Geodesic motion in cosmic string space-times

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“The higher-genus sigma function and applications”
Workshop ICMS Edinburgh
13th October 2010
Collaborations and References

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References:


Project funded by the German Research Foundation (DFG)
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6. Summary and Outlook
Cosmic strings

- Cosmic strings form when axial symmetry gets spontaneously broken during phase transitions in the early universe
- line-like defects
  (compare to vortices in superfluids)
- energy per unit length

\[ m_{(3)} \sim T_c^2 \]

\( T_c \): temperature of phase transition
- can be as heavy as \( m_{(3)} \approx 10^{12} \text{kg/m} \)
Fundamental strings (of String theory)

- Fundamental (F-) strings ...
  - have zero width
  - have tension close to the Planck scale
  - end on D-branes
  - D1-brane = D-string

Connection between cosmic strings and fundamental strings ???

- **NO**: perturbative strings as cosmic strings ruled out
  
  (Witten, 1985)
**YES**: cosmic strings are formed in **inflationary models** originating from string theory

- D-, F- and bound states of \( p \) F-strings and \( q \) D-strings (\( p-q \)-strings) are formed in **brane inflation**
  (Jones, Stoica, Tye (2002); Sarangi, Tye (2002))
... and also: Hybrid inflation (Linde (1994))

- two scalar fields
- inflation ends due to spontaneous symmetry breaking
- cosmic strings form generically at the end of hybrid inflation in Supersymmetric Grand Unified Theories (Jeannerot, Rocher, Sakellariadou (2003))
Detection of cosmic strings

- Cosmic Microwave background data can’t be explained by cosmic strings only....

... but maybe important contribution
(e.g. Bouchet, Peter, Riazuelo, Sakellariadou (2002))
Detection of cosmic strings

- Gravitational lensing

important to understand geodesic motion of massive and massless test particles
The Geodesic equation

\[ \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0 \]

\( \Gamma^\mu_{\rho\sigma} \) Christoffel symbol:

\[ \Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left( \partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma} \right) \]

\( \tau \): affine parameter (proper time for time-like geodesics)

\( g_{\mu\nu} \): metric tensor
Two approaches when describing cosmic string space–times

1. **macroscopic description**: Nambu-Goto action → infinitely thin strings
   - **Advantages**: simple to treat; analytic results possible
   - **Disadvantages**: no connection to underlying field theory

2. **microscopic description**: field theoretical models → finite core width
   - **Advantages**: “proper” description
   - **Disadvantages**: solutions only available *numerically*
The Geodesic equation

Geodesics in analytically given space-times
Geodesics in numerically given space-times
Summary and Outlook

Schwarzschild black hole pierced by cosmic string
Kerr black hole pierced by cosmic string

Schwarzschild black hole pierced by cosmic string

Ansatz for the metric in spherical coordinates \((t, r, \theta, \varphi)\)

\[
d s^2 = - \left(1 - \frac{2M}{r}\right) d t^2 + \left(1 - \frac{2M}{r}\right)^{-1} d r^2 + r^2 \left(\, d \theta^2 + \beta^2 \sin^2 \theta \, d \varphi^2 \right)
\]

\[M_{\text{phys}} = \beta M\] physical mass of black hole

\[\delta = 2\pi(1 - \beta) = 8\pi G m_{(3)} \sim 8\pi(\eta/M_{\text{Pl}})^2: \text{deficit angle}\]

\[m_{(3)}: \text{energy per unit length of the string}\]

\[\eta: \text{symmetry breaking scale at which string forms}\]

\[M_{\text{Pl}} = G^{-1/2}: \text{Planck mass}\]
Schwarzschild black holes pierced by cosmic strings

Static black hole pierced by infinitely thin cosmic string

$\beta < 1$: cosmic string is long-range *hair*

$r = 2M$: event horizon
Symmetries

- **Globally** axially symmetric
- **Locally** four Killing vectors

\[
\xi = \frac{\partial}{\partial t}
\]

\[
\chi(1) = \sin(\beta \varphi) \frac{\partial}{\partial \theta} + \frac{1}{\beta} \cos(\beta \varphi) \cot \theta \frac{\partial}{\partial \varphi}
\]

\[
\chi(2) = -\cos(\beta \varphi) \frac{\partial}{\partial \theta} + \frac{1}{\beta} \sin(\beta \varphi) \cot \theta \frac{\partial}{\partial \varphi}
\]

\[
\chi(3) = \frac{1}{\beta} \frac{\partial}{\partial \varphi}
\]
**Constants of motion**

- **Energy** $E$

  \[ \xi^\mu \frac{dx^\nu}{d\tau} g_{\mu\nu} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} =: E \]

- **Angular momenta** $L_3$ and $L^2$

  \[ \chi^{\mu(i)} \frac{dx^\nu}{d\tau} g_{\mu\nu} =: L_i, \quad i = 1, 2, 3 \]

  with

  \[ L_3 = r^2 \beta \sin^2 \theta \frac{d\varphi}{d\tau} \]

  \[ |\vec{L}|^2 \equiv L^2 = L_1^2 + L_2^2 + L_3^2 = r^4 \left(\frac{d\theta}{d\tau}\right)^2 + \frac{L_3^2}{\sin^2 \theta} \]

  and $L_1$ and $L_2$ are trivial

  - \[ \varepsilon = \frac{ds^2}{d\tau^2} = \begin{cases} 
  -1 & \text{for massive test particles} \\
  0 & \text{for massless test particles} 
\end{cases} \]
Components of Geodesic equation

\[
\left( \frac{dt}{d\tau} \right)^2 = E^2 \left( 1 - \frac{2M}{r} \right)^{-2}
\]

\[
\left( \frac{dr}{d\tau} \right)^2 = E^2 - \left( \frac{L^2}{r^2} + \epsilon \right) \left( 1 - \frac{2M}{r} \right)
\]

\[
\left( \frac{d\theta}{d\tau} \right)^2 = \frac{L^2}{r^4} - \frac{L_3^2}{r^4 \sin^2 \theta}
\]

\[
\left( \frac{d\varphi}{d\tau} \right)^2 = \frac{L_3^2}{\beta^2 r^4 \sin^4 \theta}
\]
Angular motion $\theta(\varphi)$

From the $\theta$ and $\varphi$-component

$$\cot^2 \theta = (k^2 - 1) \sin^2 (\beta \varphi) \quad , \quad k^2 = \frac{L^2}{L_3^2}$$

Turning points of $\theta$

$$\frac{d\theta}{d\tau} = 0 \Rightarrow \sin^2 \theta = \frac{1}{k^2} \Rightarrow \beta \varphi = \frac{\pi}{2} + n\pi \quad , \quad n = \pm 0, \pm 1, ...$$

For $\beta \neq 1$:
Geodesic motion in precessing plane with $\vec{L}$ as normal
$\Rightarrow$ Geodesics with $\theta \neq \frac{\pi}{2}$ are not flat
Radial motion $r(\theta)$ and $r(\varphi)$

New coordinate $z = \frac{2M}{r} - \frac{1}{3}$, $z \in [-\frac{1}{3}, \infty)$

\[
\frac{dz}{\sqrt{P(z)}} = \frac{1}{2} \left( 1 - \frac{1}{k^2 \sin^2 \theta} \right)^{-1/2} d\theta
\]

\[
\frac{dz}{\sqrt{P(z)}} = \frac{1}{2} \frac{\beta k}{\beta} \frac{1}{(k^2 - 1) \sin^2(\beta \varphi) + 1} d\varphi
\]

with

\[
P(z) = 4z^3 - 4 \left[ \frac{1}{3} - \left( \frac{4M^2}{L^2} \right) \varepsilon \right] z - 4 \left[ \frac{2}{27} + \frac{2}{3} \left( \frac{4M^2}{L^2} \right) \varepsilon - \frac{4G^2M^2}{L^2} \right] z - g_3
\]
Classification of solutions

- Need \( P(z) > 0 \) to have solutions \( \Rightarrow \) study roots of \( P(z) \)
- Discriminant \( D \) with

\[
D = g_2^3 - 27 g_3^2 \left\{ \begin{array}{ll}
> 0 & \text{three real roots } e_1 > e_2 > e_3 \\
< 0 & \text{one real root} \\
= 0 & \text{either one or two real roots}
\end{array} \right.
\]

\[
\frac{4M^2G^2}{L^2} \begin{array}{c}
\text{massive} \\
\text{massless}
\end{array}
\]

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- Geodesic motion in cosmic string space-times
Classification of solutions

- Effective potential

\[
\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + V_{\text{eff}}(r) = \frac{E^2 - \varepsilon}{2}
\]

with

\[
V_{\text{eff}}(r) = -\varepsilon \frac{M}{r} + \frac{L^2}{2r^2} - \frac{L^2 M}{r^3}
\]

- Turning points of \( r \) correspond to \( P(z) = 0 \)

\[
\frac{L^2}{32M^2} P(z(r)) = \frac{E^2 - \varepsilon}{2} - V_{\text{eff}}(r)
\]
Example for $D > 0$

Bound terminating orbit

Escape orbit

Effective potential

Characteristic polynomial

Schwarzschild black hole pierced by cosmic string
Kerr black hole pierced by cosmic string

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Geodesic motion in cosmic string space-times
Example for $D < 0$

Bound terminating orbit

Effective potential

Characteristic polynomial
Example for $D = 0$

Bound orbit
Bound terminating orbit

Effective potential
Characteristic polynomial

Schwarzschild black hole pierced by cosmic string
Kerr black hole pierced by cosmic string
Solutions to the geodesic equation

In terms of Weierstrass $\wp$-function

\[ r(\theta) = \frac{2M}{\wp(g(\theta) - c) + \frac{1}{3}} \]

\[ r(\varphi) = \frac{2M}{\wp(f(\varphi) - c) + \frac{1}{3}} \]

with

\[ g(\theta) \equiv \frac{1}{2} \left[ \arcsin \left( \frac{\cos \theta}{\sqrt{1 - k^{-2}}} \right) - \arcsin \left( \frac{\cos \theta_0}{\sqrt{1 - k^{-2}}} \right) \right] \]

\[ f(\varphi) \equiv -\frac{1}{2} \arctan \left[ k \tan(\beta(\varphi - \varphi_0)) \right] \]
Solutions to the geodesic equation

Value of constant $c$:

$$c = \int_{z_0}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$$

- $z_0 = \infty \Rightarrow c = 0$
- $z_0 = e_1 \Rightarrow c = \frac{K(\mathcal{K})}{\sqrt{e_1 - e_3}}$
- $z_0 = e_2 \Rightarrow c = \frac{K(\mathcal{K})}{\sqrt{e_1 - e_3}} + i \frac{K(\mathcal{K}')}{\sqrt{e_1 - e_3}}$
- $z_0 = e_3 \Rightarrow c = i \frac{K(\mathcal{K}')}{\sqrt{e_1 - e_3}}$

with $K$: complete elliptic integral of 1.kind with modulus

$$\mathcal{K} = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \quad \mathcal{K}' = \sqrt{1 - \mathcal{K}^2}$$
Example of geodesic: bound orbit
Example of geodesic: bound terminating orbit
Example of geodesic: escape orbit
Light deflection

For \( k = 1 \) and **massless** test particles deflection angle

\[
\Delta \varphi = \frac{1}{\beta} \left[ \frac{4}{\sqrt{e_1 - e_3}} \int_0^{\varphi^c} \frac{d\varphi}{\sqrt{1 - \kappa^2 \sin^2(\varphi)}} + 2\omega_1 \right] + \pi \left( \frac{1}{\beta} - 1 \right)
\]

\( \omega_1 \): first half period

Observational constraints with \((\Delta \varphi)_S\) Schwarzschild value

\[
\frac{\Delta \varphi - (\Delta \varphi)_S}{(\Delta \varphi)_S} \lesssim 10^{-5}
\]

\[\Rightarrow (1 - \beta) \lesssim 10^{-11} \Rightarrow m_{(3)} \lesssim 10^{16} \text{kg/m}\]
Perihelion shift

For $k = 1$ and massive test particles perihelion shift

$$\Delta \varphi = \frac{4}{\beta} \frac{K(C)}{\sqrt{e_1 - e_3}} - 2\pi$$

Observational constraints with $(\Delta \varphi)_S$ Schwarzschild value

$$\frac{\Delta \varphi - (\Delta \varphi)_S}{(\Delta \varphi)_S} \lesssim 10^{-4}$$

$$\Rightarrow (1 - \beta) \lesssim 10^{-10} \Rightarrow m_3 \lesssim 10^{17} \frac{\text{kg}}{\text{m}}$$
Kerr black hole pierced by cosmic string

Ansatz for the metric in Boyer-Lindquist coordinates \((t, r, \theta, \varphi)\)

\[
ds^2 = - \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \beta^2 \left( r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\varphi^2 - \beta \frac{4Mra \sin^2 \theta}{\rho^2} dtd\varphi
\]

with

\[
\rho^2 = r^2 + a^2 \cos^2 \theta , \quad \Delta = r^2 - 2Mr + a^2
\]

\(a = J/M\): angular momentum \(J\) per mass \(M\)
\(\delta = 2\pi (1 - \beta) = 8\pi Gm(3) \sim 8\pi (\eta/M_{Pl})^2\): deficit angle
\(m(3)\): energy per unit length of the string
\(\eta\): symmetry breaking scale at which string forms
\(M_{Pl}\): Planck mass
**Kerr black hole pierced by cosmic string**

Rotating black hole pierced by infinitely thin cosmic string

\[ \beta < 0: \text{cosmic string is long-range hair} \]
\[ r_+ = M + \sqrt{M^2 - a^2}: \text{event horizon, } r_- = M - \sqrt{M^2 - a^2}: \text{Cauchy horizon} \]
\[ 2Mr = \rho^2: \text{static limit} \]
Boyer-Lindquist coordinates

Relation to cartesian coordinates

\[
\begin{align*}
x &= \sqrt{r^2 + a^2 \sin \theta \cos(\beta \varphi)} \\
y &= \sqrt{r^2 + a^2 \sin \theta \sin(\beta \varphi)} \\
z &= r \cos \theta
\end{align*}
\]

- \( r = 0 \Rightarrow \text{disc with deficit angle } \delta = 2\pi(1 - \beta) \)
- \emph{Physical singularity} at \( r = 0, \theta = \pi/2 \)
  \( \Rightarrow \text{ring with deficit angle } \delta = 2\pi(1 - \beta) \)
- \( r < 0: \text{another conical space-time without horizons} \)
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Geodesics in numerically given space-times

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Schwarzschild black hole pierced by cosmic string

Kerr black hole pierced by cosmic string

Constants of motion

• Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$ ⇒ conserved quantities

$$\left(1 - \frac{2Mr}{\rho^2}\right) \frac{dt}{d\tau} + \beta \frac{2Mar}{\rho^2} \sin^2 \theta \frac{d\varphi}{d\tau} =: E$$

$$-\frac{2Mar}{\rho^2} \sin^2 \theta \frac{dt}{d\tau} + \beta \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\rho^2} \sin^2 \theta \frac{d\varphi}{d\tau} =: L_z$$

• Carter constant $K$: separability of Hamilton-Jacobi equations

•

$$\varepsilon = \frac{ds^2}{d\tau^2} = \left\{ \begin{array}{ll} -1 & \text{massive particles} \\ 0 & \text{massless particles} \end{array} \right.$$
Hamilton-Jacobi equations

\[
\frac{\partial S}{\partial \tau} = \frac{1}{2} g^{\mu\nu} (\partial_\mu S) (\partial_\nu S)
\]

\( S \): Hamilton function with Ansatz

\[
S = \frac{1}{2} \varepsilon \tau - Et + \beta L_z \varphi + S_r(r) + S_\theta(\theta)
\]

\( S_r(r) \): function of \( r \) only

\( S_\theta(\theta) \): function of \( \theta \) only
Introduce **Mino time** $d\lambda = \frac{d\tau}{\rho^2}$

\[
\frac{dr}{d\lambda} = \pm \sqrt{R(r)}
\]

\[
\frac{d\theta}{d\lambda} = \pm \sqrt{\Theta(\theta)}
\]

\[
\frac{d\varphi}{d\lambda} = \frac{1}{\beta} \left( \frac{L_z \csc^2 \theta - aE}{\sqrt{\Theta(\theta)}} \frac{d\theta}{d\lambda} + \frac{aP(r)}{\Delta(r) \sqrt{R(r)}} \frac{dr}{d\lambda} \right)
\]

\[
\frac{dt}{d\lambda} = \frac{a(L_z - aE \sin^2 \theta)}{\sqrt{\Theta(\theta)}} \frac{d\theta}{d\lambda} + \frac{(r^2 + a^2)P(r)}{\Delta(r) \sqrt{R(r)}} \frac{dr}{d\lambda}
\]

with

\[
\Theta(\theta) = K - (L_z - aE)^2 - \cos^2 \theta \left( L_z^2 \csc^2 \theta - a^2 (E^2 + \varepsilon) \right)
\]

\[
P(r) = E(r^2 + a^2) - L_z a
\]

\[
R(r) = P(r)^2 - \Delta(r) \left( K - \varepsilon r^2 \right)
\]
$r(\lambda)$ motion

- Need $R(r) > 0$ to have solutions with $R(r) = 0$ turning points of $r(\lambda)$ motion
- $R(r) = 0 \Rightarrow$ (a) 4 real, (b) 2 real & 2 complex, (c) 4 complex roots
- new coordinate
  
  $$z = \frac{1}{c_2} \left( \frac{1}{r - r_1} - c_1 \right)$$

  where $r_1$ largest real root and $c_1$ and $c_2$ depend on $K, L_z, E, \varepsilon$

  $$\Rightarrow d\lambda = \frac{dr}{\sqrt{R(r)}} = -\frac{dz}{\sqrt{4z^3 - g_2z - g_3}}$$

  where $g_2$ and $g_3$ depend on $K, L_z, E, \varepsilon$
Solution in terms of Weierstrass $\wp$-function

\[
    r(\lambda) = \left( \frac{1}{c_2 \wp \left( \frac{1}{c_2} (\lambda - \lambda_0) + C; g_2, g_3 \right) + c_1} + r_1 \right)
\]

with

\[
    C = \int_{z_0}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}
\]

- $z_0 = \infty \Rightarrow C = 0$
- $z_0 = e_1 \Rightarrow C = K(\mathcal{K})/\sqrt{e_1 - e_3}$
- $z_0 = e_2 \Rightarrow C = K(\mathcal{K})/\sqrt{e_1 - e_3} + iK(\mathcal{K}')/\sqrt{e_1 - e_3}$
\[ \theta(\lambda) \] motion

- Need \( \Theta(\theta) > 0 \) to have solutions with \( \Theta(\theta) = 0 \) turning points of \( \theta(\lambda) \) motion
- new coordinate
  \[ \tilde{z} = \frac{1}{c_3} \left( \sec^2 \theta - c_4 \right) \]
  where \( c_3 \) and \( c_4 \) depend on \( K, L_z, E, \varepsilon \)

\[ \Rightarrow d\lambda = \frac{d\theta}{\sqrt{\Theta(\theta)}} = \frac{d\tilde{z}}{\sqrt{4\tilde{z}^3 - \tilde{g}_2 \tilde{z} - \tilde{g}_3}} \]

where \( \tilde{g}_2 \) and \( \tilde{g}_3 \) depend on \( K, L_z, E, \varepsilon \)
\( \theta(\lambda) \) motion

- Discriminant \( D = \tilde{g}_2^3 - 27\tilde{g}_3^2 > 0 \Rightarrow 3 \) real roots \( \tilde{e}_i, i = 1, 2, 3 \)
- BUT: roots might not fulfill \( (c_3\tilde{z} + c_4)^{-1} = \cos^2 \theta \leq 1 \)
- AND: for one root \( \Rightarrow \) two values of \( \theta \) with \( \cos \theta = \pm (c_3\tilde{z} + c_4)^{-1/2} \)
- \( \Theta(\theta) = 0 \Rightarrow \) (a) 4 real, (b) 2 real, (c) no real roots

Allowed values of \( \theta \)

4 roots

2 roots

no roots
θ(λ) motion

Solution in terms of Weierstrass \( \wp \)-function

\[
\theta(\lambda) = \arccos \left[ \pm \frac{1}{\sqrt{c_3 \wp \left( \frac{1}{c_3} (\lambda - \lambda_0) + \tilde{C}; \tilde{g}_2, \tilde{g}_3 \right) + c_4}} \right]
\]

with

\[
\tilde{C} = \int_{\tilde{z}_0}^{\infty} \frac{d\tilde{z}}{\sqrt{4\tilde{z}^3 - \tilde{g}_2 \tilde{z} - \tilde{g}_3}}
\]

\[
\tilde{z}_0 = \tilde{e}_1 \Rightarrow \tilde{C} = K(K) / \sqrt{\tilde{e}_1 - \tilde{e}_3}
\]

\[
\tilde{z}_0 = \tilde{e}_3 \Rightarrow \tilde{C} = iK(K') / \sqrt{\tilde{e}_1 - \tilde{e}_3}
\]
Rewrite

\[ \beta d\varphi = \frac{L_z \csc^2 \theta - aE}{\sqrt{\Theta(\theta)}} d\theta + \frac{a\Delta^{-1}P(r)}{\sqrt{R(r)}} dr =: dl_\theta + dl_r \]

\[ dt = a(L_z - aE \sin^2 \theta) \frac{d\theta}{\sqrt{\Theta(\theta)}} + (r^2 + a^2)\Delta(r)^{-1}P(r) \frac{dr}{\sqrt{R(r)}} =: d\tilde{l}_\theta + d\tilde{l}_r \]

solutions for \( l_\theta, l_r, \tilde{l}_\theta, \tilde{l}_r \) in terms of Weierstrass \( \zeta \)- and \( \sigma \)-functions
Example: Solution for $I_\theta$

With new coordinate $\tilde{z} = (\sec^2 \theta - c_4) / c_3$:

$$dl_\theta = \frac{c_3(L_z - aE)d\tilde{z}}{\sqrt{4\tilde{z}^3 - \tilde{g}_2\tilde{z} - \tilde{g}_3}} + \frac{L_z(c_4 + c_3 c_5)d\tilde{z}}{(\tilde{z} - c_5)\sqrt{4\tilde{z}^3 - \tilde{g}_2\tilde{z} - \tilde{g}_3}}, \quad c_5 = (1 - c_4)/c_3$$

Introducing

$$x := \int_{\infty}^{\tilde{z}} \frac{d\tilde{z}}{\sqrt{4\tilde{z}^3 - \tilde{g}_2\tilde{z} - \tilde{g}_3}} \Rightarrow \tilde{z} = \varphi(x; \tilde{g}_2, \tilde{g}_3)$$

we have

$$dl_\theta = c_3(L_z - aE)dx + (c_4 + c_3 c_5)\frac{dx}{\varphi(x) - c_5}$$

with $x = \lambda / c_3$ we find

$$I_\theta = (L_z - aE)\lambda + \sum_{i=1}^{2} \frac{c_4 + c_3 c_5}{\varphi'(x_i)} \left[ \frac{\lambda}{c_3} \zeta(x_i) + \ln (\sigma(x - x_i)) \right]$$

where $\varphi(x_i) = c_5$, $i = 1, 2$
Example of geodesic: bound orbit

- Schwarzschild black hole pierced by cosmic string
- Kerr black hole pierced by cosmic string

\[ \beta = 1 \]
\[ \beta = 0.78 \]

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Geodesic motion in cosmic string space-times
Example of geodesic: escape orbit
Lense-Thirring effect

- Frame dragging effect of rotating massive body
- LAGEOS satellites:
  \[ \Omega_{LT}(\beta = 1) \approx 39 \cdot 10^{-3} \text{ arcseconds/year} \text{ (10\% accuracy)} \]
- If cosmic string present, i.e. \( \beta \neq 1 \):
  \[ \Omega_{LT}(\beta \neq 1) - \Omega_{LT}(\beta = 1) \leq 4 \cdot 10^{-3} \text{ arcseconds/year} \]
- Bound on energy per unit length \( m_{(3)} \) of cosmic string
  \[ \frac{1}{\beta} - 1 \lesssim 10^{-11} \Rightarrow m_{(3)} \lesssim 10^{16} \text{ kg/m} \]
Abelian-Higgs strings

$U(1)$ Abelian-Higgs model minimally coupled to gravity:

$$S = \int d^4 x \sqrt{-g} \left( \frac{R}{16\pi G} + \mathcal{L} \right)$$

with matter Lagrangian

$$\mathcal{L} = D_\mu \phi (D^\mu \phi)^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\phi \phi^* - \eta^2)^2$$

with

$$D_\mu \phi = \nabla_\mu \phi - ieA_\mu \phi, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$\phi$: complex scalar field

$A_\mu$: $U(1)$ gauge field

$e$: gauge coupling

$\lambda$: self-interaction coupling

$\eta \neq 0$: vacuum expectation value
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Ansatz for static, straight strings

- Matter fields \((\text{Nielsen} \& \text{Olesen, 1973})\)

\[
\phi(\rho, \varphi) = \eta h(\rho)e^{in \varphi}, \quad A_\mu dx^\mu = \frac{1}{e} (n - P(\rho)) \, d\varphi
\]

\(n: \) degree of map \(S^1 \rightarrow S^1\), homotopy group \(\pi_1(S^1) = \mathbb{Z}\)

- Metric

\[
ds^2 = N^2(\rho) dt^2 - d\rho^2 - L^2(\rho) d\varphi^2 - N^2(\rho) dz^2
\]

Four non-linear coupled 2nd order ordinary differential equations in \(h, P, N\) and \(L\) \(\Rightarrow\) have to be solved \textit{numerically}
Equations

\[
\frac{(N^2 L h')'}{N^2 L} = \frac{P^2 h}{L^2} + \frac{\alpha}{2} h(h^2 - 1)
\]

\[
\frac{L}{N^2} \left( \frac{N^2 P'}{L} \right)' = 2h^2 P
\]

\[
\frac{(LNN')'}{N^2 L} = \gamma \left[ \frac{(P')^2}{2L^2} - \frac{\alpha}{4} \left( h^2 - 1 \right)^2 \right]
\]

\[
\frac{(N^2 L')'}{N^2 L} = -\gamma \left[ \frac{2h^2 P^2}{L^2} + \frac{(P')^2}{2L^2} + \frac{\alpha}{4} \left( h^2 - 1 \right)^2 \right].
\]

with

\[
\gamma = 8\pi G\eta^2 = 8\pi \frac{\eta^2}{M_{\text{Pl}}^2}, \quad \alpha = \frac{\lambda}{e^2} = \frac{M_H^2}{M_W^2}
\]

\[
M_H = \sqrt{2\lambda \eta} \text{ Higgs boson mass}
\]

\[
M_W = \sqrt{2\epsilon \eta} \text{ gauge boson mass}
\]
Boundary conditions

- Regularity at the origin
  
  \[ h(0) = 0, \quad P(0) = n, \quad N(0) = 1, \]
  \[ N'(0) = 0, \quad L(0) = 0, \quad L'(0) = 1 \]

- Finiteness of energy
  
  \[ h(\infty) = 1, \quad P(\infty) = 0 \]
Properties of Abelian–Higgs strings

- magnetic field $\vec{B} = B_z \vec{e}_z$ and quantized magnetic flux:
  \[ B_z = -\frac{1}{e} \frac{dP/d\rho}{\rho}, \quad \Phi_M = -\frac{2\pi n}{e} \]

- scalar core width $\sim (\text{Higgs mass})^{-1} = M_H^{-1} = (\sqrt{2\lambda \eta})^{-1}$

- width of flux tubes
  $\sim (\text{gauge boson mass})^{-1} = M_W^{-1} = (\sqrt{2e\eta})^{-1}$

- $M_H = M_W$: saturate energy bound $m(3) = 2\pi \eta^2 n$
  $\Rightarrow$ BPS limit, but no analytic solutions
Geodesics: Constants of motion

\[ E := N^2 \frac{dt}{d\tau} \quad \text{energy} \]

\[ L_z := L^2 \frac{d\varphi}{d\tau} \quad \text{angular momentum} \]

\[ p_z := N^2 \frac{dz}{d\tau} \quad \text{momentum} \]

\[ \varepsilon = \frac{ds^2}{d\tau^2} = \begin{cases} 1 & \text{for massive test particles} \\ 0 & \text{for massless test particles} \end{cases} \]
The Geodesic equation

\[ \frac{1}{2} \left( \frac{d\rho}{d\tau} \right)^2 = \tilde{E} - V_{\text{eff}}(\rho) \]

with

\[ \tilde{E} = (E^2 - \varepsilon) \]

and

\[ V_{\text{eff}}(\rho) = \frac{1}{2} \left[ E^2 \left( 1 - \frac{1}{N^2} \right) + \frac{p_z^2}{N^2} + \frac{L_z^2}{L^2} \right] \]

\( V_{\text{eff}} \): effective potential
Massive particles: Effective potential

- infinite potential barrier for $L_z \neq 0$
- no bound orbits for $\alpha \geq 2$
Massive particles: Example of bound orbit

\(\gamma = 0.36\)

\(\gamma = 0.42\)
Massive particles: Example of escape orbit

\[ \gamma = 0.15 \]

\[ \gamma = 0.45 \]
Perihelion shift

For planar motion ($\rho_z = 0$):

$$\Delta \varphi = 2 \int_{\rho_{\text{min}}}^{\rho_{\text{max}}} \frac{L_z d\rho}{L(\rho)^2 \left( \frac{E^2}{N(\rho)^2} - \frac{L_z^2}{L(\rho)^2} - 1 \right)^{1/2} - 2\pi}$$
Massless particles: Effective potential

- infinite potential barrier for $L_z \neq 0$
- no bound orbits

![Graphs showing effective potential for different values of E.](image-url)
Massless particles: no bound orbits

Compare to Gibbons, 1993:

*In a general cosmic string space-time with topology* $\mathbb{R}^2 \times \Sigma$ *where* $\Sigma$ *has positive Gaussian curvature a massless test particle must move on a geodesic that escapes to infinity in both directions.*
Massless particles in $x$-$y$-plane

$$dt^2 = \frac{1}{N^2} d\rho^2 + \frac{L^2}{N^2} d\varphi^2 = \tilde{g}_{ij} dx^i dx^j, \quad i = 1, 2$$

$\tilde{g}_{ij}$ optical metric of manifold $\Sigma$ with Gaussian curvature

$$K = \frac{L'}{L} N' N - \frac{L''}{L} N^2 - (N')^2 + NN''$$

$\delta > 2\pi$

$\delta < 2\pi, K > 0$ for all $\rho$

$\delta < 2\pi, K < 0$ for some $\rho$
Massless particles: Example of escape orbit

\[ \gamma = 0.3 \]

\[ \gamma = 0.61 \]
Light deflection

For planar motion ($\rho_z = 0$):

$$\Delta \varphi = 2 \int_{\rho_{\text{min}}}^{\infty} \frac{L_z d\rho}{L(\rho)^2 \left( \frac{E^2}{N(\rho)^2} - \frac{L_z^2}{L(\rho)^2} \right)^{1/2}} - \pi$$
Summary

- Link between cosmic strings ↔ fundamental strings
- possible observation ...
  - ... in the Cosmic Microwave background (Power- and Polarization spectrum)
  - ... through motion of test particles in cosmic string space-times
- in view of this ...
  - ... found the complete set of solutions to the geodesic equation in space-time of Schwarzschild- and Kerr black hole pierced by infinitely thin cosmic string
  - ... found solutions to the geodesic equation in the space-time of an Abelian-Higgs string
Applications

- computation of gravitational wave templates for extreme mass ratio inspirals
- gravitational lensing
- test particle motion in solar system if sun is not perfectly spherically symmetric
- possible explanation of the observed alignment of polarization vectors of quasars on cosmological scales via remnants of cosmic string decay
Introduction
The Geodesic equation
Geodesics in analytically given space-times
Geodesics in numerically given space-times
Summary and Outlook
Summary and Outlook

Outlook

Work in progress...

- ... solutions to the geodesic equation in other numerically given space-times (semilocal, $p$-$q$-strings, superconducting...)

- ... solutions to the geodesic equation in space-time with cosmic string and (positive or negative) cosmological constant \(\Rightarrow\) hyperelliptic integrals

\(\Rightarrow\) compare Talks by C. Lämmerzahl and V. Kagramanova

Betti Hartmann

Geodesic motion in cosmic string space-times