Bases and addition formulae associated with higher genus Abelian functions

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The higher-genus sigma function and applications
ICMS, Edinburgh

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Outline

1. Background and motivation
   - Weierstrass elliptic function
   - Generalising to higher genus
2. Bases and addition formulae
   - Bases of Abelian functions
   - Addition formulae

Outline

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The Weierstrass $\wp$-function I

Recall the classic elliptic $\wp$-function of Weierstrass.

Karl Weierstrass
1815-1897

- It is meromorphic with two independent periods $\omega_1, \omega_2, \frac{\omega_1}{\omega_2} \notin \mathbb{R}$:

$$\wp(u+\omega_1) = \wp(u+\omega_2) = \wp(u)$$

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$$\wp(u+\omega_1) = \wp(u+\omega_2) = \wp(u)$$

- We can define using the auxiliary σ-function,

$$\wp(u) = -\frac{d^2}{du^2}\ln[\sigma(u)].$$

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The Weierstrass $\wp$-function II

- The function satisfies key differential equations,

$$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3,$$
$$\wp''(u) = 6\wp(u)^2 - \tfrac{1}{2}g_2.$$

- Consider a non-singular algebraic curve of the form,

$$y^2 = x^3 + ax + b, \qquad a, b \text{ constant.}$$

This is an elliptic curve, which is parametrised by $(\wp, \wp')$.

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The Weierstrass $\wp$-function III

- For points close to the origin we have series expansions,

$$\wp(u) = \frac{1}{u^2} + \frac{1}{20}g_2u^2 + \frac{1}{28}g_3u^4 + \dots$$
$$\sigma(u) = u - \frac{1}{240}g_2u^5 - \frac{1}{840}g_3u^7 - \dots$$

- Both $\wp(u)$ and $\sigma(u)$ satisfy addition formula.

$$\wp(u+v) = \frac{1}{4}\left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right]^2 - \wp(u) - \wp(v).$$

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(u) - \wp(v).$$

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Outline

General and cyclic (n, s)-curves

General (n,s)-curves

Let (n, s) be coprime with $n < s$. Define general (n,s)-curves as

$$y^n - x^s - \sum_{\alpha,\beta} \mu_{[ns-\alpha n-\beta s]} x^\alpha y^\beta = 0 \qquad \mu_j \text{ constants,}$$

where $\alpha, \beta \in \mathbb{Z}$ with $\alpha \in (0, s-1)$, $\beta \in (0, n-1)$ and $\alpha n + \beta s < ns$. These have genus $g = \frac{1}{2}(n-1)(s-1)$.

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They have a simpler subclass of cyclic (n,s)-curves

$$y^n = x^s + \lambda_{s-1}x^{s-1} + ... + \lambda_1 x + \lambda_0$$

These curves are invariant under

$$(x,y) \to (x,\zeta y), \qquad \zeta^n = 1.$$

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Abelian functions associated to curves

For a given (n, s)-curve, C, we can construct two standard period matrices, ω_1 and ω_2 which are associated with the curve.

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For a given (n, s)-curve, C, we can construct two standard period matrices, ω_1 and ω_2 which are associated with the curve.

Let $\mathfrak{M}(\boldsymbol{u})$ be a meromorphic function of $\boldsymbol{u} \in \mathbb{C}^g$. Then $\mathfrak{M}(\boldsymbol{u})$ is an Abelian function associated with C if

$$\mathfrak{M}(\boldsymbol{u} + \omega_1 \boldsymbol{n}^T + \omega_2 \boldsymbol{m}^T) = \mathfrak{M}(\boldsymbol{u}),$$

for all integer vectors $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}$ where $\mathfrak{M}(\boldsymbol{u})$ is defined.

We work with an Abelian functions that generalise the Weierstrass $\wp$-function and are realised in general using the higher genus σ-function, associated to an (n, s)-curve.

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The higher genus σ-function

- Function of g variables: $\sigma = \sigma(\boldsymbol{u}) = \sigma(u_1, u_2, \dots, u_g)$.
- Riemann θ-function multiplied by exponential factor.
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$$\sigma(\boldsymbol{u} + \ell) = \chi(\ell) \exp\big(\mathcal{L}(\ell)\big)\sigma(\boldsymbol{u}).$$

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- Only zeros are of order one and can be shown to form a subset of the Jacobian, $\Theta^{[g-1]}$.
- Definite parity: $\sigma(-\boldsymbol{u}) = (-1)^{\frac{1}{24}(n^2-1)(s^2-1)}\sigma(\boldsymbol{u})$.
- Expansion around the origin has leading order part given by Schur-Weierstrass polynomial.

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Kleinian $\wp$-functions I

We define $\wp$-functions associated to a given (n, s)-curve using the σ-function, (in analogy to the elliptic case).

Kleinian $\wp$-functions

Define the Kleinian $\wp$-functions as the second log derivatives.

$$\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(\mathbf{u}), \quad i \le j \in \{1, 2, ..., g\}$$

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- Imposing this notation on the elliptic case gives $\wp_{11} \equiv \wp$.

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Kleinian $\wp$-functions II

We can extend this notation to higher order derivatives. E.g.

$$\wp_{ijk} = -\frac{\partial^3}{\partial u_i \partial u_j \partial u_k} \ln \sigma(\mathbf{u}) \qquad i \leq j \leq k \in \{1, 2, ..., g\}$$

$$\wp_{ijkl} = -\frac{\partial^4}{\partial u_i \partial u_j \partial u_k \partial u_l} \ln \sigma(\mathbf{u}) \qquad i \leq j \leq k \leq l \in \{1, 2, ..., g\}$$

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- Imposing this notation on the elliptic case would show
$$\wp' \equiv \wp_{111} \qquad \wp'' \equiv \wp_{1111}$$
- A curve with $g = 3$ has 6 $\wp_{ij}$ and 10 $\wp_{ijk}$:

$$\{\wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}\}$$
$$\{\wp_{111}, \wp_{112}, \wp_{113}, \wp_{122}, \wp_{123}, \wp_{133}, \wp_{222}, \wp_{223}, \wp_{233}, \wp_{333}\}$$

Review of higher genus work I

n=2, s=3: elliptic curves

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n=2, s=3: elliptic curves

n=2, s>3: hyperelliptic curves

Felix Klein
1849-1925

H. F. Baker
1866-1956

- Klein and Baker generalised Weierstrass functions, inspiring current approach. They derived many results for those functions associated to a (2,5)-curve (genus two).

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- Klein and Baker generalised Weierstrass functions, inspiring current approach. They derived many results for those functions associated to a (2,5)-curve (genus two).
- Buchstaber, Enolski and Leykin (1997) modernised the approach, derived results for hyperelliptic curves of arbitrary genus & many details for the genus 2 & 3 cases.
- Recent progress made on addition formulae and differential equations.

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Review of higher genus work II

n=3: trigonal curves

- Considerable work has been published by authors including Baldwin, Buchstaber, Eilbeck, Enolski, Gibbons, Leykin, Matsutani, Onishi and Previato.

n=4: tetragonal curves

- The lowest genus case (g=6) was examined in detail in 2008. Solution to JIP, series expansions, PDEs, applications and an addition formula were derived.

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- The lowest genus case (g=6) was examined in detail in 2008. Solution to JIP, series expansions, PDEs, applications and an addition formula were derived.

n>4: n-gonal curves

- No specific examples have yet been studied. In theory, the techniques developed for $n = 4$ could be applied in a similar way. Computational restraints limit progress.

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Spaces of Abelian functions

We categorise the Abelian functions associated to a curve according to their pole structure. Denote by

$$\Gamma\big(J, \mathcal{O}(m\Theta^{[g-1]})\big)$$

the vector space over $\mathbb{C}$ of Abelian functions defined upon the Jacobian, J of a curve, which have poles of order at most m occurring only when $\boldsymbol{u} \in \Theta^{[g-1]}$. The Riemann-Roch theorem for Abelian varieties gives the dimension of this space as m^g.

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The n-index $\wp$-functions belong to $\Gamma(n)$. In each case there are

$$\frac{(g+n-1)!}{n!(g-1)!},$$

of these, so further classes are needed.

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The basis for $\Gamma(2)$

The simplest case is $\Gamma(2)$ since there can be no Abelian function with a pole of order 1, and an entire Abelian function must be a constant.

- $\boldsymbol{g = 1}$ **:** Dim$=2$ and space generated by $\{1, \wp\}$.
- $\boldsymbol{g = 2}$ **:** Dim$=4$ and space generated by $\{1, \wp_{11}, \wp_{12}, \wp_{22}\}$.

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When $\boldsymbol{g = 3}$ there are 6 $\wp_{ij}$ but the space has dimension 8. The form of the final basis function depends on the curve:

- $\boldsymbol{(2,7)}$**-case:** Use the function

$$\Delta = \wp_{11}\wp_{33} - \wp_{12}\wp_{23} - \wp_{13}^2 + \wp_{13}\wp_{22}.$$

- $\boldsymbol{(3,4)}$**-case:** Use the function $Q = \wp_{1333} - 6\wp_{13}\wp_{33}$.

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The Q-functions

Definition

Hirota's bilinear operator is defined as $\delta_i = \partial/\partial u_i - \partial/\partial v_i$.
It is then simple to check that

$$\wp_{ij}(\boldsymbol{u}) = -\frac{1}{2\sigma(\boldsymbol{u})^2}\delta_i\delta_j\sigma(\boldsymbol{u})\sigma(\boldsymbol{v}) \Big|_{v=u} \qquad i \leq j \in \{1,\dots,g\}.$$

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We extend this to define the *n*-index Q-functions (for n even).

$$Q_{i_1,i_2,\dots,i_n}(\boldsymbol{u}) = \frac{(-1)}{2\sigma(\boldsymbol{u})^2}\delta_{i_1}\delta_{i_2}\dots\delta_{i_n}\sigma(\boldsymbol{u})\sigma(\boldsymbol{v}) \Big|_{v=u}$$

$$i_1 \leq \dots \leq i_n \in \{1,\dots,g\}.$$

The Q-functions II

- Apply the definition with n odd and it collapses to zero.
- The Q-functions can be expressed using polynomials of $\wp$-functions. For example,

$$Q_{ijk\ell} = \wp_{ijk\ell} - 2\wp_{ij}\wp_{k\ell} - 2\wp_{ik}\wp_{j\ell} - 2\wp_{i\ell}\wp_{jk}.$$

- They are all Abelian functions with poles of order no more than two. Hence they belong to the space $\Gamma(2)$.

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The Q-functions provide an inexhaustible supply of functions for $\Gamma(2)$, allowing the derivation of this basis for all curves, (subject to computational restrictions). To find which to include we test for linear independence using the σ-expansion.

Hyperelliptic Δ-functions

It is possible to use a Q-function instead of Δ in the basis for the $(2,7)$-case. But Δ is beneficial as it allows for the theory to be complectly realised using only 2 and 3-index $\wp$-functions.

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- Similar function can be found in other hyperelliptic cases. E.g. in the $(2,9)$-case $\Gamma(2)$ is spanned by

$$\{1, \wp_{11}, \wp_{12}, \dots, \wp_{44}, \Delta_1, \Delta_2, \dots \Delta_5\}$$

  where each Δ_i is a quadratic polynomial in $\wp_{ij}$.
- It has been explicitly checked that no such functions exist in a variety of non-hyperelliptic cases.
- The Δ-functions appear to be a feature unique to hyperelliptic cases.

The basis for $\Gamma(n)$ with $n > 2$

To derive a basis for $\Gamma(n)$ start with the following steps:

- Include all the entries from $\Gamma(n-1)$. This leaves only entries with poles of order n to be identified.
- Include derivatives of the entries from $\Gamma(n-1)$. Note: these may not all be linearly independent.

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$\boldsymbol{g = 1}$ **:** here $\Gamma(3)$ has dimension 3 and is spanned by $\{1, \wp, \wp'\}$.

But in general, the derivatives of existing functions will not be sufficient to complete the next basis.

As an example, we will consider $\Gamma(3)$ in the genus 3 cases, which has dimension 27.

The basis for Γ(3) in the (3,4)-case

In EEMOP (2007) the authors derived a basis for Γ(3). They used a new class of functions, $\wp^{[ij]}$ defined as the (i,j)-minor of the matrix

$$\left[\wp_{ij}\right]_{3\times 3} = \left[\begin{array}{ccc} \wp_{11} & \wp_{12} & \wp_{13} \\ \wp_{12} & \wp_{22} & \wp_{23} \\ \wp_{13} & \wp_{23} & \wp_{33} \end{array}\right].$$

These are all the difference between two products of 2-index $\wp$-functions. For example, $\wp^{[12]} = \wp_{12}\wp_{33} - \wp_{23}\wp_{13}$.

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The basis is given by

$$\{1, \wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}, Q_{1333}, \wp_{111}, \wp_{112}, \wp_{113},$$
$$\wp_{122}, \wp_{123}, \wp_{133}, \wp_{222}, \wp_{223}, \wp_{233}, \wp_{333}, \partial_1 Q_{1333}, \partial_2 Q_{1333},$$
$$\partial_3 Q_{1333}, \wp^{[11]}, \wp^{[12]}, \wp^{[13]}, \wp^{[22]}, \wp^{[23]}, \wp^{[33]}.\}$$

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The basis for Γ(3) in the (2,7)-case I

In the (2,7)-case the functions $\wp^{[13]}$ and $\wp^{[22]}$ are linearly dependent and hence only one may be included in the basis. To complete the basis a new type of function is needed.

This problem was considered by Nakayashiki (2008) who derived properties of the missing element.

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$$\begin{aligned} T &= \wp_{222}^2 - 4\wp_{22}^3 - Q_{2222}\wp_{22} \\ &= \wp_{222}^2 + 2\wp_{22}^3 - \wp_{22}\wp_{2222}. \end{aligned}$$

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The basis for Γ(3) in the (2,7)-case II

The function T is given by $\mathcal{T}_{222222}$ where,

$$
\begin{aligned}
\mathcal{T}_{ijklmn} = {} & \wp_{ijk}\wp_{lmn} - \tfrac{2}{3}\wp_{ij}\wp_{kl}\wp_{mn} - \tfrac{2}{3}\wp_{ij}\wp_{km}\wp_{ln} - \tfrac{2}{3}\wp_{ij}\wp_{kn}\wp_{lm} \\
& - \tfrac{2}{3}\wp_{ik}\wp_{jl}\wp_{mn} - \tfrac{2}{3}\wp_{ik}\wp_{jm}\wp_{ln} - \tfrac{2}{3}\wp_{ik}\wp_{jn}\wp_{lm} - \tfrac{2}{3}\wp_{il}\wp_{jk}\wp_{mn} \\
& + \tfrac{1}{3}\wp_{il}\wp_{jm}\wp_{kn} + \tfrac{1}{3}\wp_{il}\wp_{jn}\wp_{km} - \tfrac{2}{3}\wp_{im}\wp_{jk}\wp_{ln} + \tfrac{1}{3}\wp_{im}\wp_{jl}\wp_{kn} \\
& + \tfrac{1}{3}\wp_{im}\wp_{jn}\wp_{kl} - \tfrac{2}{3}\wp_{in}\wp_{jk}\wp_{lm} + \tfrac{1}{3}\wp_{in}\wp_{jl}\wp_{km} + \tfrac{1}{3}\wp_{in}\wp_{jm}\wp_{kl} \\
& - \tfrac{2}{3}Q_{ijkl}\wp_{mn} - \tfrac{2}{3}Q_{ijkm}\wp_{ln} - \tfrac{2}{3}Q_{ijkn}\wp_{lm} + \tfrac{1}{3}Q_{ijlm}\wp_{kn} + \tfrac{1}{3}Q_{ijln}\wp_{km} \\
& + \tfrac{1}{3}Q_{ijmn}\wp_{kl} + \tfrac{1}{3}Q_{iklm}\wp_{jn} + \tfrac{1}{3}Q_{ikln}\wp_{jm} + \tfrac{1}{3}Q_{ikmn}\wp_{jl} - \tfrac{2}{3}Q_{ilmn}\wp_{jk} \\
& + \tfrac{1}{3}Q_{jklm}\wp_{in} + \tfrac{1}{3}Q_{jkln}\wp_{im} + \tfrac{1}{3}Q_{jkmn}\wp_{il} - \tfrac{2}{3}Q_{jlmn}\wp_{ik} - \tfrac{2}{3}Q_{klmn}\wp_{ij}.
\end{aligned}
$$

These belong to Γ(3), for any (n, s)-curve.

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Deriving new classes of functions

- The $\mathcal{T}$-functions were derived during a separate calculation designed to cancel the higher order poles in $\wp_{ijk}\wp_{lmn}$ **for any curve.**

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Deriving new classes of functions

- The $\mathcal{T}$-functions were derived during a separate calculation designed to cancel the higher order poles in $\wp_{ijk}\wp_{lmn}$ **for any curve**.
- This was achieved by considering arbitrary sums of functions and determining the coefficients so that the higher order poles vanish upon substitution for the definition in $\sigma(\boldsymbol{u})$.
- Similar approaches can be applied to other combinations of functions. We work systematically, considering terms with increasing numbers of indices.

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New bases

This approach has led to the derivation of many other new bases. For example:

- The basis for $\Gamma(4)$ in the (2,7)-case.
- The basis for $\Gamma(4)$ in the (3,4)-case.
- The basis for $\Gamma(3)$ in the (2,9)-case.
- The basis for $\Gamma(3)$ in the (3,5)-case.

In each case the bases were completed using functions from a general class derived using the above approach.

This approach can be applied to higher genus cases, but will be restricted by computational limits.

Notes on computation

Computing the new classes of functions is (relatively) easy.
Testing which functions are actually needed can be difficult.

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Testing which functions are actually needed can be difficult.
Use weight properties to simplify computations.

- Only use basis entries at relevant weight.
- Use cyclic σ-expansion.
- Only use sufficient σ-expansion for weight.
- When multiplying series, only perform those products which will be under the maximal weight considered.

Custom written programs used to expand the product of series.

Notes on computation

Computing the new classes of functions is (relatively) easy.
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Custom written programs used to expand the product of series.
• Can determine weight range for each basis. Minimal weight function seems to be necessary.

Outline

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Two-term two-variable addition formula I

Theorem

Every (n, s)-curve has an associated addition formula

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma(\boldsymbol{u})^2\sigma(\boldsymbol{v})^2} = \sum_i c_i A_i(\boldsymbol{u}) B_i(\boldsymbol{v})$$

where $A_i, B_i \in \Gamma(2)$ and the c_i are constants.

Follows from linear algebra after checking the LHS is Abelian.

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Follows from linear algebra after checking the LHS is Abelian.
The RHS is symmetric or anti-symmetric in $(\boldsymbol{u},\boldsymbol{v})$, when the σ-function is odd or even respectively.
These generalise the classic Weierstrass formula,

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(v) - \wp(u).$$

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Two-term two-variable addition formula II

Example: In the (3,4)-case

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma(\boldsymbol{u})^2\sigma(\boldsymbol{v})^2} = -\wp_{11}(\boldsymbol{u}) + \wp_{12}(\boldsymbol{v})\wp_{23}(\boldsymbol{u}) + \wp_{13}(\boldsymbol{v})\wp_{22}(\boldsymbol{u}) + \wp_{11}(\boldsymbol{v}) - \wp_{12}(\boldsymbol{u})\wp_{23}(\boldsymbol{v}) - \wp_{13}(\boldsymbol{u})\wp_{22}(\boldsymbol{v}) + \frac{1}{3}Q_{1333}(\boldsymbol{u})\wp_{33}(\boldsymbol{v}) - \frac{1}{3}Q_{1333}(\boldsymbol{v})\wp_{33}(\boldsymbol{u}).$$

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Automorphism addition formulae I

For the cyclic curves,

$$y^n = x^s + \lambda_{s-1}x^{s-1} + \dots + \lambda_1 x + \lambda_0$$

there are a second class of addition formulae, associated with the family of automorphisms

$$[\zeta^j] : (x, y) \to (x, \zeta^j y), \qquad \text{where } \zeta^n = 1.$$

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$$[\zeta^j] : (x, y) \to (x, \zeta^j y), \qquad \text{where } \zeta^n = 1.$$

In each case the following function should be Abelian:

$$\prod_{j=1}^{n} \frac{\sigma\left(\sum_{i=1}^{n} [\zeta^{i+j}] \boldsymbol{u}^{[i]}\right)}{\sigma((\boldsymbol{u}^{[j]})^n)}$$

Hence it is expressible as a sum of terms, each a product of n functions drawn from the basis $\Gamma(n)$, but each a function of a different variable, $\boldsymbol{u}^{[j]}$.

Automorphism addition formulae II

Example: In the cyclic (3,4)-case we have $\zeta^3 = 1$ and

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})\sigma(\boldsymbol{u}+[\zeta]\boldsymbol{v}+[\zeta^2]\boldsymbol{w})\sigma(\boldsymbol{u}+[\zeta^2]\boldsymbol{v}+[\zeta]\boldsymbol{w})}{\sigma(\boldsymbol{u})^3\sigma(\boldsymbol{v})^3\sigma(\boldsymbol{w})^3}$$
$$= f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) + f(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) + f(\boldsymbol{v},\boldsymbol{u},\boldsymbol{w}) + f(\boldsymbol{v},\boldsymbol{w},\boldsymbol{u}) + f(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) + f(\boldsymbol{w},\boldsymbol{v},\boldsymbol{u}),$$

where

$$f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \tfrac{1}{8}\wp^{[22]}(\boldsymbol{u})\wp^{[11]}(\boldsymbol{v})\wp^{[22]}(\boldsymbol{w}) + \dots$$

(Full formula available on arXiv.)

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Simplified automorphism addition formulae

Such formulae can be difficult to compute. A simplified version may be found instead, where one of the variables is set to zero.

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Example: For example, in the cyclic (3,5)-case we have

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}+[\zeta]\boldsymbol{v})\sigma(\boldsymbol{u}+[\zeta^2]\boldsymbol{v})}{\sigma(\boldsymbol{u})^3\sigma(\boldsymbol{v})^3} = f(\boldsymbol{u},\boldsymbol{v}) + f(\boldsymbol{v},\boldsymbol{u})$$

where

$$f(\boldsymbol{u},\boldsymbol{v}) = -\tfrac{1}{8}\mathcal{T}_{222222}(\boldsymbol{v}) + \tfrac{1}{4}\wp_{122}(\boldsymbol{v})\wp_{144}(\boldsymbol{u}) + \dots$$

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Automorphism addition formulae of reduced curves

We can consider reduced curves which have further automorphisms and hence extra addition formulae.

Example: The restricted (3,4)-curve, $y^3 = x^4 + \lambda_0$ has automorphisms

$$[i^j] : (x, y) \mapsto ((i)^j x, y), \qquad \text{where i is the complex variable.}$$

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The functions associated to this curve satisfy

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}+[i]\boldsymbol{v})\sigma(\boldsymbol{u}+[i^2]\boldsymbol{v})\sigma(\boldsymbol{u}+[i^3]\boldsymbol{v})}{\sigma(\boldsymbol{u})^4\sigma(\boldsymbol{v})^4} = f(\boldsymbol{u},\boldsymbol{v}) - f(\boldsymbol{v},\boldsymbol{u})$$

where $\quad f(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{6}\wp_{2222}(\boldsymbol{v})\lambda_0 - \frac{1}{6}\wp_{1111}(\boldsymbol{u}) + \dots$

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Differential Equations I

The differential equations satisfied by $\wp$-functions are of great interest. We consider several classes:

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- **4-index Equations:** We seek to express the 4-index $\wp$-functions as quadratic polynomials in 2-index $\wp$-functions, to generalise the elliptic equation

$$\wp''(u) = 6\wp(u)^2 - \frac{1}{2}g_2.$$

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  Consider $\Gamma(2)$: Express those 4-index Q-functions not in the basis as linear combination of entries.
  - Gives desired set for the hyperelliptic cases.
  - Best available set for non-hyperelliptic cases.

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Differential Equations II

- **Bilinear Equations:** This is a set of equations bilinear in 2 and 3-index $\wp$-functions. For example, in (2,7)-case:

$$
\begin{aligned}
0 &= \wp_{233}\wp_{33} + \wp_{223} - \wp_{333}\wp_{23} - \wp_{133}\\
0 &= \wp_{133}\wp_{33} + \wp_{123} - \wp_{333}\wp_{13}\\
&\vdots
\end{aligned}
$$

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  - No analogue in elliptic case.
  - Due to parity properties terms are either $\wp_{ijk}$ or $\wp_{ij}\wp_{klm}$.
  - Derive by cross-differentiating 4-index relations. E.g. $\frac{\partial}{\partial u_2}\big(\wp_{3333}\big) - \frac{\partial}{\partial u_3}\big(\wp_{2333}\big) = 0.$

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  - Derive by cross-differentiating 4-index relations. E.g.
    $$\frac{\partial}{\partial u_2}\left(\wp_{3333}\right) - \frac{\partial}{\partial u_3}\left(\wp_{2333}\right) = 0.$$
  - Derive when finding odd entries of $\Gamma(3)$: Use the class of functions where higher order poles of $\wp_{ij}\wp_{klm}$-terms cancel.
  - Useful for manipulating and deriving equations.

Differential Equations III

- **Quadratic 3-index Equations:** We seek to express the products of 3-index $\wp$-functions as cubic polynomials in 2-index $\wp$-functions, to generalise the elliptic equation

$$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3,$$

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Search for cubic relations in the entries of $\Gamma(2)$. Higher order poles can be canceled by comparing coefficients. Complete sets recently derived for both genus 3 cases. For example, in the (2,7)-case:

$$\wp_{333}^2 = 4\big(\wp_{33}^3 + \lambda_4 + \lambda_5\wp_{33} + \lambda_6\wp_{33}^2 - \wp_{13} + \wp_{22} + \wp_{33}\wp_{23}\big)$$
$$\wp_{233}\wp_{333} = 2\big(2\wp_{23}\wp_{33}^2 + \lambda_3 + \lambda_5\wp_{23} + 2\lambda_6\wp_{33}\wp_{23} + \wp_{12} + 2\wp_{33}\wp_{13} - \wp_{33}\wp_{22} + \wp_{23}^2\big)$$

$$\vdots$$

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Differential Equations IV

In the (2,7)-case, the quadratic 3-index equations can be represented using the following determinantal expression:

$$\left(\boldsymbol{l}^T A \boldsymbol{k}\right)\left(\boldsymbol{l}'^T A \boldsymbol{k}\right) = -\frac{1}{4}\begin{vmatrix} H & \boldsymbol{l}' & \boldsymbol{k}' \\ \boldsymbol{l}^T & 0 & 0 \\ \boldsymbol{k}^T & 0 & 0 \end{vmatrix},$$

where $\boldsymbol{l}, \boldsymbol{k}, \boldsymbol{l}', \boldsymbol{k}'$ are arbitrary vectors, A a 5×5 matrix of $\wp_{ijk}$ and H a 5×5 matrix of $\wp_{ij}$ and curve constants.

$$A = \begin{bmatrix} 0 & -\wp_{333} & \cdots \\ \wp_{333} & 0 & \cdots \\ -\wp_{233} & \wp_{133} & \cdots \\ \vdots & \vdots & \end{bmatrix}, \quad H = \begin{bmatrix} 4\lambda_0 & 2\lambda_1 & \cdots \\ 2\lambda_1 & 4\lambda_2 + 4\wp_{11} & \cdots \\ -2\wp_{11} & 2\lambda_3 + 2\wp_{12} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

Further Reading

- Buchstaber, Enolski, Leykin
  Kleinian functions, hyperelliptic Jacobians & applications.
  Reviews in Math. and Math. Physics, 1997, 10:1-125
- Eilbeck, Enolski, Matsutani, Ônishi and Previato
  Abelian Functions For Trigonal Curves Of Genus Three.
  Intl. Math. Research Notices, 2007, Art.ID: rnm140
- England, Eilbeck and Ônishi.
  Abelian functions associated with genus 3 curves.
  Preprint, arXiv:1008.0289v1 (2010).

Further Information

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