\{a, b\}$$

$$a \odot b = a + b$$

Georg Regensburger
Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences
georg.regensburger@oeaw.ac.at

SFB F1322

Workshop on the Algebraic Theory of Differential Equations
Heriot-Watt University, Edinburgh, 9 August, 2006

Der Wissenschaftsfonds.

RICAM

Max-plus Linear Combinations $\max(a_1 + y_1(x), a_2 + y_2(x))$

First-order differential equation:

$$f(x, y'(x)) = 0 \qquad (1)$$

$y_1(x), y_2(x)$ solutions of (1), $\quad a_1, a_2 \in \mathbb{R}$

Then

$$y(x) = \max(a_1 + y_1(x), a_2 + y_2(x))$$

is a (generalized) solution of (1) *Max-plus linear combination*

(Min-plus)

(nondifferentiable at some points)

Max-plus Interpolation $\quad y(x_1) = b_1,\ y(x_2) = b_2$

Given: $y_1(x), y_2(x), \quad x_1, x_2 \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$

Find: $a_1, a_2 \in \mathbb{R}$

such that $y(x) = \max(a_1 + y_1(x), a_2 + y_2(x))$

satisfies $y(x_1) = b_1$ and $y(x_2) = b_2$

Solve:

$$\max(a_1 + y_1(x_1), a_2 + y_2(x_1)) = b_1$$
$$\max(a_1 + y_1(x_2), a_2 + y_2(x_2)) = b_2$$

Max-plus linear system

$$\begin{pmatrix} y_1(x_1) & y_2(x_1) \\ y_1(x_2) & y_2(x_2) \end{pmatrix}$$ *Interpolation matrix*

Generalise: m points and values, and n functions

Max-plus Semiring

$\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$

$a \oplus b = \max\{a, b\}$ $\qquad$ $a \odot b = a + b$

$2 \oplus 3 = 3$ $\qquad$ $2 \odot 3 = 5$

$a \oplus -\infty = a \quad \mathbf{0} = -\infty$ $\qquad$ $a \odot 0 = a \quad \mathbf{1} = 0$

Semiring = “ring without subtraction”

Commutative additive monoid, multiplicative monoid, distributivity,

$0 \odot a = a \odot 0 = 0$

- Natural numbers $\mathbb{N}$
- Max-plus $\mathbb{R}_{\max}$ and the dual $\mathbb{R}_{\min}$ $\qquad a^{(-1)} = -a$
- Nonnegative real numbers $\mathbb{R}_+$ with usual $+, \cdot$ $\qquad$ Semifields
- Ideals of a commutative ring
- Square matrices over a semiring

Golan [Gol99]

Idempotent Semirings

$a \oplus a = a$

Max-plus: $a \oplus a = \max\{a, a\} = a$ *Idempotent* Semiring

Rings can't be idempotent: $1 + 1 = 1 \overset{(-1)}{\Longrightarrow} 1 = 0$

Idempotent Semirings:

$a \oplus b = 0 \Rightarrow a = b = 0$

Standart partial order $a \preceq b \Leftrightarrow a \oplus b = b$ lattice theory

Then $0 \preceq a$ and $a \preceq b \Rightarrow a \odot c \preceq b \odot c$

Max-plus:

$a \preceq b \Leftrightarrow a \oplus b = b \Leftrightarrow \max\{a, b\} = b \Leftrightarrow a \leq b$ usual order

Idempotent Analysis Kolokoltsov, Maslov [KM97] Litvinov [Lit05]

Tropical algebraic geometry Richter-Gebert, Sturmfels, Theobald, [RGST05]

Matrices: $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$

Max-plus Linear Algebra

$$(A \odot B)_{ij} = \bigoplus_k A_{ik} \odot B_{kj} = \mathsf{max}_k(A_{ik} + B_{kj})$$

I *identity matrix*

$$I = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}$$

D *diagonal matrix*

$$D = \begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & b \end{pmatrix} = \begin{pmatrix} a & -\infty \\ -\infty & b \end{pmatrix}$$

P *permutation matrix* permuting rows and/or columns of I

A *generalized permutation matrix*

$$A = D \odot P$$

Invertible matrices = generalized permutation matrices

In particular $A \in \mathbb{R}^{n \times n}$ not invertible

Cuninghame-Green [CG79], Butkovič [But03], Gaubert, Plus [GP97]

Max-plus Linear Systems I

$A \odot x = 0$

Linear System $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\max(-1 + x_1, 1 + x_2) = 0 \quad x_1 \le 1 \qquad x_1 \le \min(-1, 1)$

$\max(1 + x_1, -1 + x_2) = 0 \quad x_1 \le -1 \qquad = -\max \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$x_1 \le \bar{x}_1 = -1 = -\max \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$x_2 \le \bar{x}_2 = -1 = -\max \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\bar{x}$ *principal solution*

x solution of $A \odot x = 0$ iff

$x \le \bar{x}$ and

for every row i there is a column j

$a_{ij} = \max_k a_{kj} \qquad x_j = \bar{x}_j$

Solvability: test if the principal solution solves the system (O(mn))

Unique solvability: equivalent to Minimal Set Covering (NP-complete)

Cuninghame-Green [CG79], Butkovič [But03]

Max-plus Linear Systems II

Linear system $A \odot x = b$ $\quad A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

$$D = \mathrm{diag}(b_1^{-1}, \ldots, b_m^{-1}) = \mathrm{diag}(-b_1, \ldots, -b_m)$$

$$(D \odot A) \odot x = D \odot b = 0$$ *normalized system*

(not homogenous, $0 = \mathbf{1}$)

Solution set $S(A, b) = \{x \in \mathbb{R}^n : A \odot x = b\}$

As in LA the number of solutions $|S(A, b)| = \{0, 1, \infty\}$

But

$$T(A) = \{|S(A, b)| : b \in \mathbb{R}^m\} = \begin{matrix}\{0, \infty\} \\ \{0, 1, \infty\}\end{matrix}$$

Even if there is a unique solution for a RHS b
then there is a RHS $\tilde{b}$ with $|S(A, \tilde{b})| = \infty$
and one with no solutions.

Butkovič [But94,But03]

$(y'(x))^2 = 1$ Solutions: $y_1(x) = x \quad y_2(x) = -x$

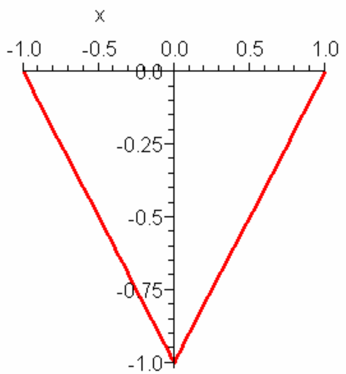
Find $a_1, a_2 \in \mathbb{R}$, $y = a_1 \odot y_1 \oplus a_2 \odot y_2$

$y(-1) = y(1) = 0$ Solve $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \odot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\mathbb{R}_{\text{max}} \quad a_1 = -1, \ a_2 = -1$

$y(x) = \max(-1 + x, -1 - x)$

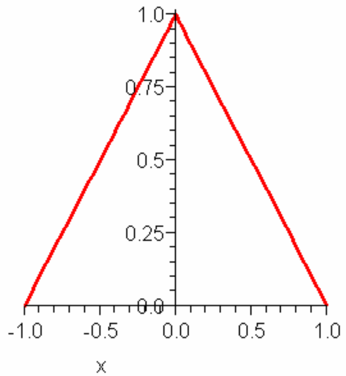
$= |x| - 1$



$\mathbb{R}_{\text{min}} \quad a_1 = 1, \ a_2 = 1$

$y(x) = \min(1 + x, 1 - x)$

$= 1 - |x|$



Max-Plus Interpolation and Multipoint BVPs $x, -x, 1/2\,x^2$

$$y_1(x) = x \quad y_2(x) = -x \qquad y_3(x) = 1/2\,x^2$$

$$(y'(x)-1)(y'(x)+1)(y'(x)-x)$$

Find $a_1, a_2, a_3 \in \mathbb{R}$, $\qquad y = a_1 \odot y_1 \oplus a_2 \odot y_2 \oplus a_3 \odot y_3$

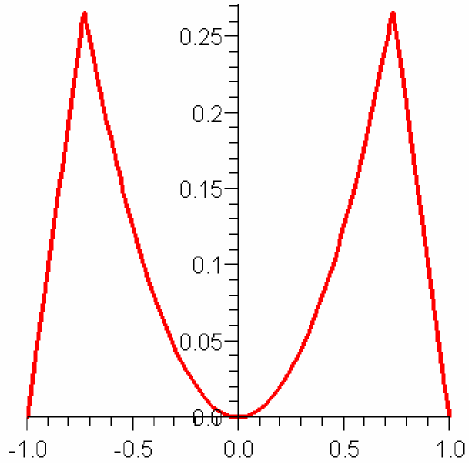
$y(-1) = y(0) = y(1) = 0$ Solve $\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 1 & -1 & \frac{1}{2} \end{pmatrix} \odot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\mathbb{R}_{\max}$ No solution

$\mathbb{R}_{\min}$ $a_1 = 1,\ a_2 = 1,\ a_3 = 0$

$$y(x) = \min(1+x, 1-x, 1/2\,x^2)$$

$$= \frac{1}{4}x^2 - \frac{1}{2}|x| + \frac{1}{2} - \frac{1}{2}\left|\frac{1}{2}x^2 - 1 + |x|\right|$$



`> LinearMaxSolve(A,b)`

Maple implementation:

- Solve max(min)-plus linear systems
- Basic matrix vector operations, generate equations, conversions
- Based on **`LinearAlgebra`** package
- Max-plus interpolation
- Use **`dsolve`** to solve differential equations

Maple **`solve`** gives not all, wrong solutions of max-plus linear systems

Convert max to abs $\max(a,b) = \dfrac{a+b+|a-b|}{2}$

Solutions of BVPs can be expressed with nested absolute values (advantage for symbolic differentiation)

Conclusion and Outlook

• Solve nonlinear first-order ordinary BVPs given symbolic solutions to the initial value problem via Max-plus interpolation

• To decide (unique) solvability and compute Max-plus solutions we only need evaluation of the solution at “boundary” points

• Use numerical solutions of nonlinear ODEs

• Relate Max-plus solutions to known solution concepts with Martin Burger: viscosity solutions

• Consider PDEs (Hamilton-Jacobi equations)

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