Hyperelliptic curve of arbitrary genus in geodesic equations of higher dimensional space-times

Valeria Kagramanova[1]

V. Z. Enolski[3], E. Hackmann[2], J. Kunz[1], C. Lämmerzahl[2]

[1] Oldenburg University, Germany

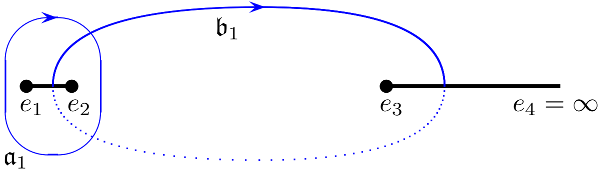
[2] ZARM, Bremen University, Germany

[3] Institute of Magnetism, Kiev

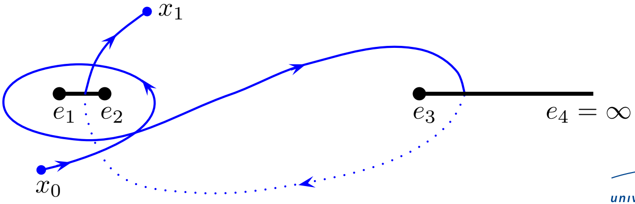
The higher-genus sigma functions and applications, Edinburgh 2010

$$t + 2n\omega + 2m\omega' = \int_\infty^x \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} \ ,$$

where $x = \wp(t) = \wp(t + 2n\omega + 2m\omega')$ is an elliptic function. **Inversion possible!**



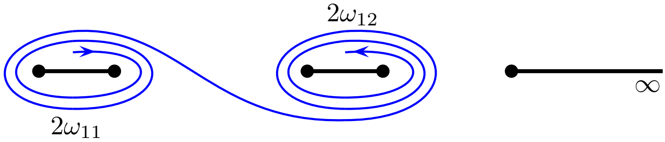
Homology basis on the Riemann surface of the curve $y^2 = \prod_{i=1}^4 (x - e_i)$ with real branch points $e_1 < e_2 < \ldots < e_4 = \infty$ (upper sheet). The cuts are drawn from e_{2i-1} to e_{2i}, $i = 1, 2$. The b–cycles are completed on the lower sheet (dotted lines).



Elliptic function does not depend on the way of integration!

Inversion of hyperelliptic integral P_n, $n > 4$:

does not work. Reason: infinitely small periods appear



For hyperelliptic curve of genus 2 a combination of periods is possible such that $2\omega_{11}n + 2\omega_{12}m \propto 0$.

Jacobi: $2g$-periodic functions of one complex variable do not exist for $g > 1$.
Jacobi's solution for $g = 2$, $y^2 = \prod_{i=1}^{5}(x - a_i)$:
correct formulation of inversion problem for genus 2

$$\int_{x_0}^{x_1} \frac{dx}{y} + \int_{x_0}^{x_2} \frac{dx}{y} = u_1 \,, \quad \int_{x_0}^{x_1} \frac{xdx}{y} + \int_{x_0}^{x_2} \frac{xdx}{y} = u_2 \,,$$

with holomorphic diferentials

$$2\omega = \left(\oint_{\mathfrak{a}_k} \mathrm{d}u_i \right)_{i,k=1,\ldots,g} \qquad 2\omega' = \left(\oint_{\mathfrak{b}_k} \mathrm{d}u_i \right)_{i,k=1,\ldots,g}$$

Inversion of hyperelliptic integral P_n, $n > 4$:

Only symmetric functions of upper bounds (x_1, x_2) make sence (exchange of x_1 and x_2 changes nothing)

$$x_1 + x_2 = F(u_1, u_2)$$
$$x_1 x_2 = G(u_1, u_2) \ ,$$

with $F(\vec{u} + 2n_1\vec{\omega}_1 + 2n_2\vec{\omega}_2 + 2m_1\vec{\omega}'_1 + 2m_2\vec{\omega}'_2) = F(\vec{u})$ where F is a 4-periodic Abelian function (function of g complex variables with $2g$ periods being the columns of the period matrix).

Applications in physics

The goal 1 is using the theory of Abelian functions and Jacobi inversion problem to describe the multivalued functions which appear in the inversion of a hyperelliptic integral. That will be achieved by restriction of the θ-divisor in the Jacobi variety.

Motion of neutral or charged test particles in

- **spherically symmetric space-times:**
  - Schwarzschild space-time: mass
  - Schwarzschild-de Sitter: mass, cosmological constant
  - Reissner-Nordström space-time: mass, electric and magnetic charges
  - Reissner-Nordström-de Sitter space-time: mass, electric and magnetic charges, cosmological constant
- **axial symmetric space-times**
  - Taub-NUT space-time: mass (gravitoelectric charge), NUT parameter (gravitomagnetic charge)
  - Kerr space-time: mass, rotation (Kerr) parametter
  - Myers-Perry space-times (higher dimensional Kerr space-times): mass, rotation parameters
  - Plebański and Demiański space-time: mass, electric and magnetic charges, rotation parameter, NUT parameter, cosmological constant

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Physical applications in tables

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Reissner–Nordström–de Sitter	**+**	+	*	**+**	*	*	*	*	*

+ integration by elliptic functions

+ integration by hyperelliptic functions

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Necessary calculations

The goal 2 is to provide effective calculation of hyperelliptic functions using maple routines (package alcurves).

- calculation of the matrix of periods of holomorphic differentials
- calculation of the matrix of periods of meromorphic differentials
- calculation of characteristics of abelian images of branch points in a given basis

$$\mathfrak{A}_k = \int_{\infty}^{e_k} \mathrm{d}\boldsymbol{u} = \omega\boldsymbol{\varepsilon}_k + \omega'\boldsymbol{\varepsilon}'_k, \quad k = 1, \ldots, 2g+2\,,$$

- calculation of the vector of Riemann constant in a given basis

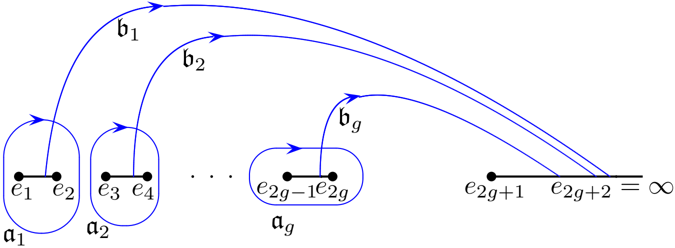
Hyperelliptic functions

Hyperelliptic curve X_g of genus g is given by the equation

$$w^2 = P_{2g+1}(z) = \sum_{i=0}^{2g+1} \lambda_i z^i = 4 \prod_{k=1}^{2g+1} (z - e_k) \; .$$

Equip the curve with a canonical homology basis

$$(\mathfrak{a}_1, \ldots, \mathfrak{a}_g; \mathfrak{b}_1, \ldots, \mathfrak{b}_g), \qquad \mathfrak{a}_\mathfrak{i} \circ \mathfrak{b}_\mathfrak{j} = -\mathfrak{b}_\mathfrak{i} \circ \mathfrak{a}_\mathfrak{j} = \delta_{i,j}, \; \mathfrak{a}_\mathfrak{i} \circ \mathfrak{a}_\mathfrak{j} = \mathfrak{b}_\mathfrak{i} \circ \mathfrak{b}_\mathfrak{j} = 0$$



A homology basis on a Riemann surface of the hyperelliptic curve of genus g with real branch points $e_1, \ldots, e_{2g+2} = \infty$ (upper sheet). The cuts are drawn from e_{2i-1} to e_{2i} for $i = 1, \ldots, g+1$. The $\mathfrak{b}$-cycles are completed on the lower sheet (the picture on lower sheet is just flipped horizontally).

Canonical differentials

Choose canonical holomorphic differentials (first kind) $\mathrm{d}\boldsymbol{u}^t = (\mathrm{d}u_1, \ldots, \mathrm{d}u_g)$ and associated meromorphic differentials (second kind) $\mathrm{d}\boldsymbol{r}^t = (\mathrm{d}r_1, \ldots, \mathrm{d}r_g)$ in such a way that their periods

$$2\omega = \left(\oint_{\mathfrak{a}_k} \mathrm{d}u_i \right)_{i,k=1,\ldots,g} \qquad 2\omega' = \left(\oint_{\mathfrak{b}_k} \mathrm{d}u_i \right)_{i,k=1,\ldots,g}$$

$$2\eta = \left(-\oint_{\mathfrak{a}_k} \mathrm{d}r_i \right)_{i,k=1,\ldots,g} \qquad 2\eta' = \left(-\oint_{\mathfrak{b}_k} \mathrm{d}r_i \right)_{i,k=1,\ldots,g}$$

satisfy the generalized Legendre relation

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^t = -\frac{1}{2}\pi \mathrm{i} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}.$$

Such a basis of differentials can be realized as follows (see Baker (1897), p. 195):

$$\mathrm{d}\boldsymbol{u}(z,w) = \frac{\boldsymbol{\mathcal{U}}(z)\mathrm{d}z}{w}, \quad \mathcal{U}_i(z) = x^{i-1}, \qquad i = 1\ldots,g,$$

$$\mathrm{d}\boldsymbol{r}(z,w) = \frac{\boldsymbol{\mathcal{R}}(z)\mathrm{d}z}{4w}, \quad \mathcal{R}_i(z) = \sum_{k=i}^{2g+1-i} (k+1-i)\lambda_{k+1+i} z^k, \qquad i = 1\ldots,g.$$

Jacobi variety $\mathrm{Jac}(X_g) = \mathbb{C}^g/2\omega \oplus 2\omega'$, $\widetilde{\mathrm{Jac}}(X_g) = \mathbb{C}^g/1_g \oplus \tau$.

θ-functions

The hyperelliptic θ–function, $\theta : \widetilde{\mathrm{Jac}}(X_g) \times \mathcal{H}_g \to \mathbb{C}^g$, with characteristics $[\varepsilon]$ is defined as the Fourier series

$$\theta[\varepsilon](\boldsymbol{v}|\tau) = \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp \pi \mathrm{i} \left\{ (\boldsymbol{m}+\boldsymbol{\varepsilon}')^t \tau (\boldsymbol{m}+\boldsymbol{\varepsilon}') + 2(\boldsymbol{v}+\boldsymbol{\varepsilon})^t(\boldsymbol{m}+\boldsymbol{\varepsilon}') \right\}$$

In the following, the values ε_k, ε'_k will either be 0 or $\frac{1}{2}$. The equation

$$\theta[\varepsilon](-\boldsymbol{v}|\tau) = \mathrm{e}^{-4\pi\mathrm{i}\boldsymbol{\varepsilon}^t\boldsymbol{\varepsilon}'}\theta[\varepsilon](\boldsymbol{v}|\tau),$$

implies that the function $\theta[\varepsilon](\boldsymbol{v}|\tau)$ with characteristics $[\varepsilon]$ of only half-integers is even if $4\boldsymbol{\varepsilon}^t\boldsymbol{\varepsilon}'$ is an even integer, and odd otherwise. Correspondingly, $[\varepsilon]$ is called even or odd, and among the 4^g half-integer characteristics there are $\frac{1}{2}(4^g + 2^g)$ even and $\frac{1}{2}(4^g - 2^g)$ odd characteristics.

Characteristics

Every abelian image of a branch point is given by its characteristic

$$\mathfrak{A}_k = \int_{\infty}^{e_k} \mathrm{d}\boldsymbol{u} = \omega\boldsymbol{\varepsilon}_k + \omega'\boldsymbol{\varepsilon}'_k, \quad k = 1, \ldots, 2g+2\,,$$

or

$$[\mathfrak{A}_i] = \left[\int_{\infty}^{e_i} \mathrm{d}\boldsymbol{u}\right] = \left[\begin{array}{c} \boldsymbol{\varepsilon}'^T_i \\ \boldsymbol{\varepsilon}_i \end{array}\right] = \left[\begin{array}{cc} \varepsilon'_{i,1} & \varepsilon'_{i,2} \\ \varepsilon_{i,1} & \varepsilon_{i,2} \end{array}\right],$$

The $2g+2$ characteristics $[\mathfrak{A}_i]$ serve as a basis for the construction of all 4^g possible half integer characteristics $[\varepsilon]$. There is a one-to-one correspondence between these $[\varepsilon]$ and partitions of the set $\mathcal{G} = \{1, \ldots, 2g+2\}$ of indices of the branch points (Fay (1973), p. 13, Baker (1897) p. 271).

Characteristics

The partitions of interest are

$$\mathcal{I}_m = \{i_1, \ldots, i_{g+1-2m}\}, \qquad \mathcal{J}_m = \{j_1, \ldots, j_{g+1+2m}\},$$

where m is any integer between 0 and $\left[\frac{g+1}{2}\right]$. The corresponding characteristic $[\varepsilon_m]$ is defined by the vector

$$\boldsymbol{E}_m = (2\omega)^{-1} \sum_{k=1}^{g+1-2m} \mathfrak{A}_{i_k} + \boldsymbol{K}_\infty =: \boldsymbol{\varepsilon}_m + \tau \boldsymbol{\varepsilon}'_m.$$

Characteristics with even m are even, and with odd m odd. There are $\frac{1}{2}\binom{2g+2}{g+1}$ different partitions with $m=0$, $\binom{2g+2}{g-1}$ different with $m=1$, and so on, down to $\binom{2g+2}{1} = 2g+2$ if g is even and $m = g/2$, or $\binom{2g+2}{0} = 1$ if g is odd and $m = (g+1)/2$. According to the Riemann theorem on the zeros of θ-functions, $\theta(\boldsymbol{E}_m + \boldsymbol{v})$ vanishes to order m at $\boldsymbol{v} = 0$.

The fundamental σ-function of the curve X_g is defined as

$$\sigma(\boldsymbol{u}) = C(\tau)\theta[\boldsymbol{K}_\infty]((2\omega)^{-1}\boldsymbol{u};\tau)\exp\left\{\boldsymbol{u}^T\varkappa\boldsymbol{u}\right\}.$$

Here $\tau = \omega^{-1}\omega'$, $\varkappa = \eta(2\omega)^{-1}$ and $C(\tau)$ is given by the formula

$$C(\tau) = \sqrt{\frac{\pi^g}{\det(2\omega)}}\left(\prod_{1\le i<j\le 2g+1}(e_i - e_j)\right)^{-1/4}.$$

That's natural generalization of the Weierstrass σ-function

$$\sigma(u) = \sqrt{\frac{\pi}{2\omega}}\frac{\epsilon}{\sqrt[4]{(e_i - e_2)(e_1 - e_3)(e_2 - e_3)}}\vartheta_1\left(\frac{u}{2\omega}\right)\exp\left\{\frac{\eta u^2}{2\omega}\right\}, \quad \epsilon^8 = 1.$$

Properties of sigma functions

- it is an entire function on $\mathrm{Jac}(X_g)$,
- it satisfies the two sets of functional equations

$$\begin{aligned} \sigma(\boldsymbol{u}+2\omega\boldsymbol{k}+2\omega'\boldsymbol{k};\mathfrak{M}) &= \exp\{2(\eta\boldsymbol{k}+\eta'\boldsymbol{k}')(\boldsymbol{u}+\omega\boldsymbol{k}+\omega'\boldsymbol{k}')\}\sigma(\boldsymbol{u};\mathfrak{M}) \\ \sigma(\boldsymbol{u};\gamma\mathfrak{M}) &= \sigma(\boldsymbol{u};\mathfrak{M}), \gamma\in \mathrm{Sp}(2g,\mathbb{Z}) \end{aligned}$$

the first of these equations displays the *periodicity property*, while the second one the *modular property*.

Here $\mathfrak{M}$-modules, i.e. matrices of periods 2ω, $2\omega'$, 2η, $2\eta'$.

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \det(\gamma)=1 , \quad A,B,C,D\in\mathbb{Z}^g$$

Action of γ on period matrix is defined as

$$\begin{aligned} \gamma\circ\omega &= A\omega + B\omega' \\ \gamma\circ\omega' &= C\omega + D\omega' \end{aligned}$$

Jacobi inversion problem in general case

Jacobi's inversion problem in coordinate notation is

$$\begin{aligned}
&\int_{P_0}^{P_1} \frac{\mathrm{d}x}{y} + \ldots + \int_{P_0}^{P_g} \frac{\mathrm{d}x}{y} = u_1 \,, \\
&\int_{P_0}^{P_1} \frac{x\mathrm{d}x}{y} + \ldots + \int_{P_0}^{P_g} \frac{x\mathrm{d}x}{y} = u_2 \,, \\
&\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\int_{P_0}^{P_1} \frac{x^{g-1}\mathrm{d}x}{y} + \ldots + \int_{P_0}^{P_g} \frac{x^{g-1}\mathrm{d}x}{y} = u_g \,,
\end{aligned}$$

and solved in terms of Kleinian $\wp$-functions as follows

$$\begin{aligned}
&x^g - \wp_{g,g}(\boldsymbol{u})x^{g-1} - \ldots - \wp_{g,1}(\boldsymbol{u}) = 0 \,, \\
&y_k = -\wp_{g,g,g}(\boldsymbol{u})x_k^{g-1} - \ldots - \wp_{g,g,1}(\boldsymbol{u}) \,,
\end{aligned}$$

where $P_k = (x_k, y_k)$.

Relation between the matrices of holomorphic and meromorphic differentials

Proposition

Let $\mathfrak{P}(\boldsymbol{\Omega})$ *denote* $g \times g$*- symmetric matrix whose elements are symmetric functions of* $(e_{i_1}, \ldots, e_{i_g})$

$$\mathfrak{P}(\boldsymbol{\Omega}) = (\wp_{i,j}(\boldsymbol{\Omega}))_{i,j=1,\ldots,g} \ , \boldsymbol{\Omega} = \int_{\infty}^{e_{i_1}} d\boldsymbol{u} + \ldots + \int_{\infty}^{e_{i_g}} d\boldsymbol{u} \, ,$$

let $(2\omega)^{-1}\boldsymbol{\Omega} + \boldsymbol{K}_\infty$ *be an arbitrary non-singular even half-period, and* $\mathfrak{T}(\boldsymbol{\Omega})$ *the* $g \times g$ *matrix*

$$\mathfrak{T}(\boldsymbol{\Omega}) = \left(-\frac{\partial^2}{\partial z_i \partial z_j} \log \theta[\boldsymbol{K}_\infty]((2\omega)^{-1}\boldsymbol{\Omega}; \tau) \right)_{i,j=1,\ldots,g} \, .$$

Then the $\varkappa$*-matrix is given by the formula*

$$\varkappa = -\frac{1}{2}\mathfrak{P}(\boldsymbol{\Omega}) - \frac{1}{2}(2\omega)^{-1\,T}\mathfrak{T}(\boldsymbol{\Omega})(2\omega)^{-1}$$

and the half-periods of the meromorphic differentials η *and* η' *are given as*

$$\eta = 2\varkappa\omega, \qquad \eta' = 2\varkappa\omega' - \frac{\mathrm{i}\pi}{2}(\omega^{-1})^T \, .$$

Relation between the matrices of holomorphic and meromorphic differentials

To calculate missing $\wp_{i,j}$ use the following differential cubic relation

$$\wp_{ggi}\wp_{ggk} = 4\wp_{gg}\wp_{gi}\wp_{gk} - 2(\wp_{gi}\wp_{g-1,k} + \wp_{gk}\wp_{g-1,i}) + 4(\wp_{gk}\wp_{g,i-1} + \wp_{gi}\wp_{g,k-1}) + 4\wp_{k-1,i-1} - 2(\wp_{k,i-2} + \wp_{i,k-2}) + \lambda_{2g}\wp_{gk}\wp_{gi} + \frac{\lambda_{2g-1}}{2}(\delta_{ig}\wp_{gk} + \delta_{kg}\wp_{gi}) + \lambda_{2i-2}\delta_{ik} + \frac{1}{2}(\lambda_{2i-1}\delta_{k,i+1} + \lambda_{2k-1}\delta_{i,k+1}) \ ,$$

with $\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

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Relation between the matrices of holomorphic and meromorphic differentials

The Proposition represents the natural generalization of the Weierstrass formulae, see e.g. the Weierstrass-Schwarz lectures, p. 44

$$2\eta\omega = -2e_1\omega^2 - \frac{1}{2}\frac{\vartheta_2''(0)}{\vartheta_2(0)}\,, \quad 2\eta\omega = -2e_2\omega^2 - \frac{1}{2}\frac{\vartheta_3''(0)}{\vartheta_3(0)}\,, \quad 2\eta\omega = -2e_3\omega^2 - \frac{1}{2}\frac{\vartheta_4''(0)}{\vartheta_4(0)}$$

Therefore the Proposition allows to reduce the variety of modules necessary for calculations of σ and $\wp$-functions to the first period matrix.

Strata of theta-divisor

The subset $\widetilde{\Theta}_k \subset \widetilde{\Theta}$ $k \geq 1$ is called k-th stratum if each point $\boldsymbol{v} \in \widetilde{\Theta}$ admits a parametrization

$$\widetilde{\Theta}_k : \quad \boldsymbol{v} = \sum_{j=1}^{k} \int_{\infty}^{P_j} \mathrm{d}\boldsymbol{v} + \boldsymbol{K}_{\infty}, \quad 0 < k < g\,.$$

Orders $m(\Theta_k)$ of vanishing of $\theta(\Theta_k + \boldsymbol{v})$ along stratum Θ_k for small genera are given in the Table

g	$m(\Theta_0)$	$m(\Theta_1)$	$m(\Theta_2)$	$m(\Theta_3)$	$m(\Theta_4)$	$m(\Theta_5)$	$m(\Theta_6)$
1	1	0	-	-	-	-	-
2	1	1	0	-	-	-	-
3	2	1	1	0	-	-	-
4	2	2	1	1	0	-	-
5	3	2	2	1	1	0	-
6	3	3	2	2	1	1	0

Orders $m(\Theta_k)$ of zeros $\theta(\Theta_k + \boldsymbol{v})$ at $\boldsymbol{v} = 0$ on strata Θ_k

Solution for genus 2

The Jacobi inversion problem can be reduced to the quadratic equation

$$x^2 - \wp_{22} x - \wp_{12} = 0$$

with the solution

$$x_1 + x_2 = \wp_{22}$$
$$x_1 x_2 = -\wp_{12}$$

Now choose $x_2 = \infty$: $x_1 = -\lim_{x_2 \to \infty} \frac{\wp_{12}}{\wp_{22}}$. We take away one point and this allows us to use the Riemann theorem $\theta\left(\sum_{k=1}^{N} \int_{P_0}^{P_k} \frac{dx}{\sqrt{P(x)}} + K_\infty\right) \equiv 0$ if $N < g$. K_∞ is a Riemann constant.

With $\wp_{ij}(\vec{u}) = -\frac{\partial^2 \ln \sigma(\vec{u})}{\partial \vec{u}_i \partial \vec{u}_j}$, $i, j = 1, \ldots, g$ the final solution is

$$x_1 = -\frac{\sigma_{12}}{\sigma_{22}}.$$

This is **Grant-Jorgenson formula**.

Solution for genus 2

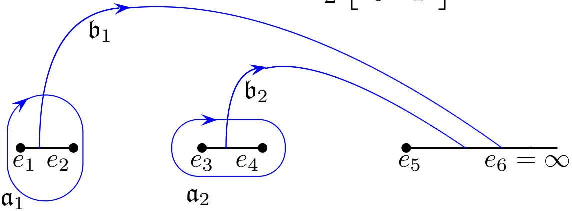
In the homology basis with $e_6 = +\infty$ the characteristics are:

$$[\mathfrak{A}_1] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [\mathfrak{A}_2] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad [\mathfrak{A}_3] = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$[\mathfrak{A}_4] = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad [\mathfrak{A}_5] = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad [\mathfrak{A}_6] = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

and the characteristic of the vector of Riemann constants $\boldsymbol{K}_\infty$ is

$$[\boldsymbol{K}_\infty] = [\mathfrak{A}_2] + [\mathfrak{A}_4] = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$



A homology basis on a Riemann surface of the hyperelliptic curve of genus 2 with real branch points $e_1, \ldots, e_6 = \infty$ (upper sheet). The cuts are drawn from e_{2i-1} to e_{2i} for $i = 1, \ldots, 3$. The $\mathfrak{b}$-cycles are completed on the lower sheet (the picture on lower sheet is just flipped horizontally).

The expression for the matrix $\varkappa$ is

$$\varkappa = -\frac{1}{2}\begin{pmatrix} e_1e_2(e_3+e_4+e_5)+e_3e_4e_5 & -e_1e_2 \\ -e_1e_2 & e_1+e_2 \end{pmatrix} - \frac{1}{2}(2\omega)^{-1\,T}\mathfrak{T}(\mathbf{\Omega_{1,2}})(2\omega)^{-1},$$

where $\mathfrak{T}$ is the 2×2-matrix and 10 half-periods for $i \neq j = 1,\ldots,5$ that are images of two branch points are

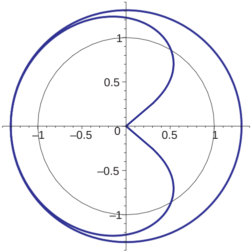
$$\mathbf{\Omega}_{i,j} = \omega(\boldsymbol{\varepsilon}_i + \boldsymbol{\varepsilon}_j) + \omega'(\boldsymbol{\varepsilon}'_i + \boldsymbol{\varepsilon}'_j), \quad i = 1,\ldots,6\,.$$

and the characteristics of the 10 half-periods

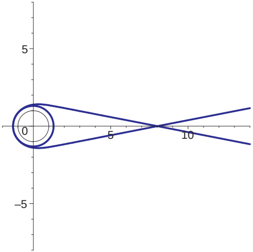
$$[\varepsilon_{i,j}] = \left[(2\omega)^{-1}\mathbf{\Omega}_{i,j} + \boldsymbol{K}_\infty\right], \quad 1 \le i < j \le 5$$

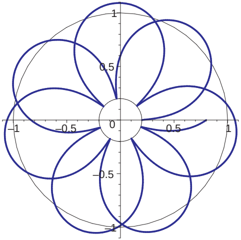
are non-singular and even

Examples 2D

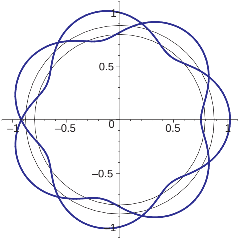


(1) Schwarzschild-de Sitter, 9D

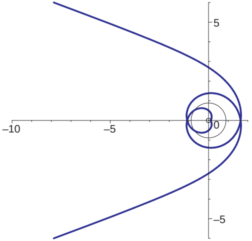




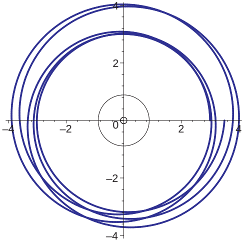
(2) Reissner-Nordström, 7D



(3) Reissner-Nordström, 7D

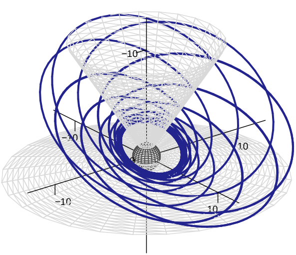


(4) Reissner-Nordström-de Sitter, 4D

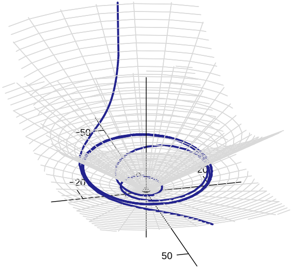


(5) Reissner-Nordström-de Sitter, 4D

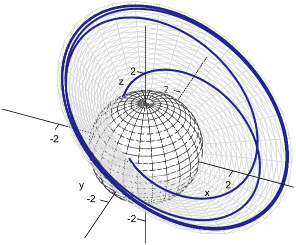
Examples 3D



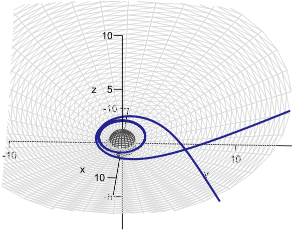
NUT-de Sitter bound orbit



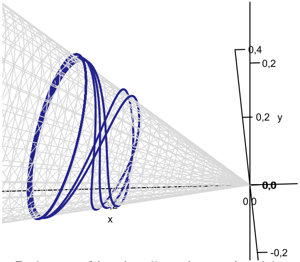
NUT-de Sitter, escape orbit



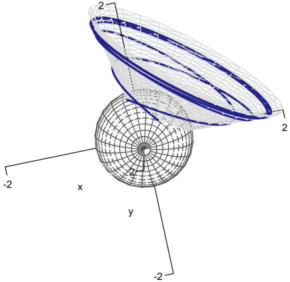
NUT, crossover bound orbit



NUT, escape orbit



Reissner-Nordström, bound orbit



and many-world bound orbit

Solution for genus 3

Solution in this case is (**Onishi formula**)

$$x_1 = -\frac{\sigma_{13}}{\sigma_{23}}\Bigg|_{\sigma(\vec{u})=0,\sigma_3(\vec{u})=0}$$

Characteristics for genus 3

Let $\mathfrak{A}_k$ be the Abelian image of the k-th branch point, namely

$$\mathfrak{A}_k = \int_\infty^{e_k} \mathrm{d}\boldsymbol{u} = \omega\boldsymbol{\varepsilon}_k + \omega'\boldsymbol{\varepsilon}'_k, \quad k = 1, \ldots, 8\,,$$

where $\boldsymbol{\varepsilon}_k$ and $\boldsymbol{\varepsilon}'_k$ are column vectors whose entries $\varepsilon_{k,j}$, $\varepsilon'_{k,j}$, are 1 or zero for all $k = 1, \ldots, 8$, $j = 1, 2, 3$.

The correspondence between branch points and characteristics in the fixed homology basis is given as

$$[\mathfrak{A}_1] = \frac{1}{2}\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], [\mathfrak{A}_2] = \frac{1}{2}\left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right], [\mathfrak{A}_3] = \frac{1}{2}\left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right],$$

$$[\mathfrak{A}_4] = \frac{1}{2}\left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right], \quad [\mathfrak{A}_5] = \frac{1}{2}\left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right], \quad [\mathfrak{A}_6] = \frac{1}{2}\left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}\right],$$

$$[\mathfrak{A}_7] = \frac{1}{2}\left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}\right], \quad [\mathfrak{A}_8] = \frac{1}{2}\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Solution for genus 3

The vector of Riemann constant $\boldsymbol{K}_\infty$ with the base point at infinity is given by the sum of even characteristics,

$$[\boldsymbol{K}_\infty] = [\mathfrak{A}_2] + [\mathfrak{A}_4] + [\mathfrak{A}_6] = \frac{1}{2}\left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right].$$

From the above characteristics 64 half-periods can be build:

- 7 odd $[(2\omega)^{-1}\boldsymbol{\Omega}_i + \boldsymbol{K}_\infty]$, where $\boldsymbol{\Omega}_i = \mathfrak{A}_i$
- 21 odd $[(2\omega)^{-1}\boldsymbol{\Omega}_{i,j} + \boldsymbol{K}_\infty]$, where $\boldsymbol{\Omega}_{i,j} = \mathfrak{A}_i + \mathfrak{A}_j$
- 36 even $[(2\omega)^{-1}\boldsymbol{\Omega}_{i,j,k} + \boldsymbol{K}_\infty]$, where $\boldsymbol{\Omega}_{i,j,k} = \mathfrak{A}_i + \mathfrak{A}_j + \mathfrak{A}_k$ and $\boldsymbol{K}_\infty$

where $1 \leq i < j < k \leq 7$ and $\boldsymbol{K}_\infty$ is singular characteristic ($\theta(\boldsymbol{K}_\infty) = 0$).

Analog of Thomae formula: all period systems

For the branch points $e_1, \dots, e_8$ the following formulae are valid

$$e_i = -\frac{\sigma_{13}(\boldsymbol{\Omega}_i)}{\sigma_{23}(\boldsymbol{\Omega}_i)}, i = 1, \dots, 8, \quad \text{where } \boldsymbol{\Omega}_i \in \Theta_1 : \sigma(\boldsymbol{\Omega}_i) = 0, \sigma_3(\boldsymbol{\Omega}_i) = 0$$

For the branch points $e_1, \dots, e_8$ the following set of formulas is valid

$$\begin{aligned} e_i + e_j &= -\frac{\sigma_2(\boldsymbol{\Omega}_{i,j})}{\sigma_3(\boldsymbol{\Omega}_{i,j})}, \\ e_i e_j &= \frac{\sigma_1(\boldsymbol{\Omega}_{i,j})}{\sigma_3(\boldsymbol{\Omega}_{i,j})} \end{aligned} \qquad i \neq j = 1, \dots, 8$$

where $\boldsymbol{\Omega}_{i,j} \in \Theta_2$: $\sigma(\boldsymbol{\Omega}_{i,j}) = 0$.
From the solution of the Jacobi inversion problem follows for any $i \neq j = 1 \dots, 3$

$$e_i + e_j + e_k = \wp_{33}(\boldsymbol{\Omega}_{i,j,k}), \quad -e_i e_j - e_i e_k - e_j e_k = \wp_{23}(\boldsymbol{\Omega}_{i,j,k}), \quad e_i e_j e_k = \wp_{13}(\boldsymbol{\Omega}_{i,j,k})$$

Solution for arbitrary genus

Solution is (**Matsutani, Previato**)

$$x_1 = -\left.\frac{\frac{\partial^{M+1}}{\partial u_1 \partial u_g^M}\sigma(\vec{u})}{\frac{\partial^{M+1}}{\partial u_2 \partial u_g^M}\sigma(\vec{u})}\right|_{\vec{u}\in\Theta_1}, \quad M = \frac{(g-2)(g-3)}{2} + 1$$

with $\boldsymbol{u} = (u_1, \ldots, u_g)^T$ and

$$\Theta_1 : \quad \sigma(\boldsymbol{u}) = 0, \qquad \frac{\partial^j}{\partial u_g^j}\sigma(\boldsymbol{u}) = 0, \quad j = 1, \ldots, g-2\,.$$

Remark: the half-periods associated to branch points $e_1, \ldots, e_{2g+1}$ are elements of the first stratum,

$$\boldsymbol{\Omega}_i = \int_{e_{2g+2}}^{(e_i,0)} \mathrm{d}\boldsymbol{u} \quad \in \Theta_1;\ e_i \neq e_{2g+2}$$

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Solution for arbitrary genus

Proposition

Let $\boldsymbol{\Omega}_i$ be the half-period that is the Abelian image with the base point $P_0 = (\infty, \infty)$ of a branch point e_i. Then

$$e_i = -\frac{\frac{\partial^{M+1}}{\partial u_1 \partial u_g^M}\sigma(\boldsymbol{\Omega}_i)}{\frac{\partial^{M+1}}{\partial u_2 \partial u_g^M}\sigma(\boldsymbol{\Omega}_i)}, \qquad M = \frac{(g-2)(g-3)}{2} + 1\,.$$

In the case of genus $g = 2$ such a representation of branch points, which is equivalent to the Thomae formulas, was mentioned by Bolza

$$e_i = -\frac{\sigma_1(\boldsymbol{\Omega}_i)}{\sigma_2(\boldsymbol{\Omega}_i)}\,.$$

Similar formulas can be written on other strata Θ_k.

Solution for arbitrary genus

Proposition

Let X_g be a hyperelliptic curve of genus g and consider a partition

$$\mathcal{I}_1 \cup \mathcal{J}_1 = \{i_1, \ldots, i_{g-1}\} \cup \{j_1, \ldots, j_{g+2}\}$$

of branch points such that the half-periods

$$(2\omega)^{-1}\boldsymbol{\Omega}_{\mathcal{I}_1} + \boldsymbol{K}_\infty \in \Theta_{g-1} \cup \Theta_{g-2}$$

are non-singular odd half-periods. Denote by $s_k(\mathcal{I}_1)$ the elementary symmetric function of order k built by the branch points $e_{i_1}, \ldots, e_{i_{g-1}}$. Then the following formula are valid

$$s_k(\mathcal{I}_1) = (-1)^{k+1} \frac{\sigma_{g-k}(\boldsymbol{\Omega}_{\mathcal{I}_1})}{\sigma_g(\boldsymbol{\Omega}_{\mathcal{I}_1})}\,.$$

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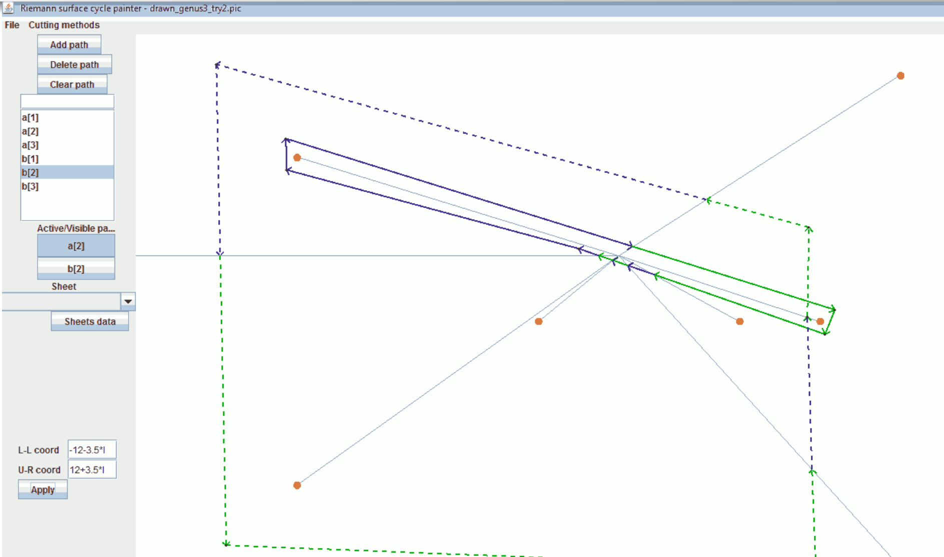
Possibility I: Tim Northover's routine



aim: calculate the transition matrix from the period matrix
in Tretkoff basis to the period matrix in the basis of your choice

Tim Northover's routine

```
> with(LinearAlgebra):
> march('open',"D:/My Maple/CyclePainter/extcurves.mla");
> with(extcurves);
> f:=y^2-4*(mul( x-zeros[i], i=1..2*g+1 ))); curve := Record('f'=f,
'x'=x, 'y'=y):
> hom:=all_extpaths_from_homology(curve):
> PI:=periodmatrix(curve,hom);
> A:=PI[1..g,1..g]; B:=PI[1..g,g+1..2*g]; tau:=A^(-1).B;
> curve, homDrawn, names := read_pic("D:/My Maple/CyclePainter/drawn.pic"):
> T1:=from_algcurves_homology(curve, homDrawn);
> tau_basis:=PI.Transpose(T1);
> A_basis:=tau_basis[1..g,1..g]; B_basis:=tau_basis[1..g,g+1..2*g];
```

Possibility II: Correspondence between branch points and half-periods in Tretkoff basis

Step 1. For the given curve compute first period of matrices $(2\omega, 2\omega')$ and $\tau = \omega^{-1}\omega'$ by means of Maple/algcurves code. Compute then winding vectors, i.e. columns of the inverse matrix

$$(2\omega)^{-1} = (\boldsymbol{U}_1, \dots, \boldsymbol{U}_g).$$

Step 2. There are two sets of non-singular odd characteristics:

$$\int_{\infty}^{e_{i_1}} \mathrm{d}\boldsymbol{v} + \ldots + \int_{\infty}^{e_{i_{g-1}}} \mathrm{d}\boldsymbol{v} + \boldsymbol{K}_\infty \subset \Theta_{g-1}, \quad i_1, \dots, i_{g-1} \neq 2g+2$$

and

$$\int_{\infty}^{e_{i_1}} \mathrm{d}\boldsymbol{v} + \ldots + \int_{\infty}^{e_{i_{g-2}}} \mathrm{d}\boldsymbol{v} + \boldsymbol{K}_\infty \subset \Theta_{g-2}$$

Correspondence between branch points and half-periods in Tretkoff basis

Find the correspondence between sets of branch points

$$\{e_{i_1}, \dots, e_{i_{g-1}}\}, \qquad \{e_{i_1}, \dots, e_{i_{g-2}}\}$$

and non-singular odd characteristics $[\delta_{i_1,\dots,i_{g-1}}]$, $[\delta_{i_1,\dots,i_{g-2}}]$ one can add $[\delta_{i_1,\dots,i_{g-1}}] + [\delta_{i_1,\dots,i_{g-2}}]$ and find correspondence,

$$\int_{\infty}^{e_{i_{g-1}}} \mathrm{d}\boldsymbol{v} \leftrightarrows [\varepsilon_{i_{g-1}}], \quad i = 1, \dots, 2g+2$$

Step 3. Among $2g+2$ characteristics should be precisely g odd and $g+2$ even characteristics. Sum of all odd characteristic gives the vector of Riemann constants with base point at the infinity. Check that this characteristic is singular of order $\left[\frac{g+1}{2}\right]$
Step 4. Calculate symmetric matrix $\varkappa$ and then second period matrices $2\eta, 2\eta'$ following to the Proposition 1.

Outlook

- effective one body problem
- test particles with spin
- test particles with multipole moments
- ...

THANK YOU!