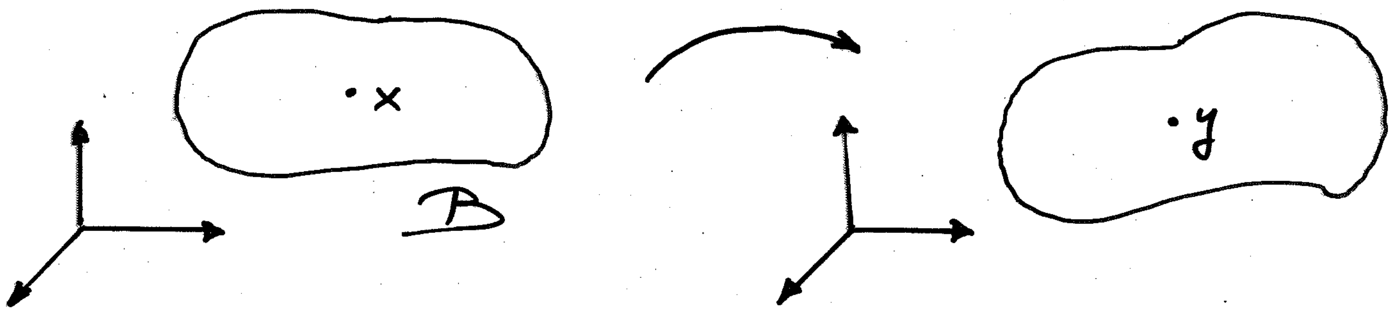
ELASTOSTATICS



placement: $\quad y = y(x)$

deformation gradient: $\quad F = Dy$

strain energy: $\quad W = W(F)$

MINIMIZE: $\quad I[y] = \int_{\mathcal{B}} W(F)\,dx$

$$\frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}} G_{i\alpha} G_{j\beta} > 0 \qquad \text{convexity}$$

$$\frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}} \xi_i \xi_j \nu_\alpha \nu_\beta > 0 \qquad \text{rank-one convexity}$$

convexity: sufficient but not allowable

rank-one convexity: allowable but not sufficient

POLYCONVEXITY

$$\partial_\alpha F_{i\beta} - \partial_\beta F_{i\alpha} = 0, \qquad i, \alpha, \beta = 1, 2, 3$$

$\Longrightarrow$ $\det F$, $F^* = (\det F) F^{-1}$ wk. continuous

$$W(F) = \sigma(F, F^*, \det F)$$

W polyconvex $\Longleftrightarrow$ σ convex

polyconvexity $\Rightarrow$ rank-one convexity

JOHN BALL

ELASTODYNAMICS

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0 & i,\alpha = 1,2,3 \\ \partial_t v_i - \partial_\alpha T_{i\alpha}(F) = 0 & i = 1,2,3 \end{cases}$$

$$T_{i\alpha}(F) = \frac{\partial W(F)}{\partial F_{i\alpha}}$$

$$\partial_t \left[\tfrac{1}{2} |v|^2 + W(F) \right] - \partial_\alpha \left[v_i T_{i\alpha}(F) \right] = 0$$

rank-one convexity $\Leftrightarrow$ hyperbolicity

$$\partial_\alpha F_{i\beta} - \partial_\beta F_{i\alpha} = 0$$

KINEMATIC CONSERVATION LAWS

$$\partial_\alpha F_{i\beta} - \partial_\beta F_{i\alpha} = 0$$

imply:

$$\partial_t \det F - \partial_\alpha \left(v_j \frac{\partial \det F}{\partial F_{j\alpha}} \right) = 0$$

$$\partial_t F^*_{i\alpha} - \partial_\beta \left(v_j \frac{\partial F^*_{i\alpha}}{\partial F_{j\beta}} \right) = 0$$

Qin, Dafermos, Demoulini–Stuart–Tzavaras

HYPERBOLIC CONSERVATION LAWS

$$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = 0$$

$x \in \mathbb{R}^m$, $t \in \mathbb{R}$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$, $\alpha = 1, \dots, m$

$U \in \mathbb{R}^n$, $G_\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\alpha = 1, \dots, m$

Hyperbolic:

For all $U \in \mathbb{R}^n$ and $\nu \in S^{m-1}$, the matrix

$$\Lambda(U;\nu) = \sum_{\alpha=1}^{m} \nu_\alpha \, DG_\alpha(U)$$

has real eigenvalues and n linearly independent eigenvectors

ENTROPY

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

entropy - entropy flux pair: $(\eta(U), q_\alpha(U))$

$$Dq_\alpha(U) = D\eta(U) DG_\alpha(U), \quad \alpha = 1, \dots, m$$

U classical (=Lipschitz) solution:

$$\partial_t \eta(U) + \sum_{\alpha=1}^{m} \partial_\alpha q_\alpha(U) = 0$$

$$D^2\eta(U) DG_\alpha(U) = DG_\alpha(U)^T D^2\eta(U), \quad \alpha = 1, \dots, m$$

Symmetric: $DG_\alpha^T = DG_\alpha \qquad \eta(U) = |U|^2$

Lax, Friedrichs, Godunov, Kruzkov, ...

CONVEX ENTROPY $\Rightarrow$ LOCAL EXISTENCE

For the Cauchy problem

$$\begin{cases} \partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0 & \text{on } \mathbb{R}^m \times [0,T) \\ U(x,0) = U_0(x) & \text{on } \mathbb{R}^m \end{cases}$$

with $\nabla_x U_0 \in H^\ell(\mathbb{R}^m)$, $\ell > \frac{m}{2}$

there exists unique classical solution U:

$$\nabla_x U(\cdot,t) \in C^0([0,T_\infty); H^\ell(\mathbb{R}^m))$$

$[0,T_\infty)$ is maximal: if $T_\infty < \infty$ then

$$\int_0^{T_\infty} \| \nabla_x U(\cdot,t)\|_{L^\infty} \, dt = \infty$$

Schauder, Friedrichs, Lax, Kato, Majda ...

ADMISSIBLE WEAK SOLUTIONS

$$\begin{cases} \partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0 & \text{on } \mathbb{R}^m \times [0,\infty) \\ U(x,0) = U_0(x) & \text{on } \mathbb{R}^m \end{cases}$$

- Classical solutions blow up in finite time.
- Weak solutions $U \in L^\infty(\mathbb{R}^m \times [0,\infty))$.
- Nonuniqueness.
- Entropy admissibility condition

$$\partial_t \eta(U) + \sum_{\alpha=1}^{m} \partial_\alpha q_\alpha(U) \leq 0 \qquad \text{on } \mathbb{R}^m \times [0,\infty)$$

CONVEX ENTROPY $\Rightarrow$ CLASSICAL SOLUTIONS ARE UNIQUE AND STABLE WITHIN THE CLASS OF ADMISSIBLE WEAK SOLUTIONS

Assume :

$\bar{U}$ classical solution on $\mathbb{R}^m \times [0,T)$

U admissible weak solution on $\mathbb{R}^m \times [0,T)$

Then :

$$\int_{|x|<R} |U(x,t) - \bar{U}(x,t)|^2 \, dx$$

$$\leq C e^{\alpha t} \int_{|x|<R+st} |U(x,0) - \bar{U}(x,0)|^2 \, dx$$

Dafermos, DiPerna

"PROOF":

$$\partial_t \bar{U} + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(\bar{U}) = 0 \qquad \partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

$$\partial_t \eta(\bar{U}) + \sum_{\alpha=1}^{m} \partial_\alpha q_\alpha(\bar{U}) = 0 \qquad \partial_t \eta(U) + \sum_{\alpha=1}^{m} \partial_\alpha q_\alpha(U) \leq 0$$

$$h(U, \bar{U}) = \eta(U) - \eta(\bar{U}) - D\eta(\bar{U})[U - \bar{U}]$$

$$f_\alpha(U, \bar{U}) = q_\alpha(U) - q_\alpha(\bar{U}) - D\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})]$$

$$Z_\alpha(U, \bar{U}) = D^2\eta(\bar{U})\{G_\alpha(U) - G_\alpha(\bar{U}) - DG_\alpha(\bar{U})[U - \bar{U}]\}$$

$$\partial_t h(U, \bar{U}) + \sum_{\alpha=1}^{m} \partial_\alpha f_\alpha(U, \bar{U})$$

$$\leq - \sum_{\alpha=1}^{m} \partial_\alpha \bar{U}^T Z_\alpha(U, \bar{U})$$

INVOLUTIONS

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

$$A_\alpha G_\beta(U) + \overset{k\times n}{A_\beta}\, G_\alpha(U) = 0$$

$$\sum_{\alpha=1}^{m} A_\alpha \partial_\alpha U = 0$$

$$\Lambda(U;\nu) = \sum_{\alpha=1}^{m} \nu_\alpha \, DG_\alpha(U)$$

$$N(\nu) = \sum_{\alpha=1}^{m} \nu_\alpha A_\alpha$$

$$N(\nu)\,\Lambda(U;\nu) = 0$$

$$\operatorname{rank} N(\nu) = \dim\ker \Lambda(U;\nu)$$

NVOLUTION CONE: $\mathscr{C} = \bigcup_{\nu\in S^{m-1}} \ker N(\nu)$

CAUCHY PROBLEM

$$\begin{cases} \partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0 \\ U(x,0) = U_0(x) \end{cases}$$

$$\sum_{\alpha=1}^{m} A_\alpha \partial_\alpha U = 0$$

$(\eta(U), q(U))$ entropy-entropy flux pair

$D^2\eta(U)$ + definite in direction of $\mathscr{C}$

THEOREM: For $\nabla U_0 \in H^\ell$, $\ell > \frac{m}{2}$, $\exists$ solution

$$\nabla U(\cdot,t) \in C^0([0,T_\infty); H^\ell)$$

$$T_\infty < \infty \Rightarrow \|\nabla U(\cdot,t)\|_{L^\infty} \to \infty \text{ as } t \uparrow T_\infty$$

CONSTRUCTION OF SOLUTION

Standard method

$$\partial_t U + \sum_{\alpha=1}^{m} DG_\alpha(U)\,\partial_\alpha U = 0$$

$$\partial_t U + \sum_{\alpha=1}^{m} DG_\alpha(V)\,\partial_\alpha U = 0$$

$$V \longmapsto U$$

may fail.

Vanishing viscosity approach:

$$\partial_t U + \sum_{\alpha=1}^{m} DG_\alpha(U)\,\partial_\alpha U = \varepsilon \Delta U$$

UNIQUENESS

Assume:

- $\bar{U}$ classical solution
- U admissible weak solution
- $$U(x,0) = \bar{U}(x,0)$$
- $$U(x,t) - \bar{U}(x,t) \longrightarrow 0 \,, \quad \text{as } |x| \to \infty$$
- $$\limsup_{x \to y,\, t \to \tau} |U(x,t) - \bar{U}(y,\tau)| << 1$$

Then $U \equiv \bar{U}$

CONTINGENT ENTROPY–ENTROPY FLUX PAIRS

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

$$\sum_{\alpha=1}^{m} A_\alpha \partial_\alpha U = 0$$

$(\eta(U), q(U))$ contingent entropy-entropy flux pair

$$D q_\alpha(U) = D\eta(U) D G_\alpha(U) + R(U)^T A_\alpha$$

U classical solution satisfying the involution:

$$\partial_t \eta(U) + \sum_{\alpha=1}^{m} \partial_\alpha q_\alpha(U) = 0$$

D. Serre

POLYCONVEXITY

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

$$\sum_{\alpha=1}^{m} A_\alpha \partial_\alpha U = 0$$

Principal contingent pair: $(\eta(U), q(U))$

Supplementary contingent pairs: $(\Phi(U), \Psi(U))$

$$D\Psi_\alpha(U) = D\Phi(U) DG_\alpha(U) + S(U)^T A_\alpha$$

Completeness: $\operatorname{rank} D\Phi(U) = n$

$\eta(U)$ polyconvex if

$$\eta(U) = \overset{\text{CONVEX}}{\theta}(\Phi(U))$$

STABILITY

THEOREM Let $\bar{U}$ be a classical solution with initial data $\bar{U}_0$ and U an admissible weak solution with initial data U_0. Assume $\bar{U}_0$ and U_0 satisfy the involution. Then

$$\int_{|x|<R} |U(x,t) - \bar{U}(x,t)|^2 \, dx$$

$$\le \alpha e^{\beta t} \int_{|x|<R+st} |U_0(x) - \bar{U}_0(x)|^2 \, dx$$

"PROOF"

$$h(U,\bar{U}) = \eta(U) - \eta(\bar{U}) - \theta_\Phi(\Phi(\bar{U}))[\Phi(U) - \Phi(\bar{U})]$$

$$f_\alpha(U,\bar{U}) = q_\alpha(U) - q_\alpha(\bar{U}) - \theta_\Phi(\Phi(\bar{U}))[\Psi_\alpha(U) - \Psi_\alpha(\bar{U})]$$
$$+[\theta_\Phi(\Phi(\bar{U}))S(\bar{U})^T - R(\bar{U})^T]A_\alpha[U-\bar{U}]$$

$$Z_\alpha(U,\bar{U}) = -DG_\alpha(\bar{U})^T D\Phi(\bar{U})^T \theta_{\Phi\Phi}(\Phi(\bar{U}))[\Phi(U) - \Phi(\bar{U})]$$
$$+D\Phi(\bar{U})^T\theta_{\Phi\Phi}(\Phi(\bar{U}))[\Psi_\alpha(U) - \Psi_\alpha(\bar{U})]$$
$$-D\Phi(\bar{U})^T\theta_{\Phi\Phi}(\Phi(\bar{U}))S(\bar{U})^T A_\alpha[U-\bar{U}]$$
$$+\Xi(\bar{U})^T A_\alpha[U-\bar{U}]$$

$$\Xi(U) = DR(U) - \sum_{I=1}^{N} \theta_{\Phi_I}(\Phi(U))DS_I(U)$$

$$\partial_t h(U,\bar{U}) + \sum_{\alpha=1}^{m} \partial_\alpha f_\alpha(U,\bar{U})$$
$$\le -\sum_{\alpha=1}^{m} \partial_\alpha \bar{U}^T Z_\alpha(U,\bar{U})$$

EXTENDED SYSTEMS

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0$$

$$\sum_{\alpha=1}^{m} A_\alpha \partial_\alpha U = 0$$

$(\Phi(U), \Psi(U))$ complete contingent entropy pairs

$\eta(U) = \Theta(\Phi(U))$ polyconvex entropy

$$\partial_t \Phi + \sum_{\alpha=1}^{m} \partial_\alpha X_\alpha(\Phi) = 0$$

$$X(\Phi(U)) = \Psi(U)$$

$\Theta(\Phi)$: entropy for extended system?

Brenier, Serre, Demoulini-Stuart-Tzavaras

Lattanzio-Tzavaras