Geodesic equation in axially symmetric space–times
— Analytic solutions and observables —

Claus Lämmerzahl
with Eva Hackmann, Valeria Kagramanova, and Jutta Kunz

Centre for Applied Space Technology and Microgravity (ZARM),
University of Bremen, 28359 Bremen, Germany

The higher–genus sigma function and applications
Edinburgh 11. – 15. 10. 2010

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Where we are


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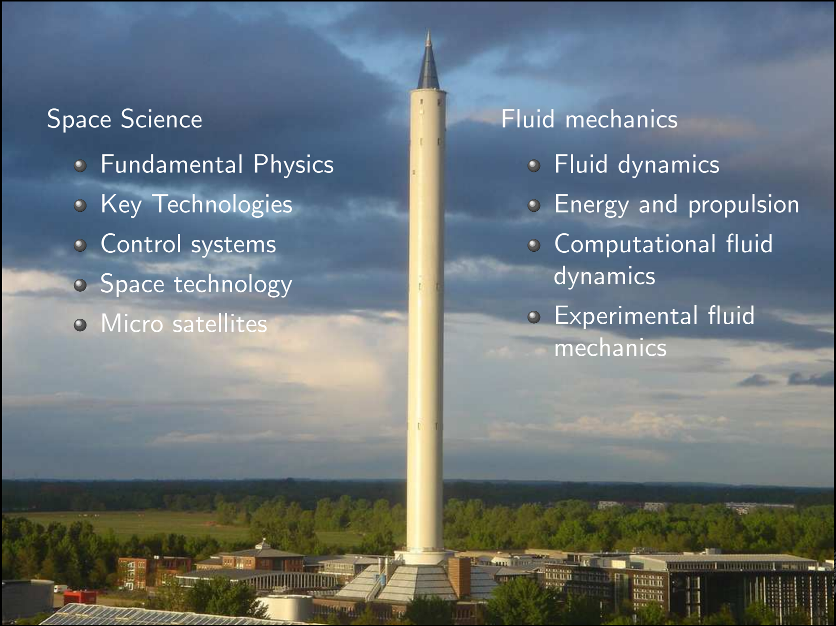
OSTSEE
NORDSEE
SCHLESWIG-
HOLSTEIN
MECKLENBURG-
VORPOMMERN
HAMBURG
BREMEN
NIEDERSACHSEN
SACHSEN-
ANHALT
BERLIN
BRANDENBURG
HANNOVER
MAGDEBURG
NORDRHEIN-
WESTFALEN
DÜSSELDORF
THÜRINGEN
ERFURT
SACHSEN
DRESDEN
HESSEN
WIESBADEN
RHEINLAND-
PFALZ
SAARBRÜCKEN
PRAHA
BAYERN
MÜNCHEN
STUTTGART
BADEN-
WÜRTTEMBERG
DK
PL
NL
B
L
F
CZ
A
CH

Where we are

Where we are

The Bremen drop tower

The Bremen drop tower



Space Science

- Fundamental Physics
- Key Technologies
- Control systems
- Space technology
- Micro satellites

Fluid mechanics

- Fluid dynamics
- Energy and propulsion
- Computational fluid dynamics
- Experimental fluid mechanics

Fundamental Physics at ZARM: Scope

Scope of Fundamental Physics at ZARM

- Development of new technologies
  - for microgravity experiments (drop tower, ISS, satellite)
  - for applications in space
- Accompanying theoretical investigations
  - motivation for experiments and missions
  - theory for experiments and applications
- High precision modeling
  - experimental devices
  - whole spacecraft
  - whole missions
  - quantum modeling

23 members

3 Professors, 3 post–docs, 14 PhD students, 1 diploma student, 2 technicians

Center of Applied Space Technology and Microgravity

Research areas

Fundamental Physics at ZARM

Center of Applied Space Technology and Microgravity

Research areas

- Satellite dynamics
  - modeling
  - disturbance forces
  - thermal and stress analysis
  - HPS (High Performance satellite dynamics Simulator)

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  - Bose–Einstein Condensate, BEC (exp, theory & modeling)
  - atom interferometry (exp & theory)
  - quantum tests (equivalence principle, decoherence, linearity, ...)
  - development of corresponding space technology

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- Gravitational physics
  - tests of Equivalence Principle
  - analytical and numerical solutions for orbits
  - quantum gravity phenomenology
  - theoretical description of experiments testing SR and GR

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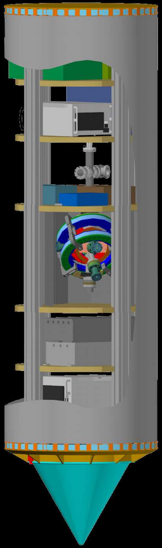
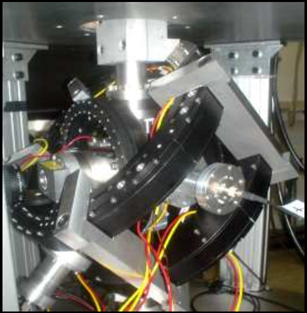
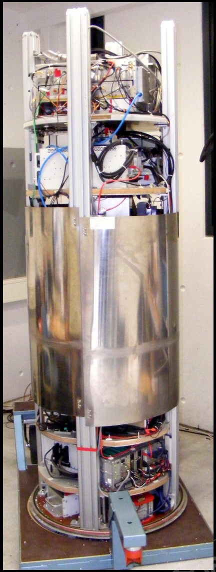
BEC in microgravity



design of capsule



vacuum chamber



capsule

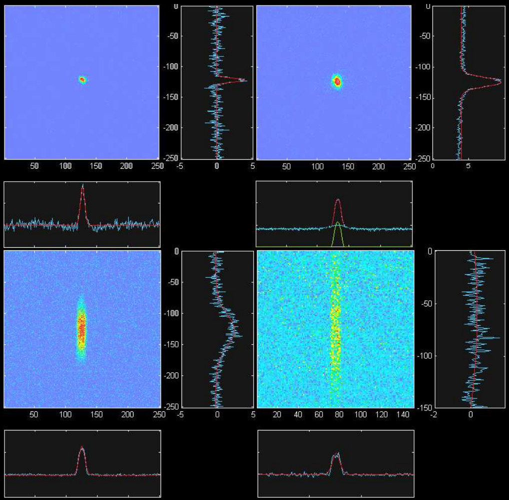
First BEC in microgravity / extended free fall

LU Hannover, ZARM, MPQ Munich, U Hamburg, HU Berlin, U Ulm

BEC in microgravity – long free evolution

50 ms

100 ms



500 ms

1000 ms

10^4 atoms, 1 s free evolution time (not possible on ground)

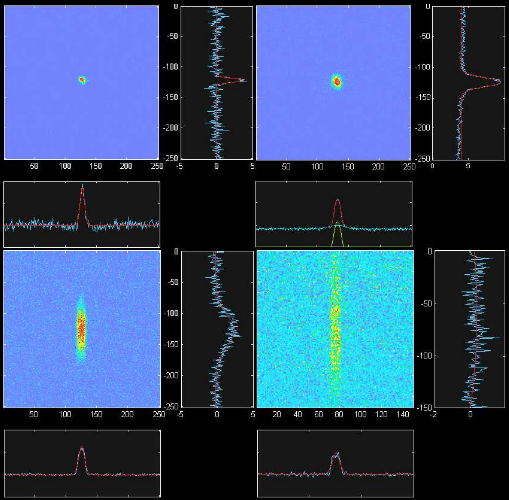
van Zoest et al, Science 2010

50 ms

100 ms

500 ms

1000 ms



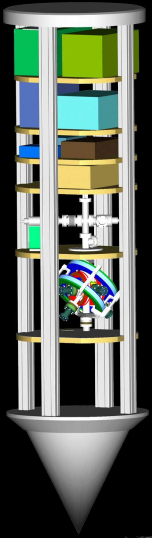
10^4 atoms, 1 s free evolution time (not possible on ground)

van Zoest et al, Science 2010

BEC in free fall

- Status
  - until now almost 200 drops
  - BEC is created regularly
  - extremely robust (survives $\sim 50\,g$)

  Worldwide most advanced technology towards space application and fundamental quantum physics in μg
- Ongoing work
  - PRIMUS (PRäzisions–Interferometrie mit Materiewellen Unter Schwerelosigkeit)
  - FOKUS (FaserOptischer FrequenzKamm Unter Schwerelosigkeit)
  - ATUS (Atom Interferometer Modeling)
  - Fluctuations in Quantum Systems
- In future
  - Fundamental Physics experiments
  - Drop tower — Texus — ISS
  - Inertial sensors
  - High precision clocks



High precision quantum modeling

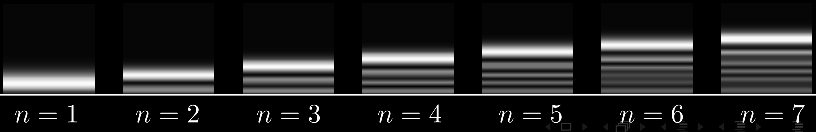
GOST: trampolin with a Bose–Einstein condensate

BEC

BEC described by Gross–Pitaevskii equation

$$i\partial_t \psi = \Delta\psi + V(\boldsymbol{x})\psi + g|\psi|^2\psi$$

- wide class of solutions known
- dynamical solutions in gravitational field $V(\boldsymbol{x}) = \boldsymbol{g}\cdot\boldsymbol{x}$ (Chen & Lee, PRL 1976)
- stationary and dynamical solutions in gravito–optical surface trap: boundary condition $\psi = 0$ for $z = 0$
- nonlinear hydrogen atom $V(\boldsymbol{x}) = 1/r$
- solutions in periodic potentials
- solutions in gravitational waves
- solitons, vortices



Multi–component BEC

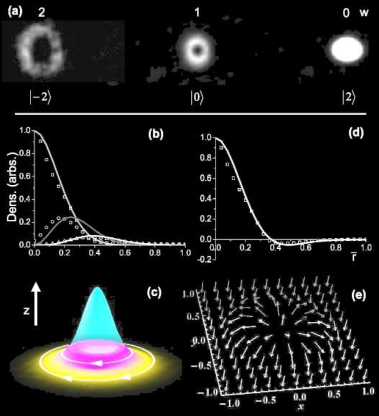
for testing the Universality of Free Fall:
two BECs in the same trap
→ multicomponent BEC
multi–component BEC described by

$$i\partial_t \Psi_a = \Delta\Psi_a + V(\boldsymbol{x})\Psi_a + g\|\Psi\|^2\Psi_a$$

with

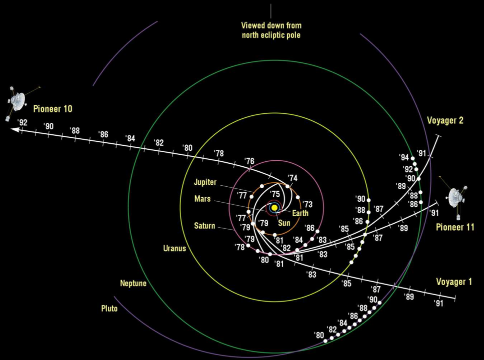
$$\Psi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix}$$

- vector Schrödinger equation: analytical solutions? In gravitational fields?
- skyrmions
- ...



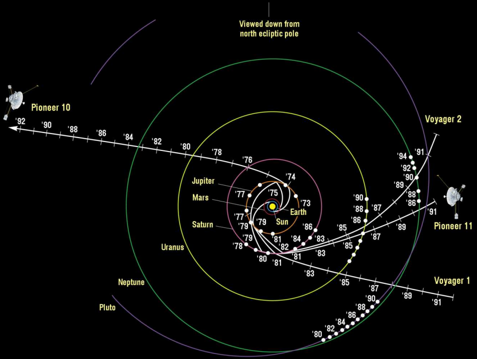
Leslie et al, PRL 2009

The Pioneer anomaly



Anomalous acceleration toward Sun

- anomalous gravity?
- systematics?



Anomalous acceleration toward Sun

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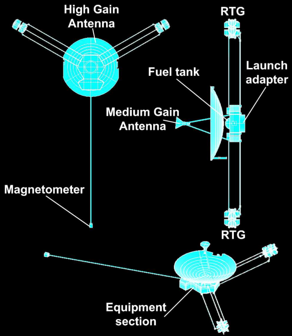
High precision modeling

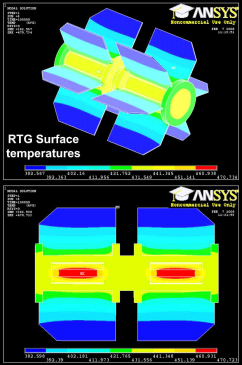
New method for high precision modeling of

- experimental devices
- spacecraft

Needed for

- ground experiments (Michelson–Morley, gravitational wave interferometers, etc.)
- analysis of Pioneer anomaly
- LISA, LISA pathfinder, geodesy missions, ...





Rievers, C.L., List, Bremer & Dittus, NJP 2009

High precision modeling – FE

Main topics

- Thermal modeling
- Strains and stresses
- Drag and recoil forces
- Analytical solutions

Aim

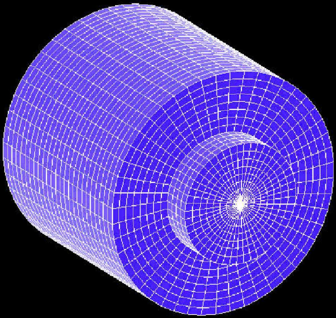
- Accuracy $\sim 10^{-20}$
- motivated by cavities and optical clocks

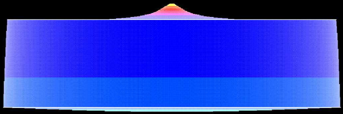
$$\nu = \frac{2n\pi}{L} c \qquad \Rightarrow \qquad \frac{\delta c}{c} = \sqrt{\left(\frac{\delta\nu}{\nu}\right)^2 + \left(\frac{\delta L}{L}\right)^2}$$

future frequency stability $\sim 10^{-18}$ requires same mechanical stability

High precision modeling – FE

Modeling an optical resonator





applications to gravitational wave interferometers, LISA, LISA Pathfinder, clock missions, ...

High precision modeling – FE

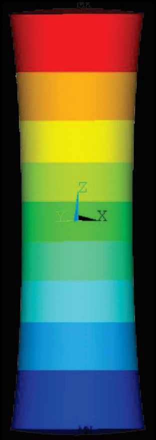
Analytic solution of Lamé-Navier equation in gravity gradient

$$\mu \Delta u^i + (\lambda + \mu)(\mathrm{grad}\,\mathrm{div}\, u)^i + K^i = 0$$

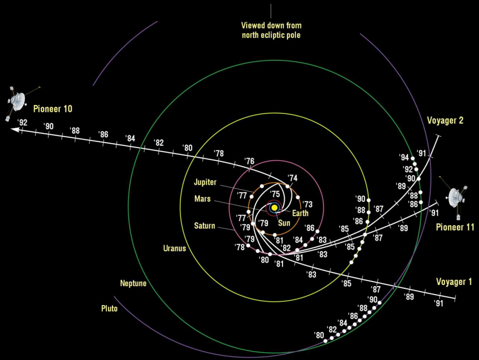
Homogenous part implies biharmonic equation

$$\Delta \Delta u = 0$$

Scheithauer & C.L., CQG 2006

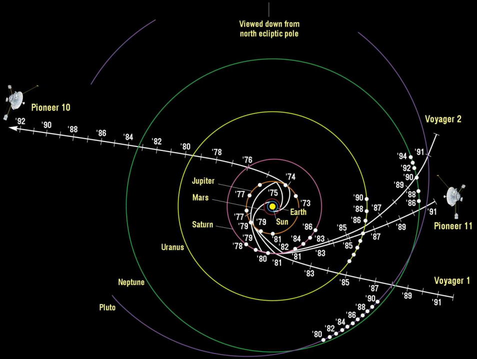


The Pioneer anomaly



Anomalous acceleration toward Sun

- anomalous gravity?
- systematics?



Anomalous acceleration toward Sun

- anomalous gravity?
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Orbits in gravitational fields

- New analytic solutions, based on algebra–geometric methods
- Analytic solution of differential equations of the form

$$\left(\frac{dx}{ds}\right)^2 = P_n(x) \qquad \text{and} \qquad \left(x\frac{dx}{ds}\right)^2 = P_n(x)$$

  P_n = polynomial of degree n
- Application: influence of cosmological constant on motion
  - Plebański–Demiański space–times without acceleration in 4D (Petrov D)
  - higher dimensions
  - in space–times with mass multipoles
- Practical application
  - Pioneer anomaly, dark matter problems
  - geodesy
  - clocks in space

Hackmann & Lämmerzahl, PRL 2008, PRD 2008,
Hackmann, Kagramanova, Kunz & Lämmerzahl, PRD 2008, 2009, 2010, EPL 2009

Outline

1. Introduction and motivation

Outline

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2. General Relativity

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3. Space–times
   - Vacuum space–times: Plebański–Demiański space-time
   - String space–times
   - Higher dimensions
   - PPN space–times
4. Solutions of the geodesic equation
   - The solutions
   - Examples for orbits
   - Classification of orbits

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5. Observables
   - Observables for bound orbits
   - Observables for escape orbits

Outline

Outline

The problem

gravity can only be explored through the motion of particles and light

- particles (spacecraft, stars, pulsars, black holes, ...) and light
  - point particles and light rays $\longrightarrow$ geodesic equation
  - spinning particles and polarized light $\longrightarrow$ MPD equation
  - particles with mass multipoles $\longrightarrow$ MPD equation

Known solutions

gravity can only be explored through the motion of particles and light

- analytic solutions for geodesic equations in vacuum space–times
  - Schwarzschild (Hagihara, JJGA 1931)
  - Reissner–Nordström (Chandrasekhar 1983)
  - Kerr (Carter 1968, Chandrasekhar 1983)
  - Schwarzschild–de Sitter (Hackmann & C.L. PRL 2008, PRD 2008)
  - spherically symmetric space–times in higher dimensions (Hackmann, Kagramanova, Kunz, C.L., PRD 2008)
  - Plebański–Demiański (Hackmann, Kagramanova, Kunz, C.L., EPL 2009)
  - Kerr–de Sitter (Hackmann, Kagramanova, Kunz, C.L., PRD 2010)
  - Taub–NUT (Kagramanova, Kunz, Hackmann, C.L., PRD 2010)
  - Taub–NUT–de Sitter (Hackmann, Kagramanova, Kunz, C.L., in preparation)
- analytic solutions for geodesic equations in nonvacuum space–times
  - Schwarzschild–string (Hackmann, Hartmann, C.L., Sirimachan, PRD 2010)
  - Kerr–string (Hackmann, Hartmann, C.L., Sirimachan, PRD 2010)
  - Plebański–Demiański–string (Hackmann, Hartmann, C.L., Sirimachan, in prep.)

Known solutions

gravity can only be explored through the motion of particles and light

- analytical solutions for extended particles
  - spinning particles in Schwarzschild (Micolaut, ZP 1967)
  - spinning particle in spherically symmetric space–times (C.L. & Schaffer, in prep.)

Applications

- analytical calculation of satellite orbits
- analytical calculation of general relativistic effects
  huge perihelion shift (binary black holes), Lense–Thirring effect, etc
- analytical calculation of effects of generalized gravity theories
- tests of numerical codes (for gravitational wave templates)
- binary systems and gravitational waves
  - calculation of gravitational wave templates for EMRIs
  - technique can be applied to effective one–body formalism
  - self force calculation
- accretion discs
- further application: motion in mass multipole fields
- pulsar timing formula

inclusion of spin / quadrupole $\rightarrow$ modification, in particular enhancement, of effects
further issue: solutions of field equations

Outline

The geometry and the Einstein equations

All foundations and predictions of GR are experimentally well tested and confirmed

Foundations

The Einstein Equivalence Principle

- Universality of Free Fall
- Universality of Gravitational Redshift
- Local Lorentz Invariance

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$\Downarrow$

Implication

Gravity is a metrical theory

The geometry and the Einstein equations

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Implication

Gravity is a metrical theory

$\Rightarrow$

Predictions for metrical theories

- Solar system effects
  - Perihelion shift
  - Gravitational redshift
  - Deflection of light
  - Gravitational time delay
  - Lense–Thirring effect
  - Schiff effect
- Strong gravitational fields
  - Binary systems
  - Black holes
- Gravitational waves

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All foundations and predictions of GR are experimentally well tested and confirmed

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$\Downarrow$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

Equations of motion: Point particles and light rays

- Geodesic equation

$$0 = (D_u u)^\mu = \frac{d^2 x^\mu}{ds^2} + \left\{ {}^{\mu}_{\rho\sigma} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}$$

with

$$\left\{ {}^{\mu}_{\rho\sigma} \right\} = \frac{1}{2} g^{\mu\nu} \left(\partial_\rho g_{\mu\sigma} + \partial_\sigma g_{\mu\rho} - \partial_\mu g_{\rho\sigma} \right)$$

and

$$\begin{aligned} g_{\mu\nu} u^\mu u^\nu &= 1 \qquad \text{for point particles} \\ g_{\mu\nu} u^\mu u^\nu &= 0 \qquad \text{for light rays} \end{aligned} \qquad \text{with} \qquad u^\mu = \frac{dx^\mu}{ds}$$

where $g_{\mu\nu}$ is the pseudo–Riemannian space–time metric

- reading of clocks = proper time of massive particles

$$s = \int_{\text{orbit}} ds = \int_{\text{orbit}} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \int_{\text{orbit}} \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} dt$$

Equations of motion: Extended bodies

Equation of motion for matter field

$$D_\nu T^{\mu\nu} = 0$$

is partial differential equation: difficult to solve
task: reduction of this PDE to a set of ordinary differential equations
$\rightarrow$ Mathisson–Papapetrou–Dixon

Equations of motion: Extended bodies

Mathisson–Papapetrou–Dixon: equation of particle with mass multipole and spin

$$
\begin{aligned}
v^\mu &= \frac{dx^\mu}{ds} \\
D_v p_\mu &= R_{\mu\nu\rho\sigma} v^\nu S^{\rho\sigma} + D_\mu R_{\rho\sigma\tau\kappa} J^{\rho\sigma\tau\kappa} \\
D_v S^{\mu\nu} &= v^\mu p^\nu - v^\nu p^\mu \\
S^{\mu\nu} p_\nu &= 0
\end{aligned}
$$

Quantities involved

- v^μ gives the geometric trajectory of the body
- p_μ is an auxiliary quantity describing the momentum (if p_μ is considered as derived from $T^\mu{}_\nu$ then p_μ is the primary quantity and v^μ is an auxiliary quantity which however possesses the same interpretation as geometric orbit)
- $S^{\mu\nu}$ spin of particle
- $J^{\rho\sigma\tau\kappa\dots}$ mass multipole moments
  special case: standard mass quadrupole $J^{\rho\sigma\tau\kappa} = -3p^{[\rho}Q^{\sigma][\tau}p^{\kappa]}$ with $Q^{\mu\nu}p_\nu = 0$

Equations of motion: Extended bodies

Mathisson–Papapetrou–Dixon: equation of particle with mass multipole and spin

$$\begin{aligned} v^\mu &= \frac{dx^\mu}{ds} \\ D_v p_\mu &= R_{\mu\nu\rho\sigma} v^\nu S^{\rho\sigma} + D_\mu R_{\rho\sigma\tau\kappa} J^{\rho\sigma\tau\kappa} \\ D_v S^{\mu\nu} &= v^\mu p^\nu - v^\nu p^\mu \\ S^{\mu\nu} p_\nu &= 0 \end{aligned}$$

Meaning

- direct access to curvature
- quadrupole motion in quadrupole space–time
  - special case: aligned quadrupoles
  - equatorial orbit possible: analytic solution?
  - exact quadrupole–quadrupole interaction in GR

Outline

Outline

Plebański–Demiański space-time

stationary axially symmetric metric

$$ds^2 = \frac{\Delta_r}{p^2}\left(dt - A_\vartheta d\varphi\right)^2 - \frac{p^2}{\Delta_r}dr^2 - \frac{\Delta_\vartheta}{p^2}\sin^2\vartheta(a dt - A_r d\varphi)^2 - \frac{p^2}{\Delta_\vartheta}d\vartheta^2$$

where

$$\begin{aligned} p^2 &= r^2 + (n - a\cos\vartheta)^2 \\ \Delta_\vartheta &= 1 + \tfrac{1}{3}a^2\Lambda\cos^2\vartheta - \tfrac{4}{3}\Lambda a n\cos\vartheta \\ \Delta_r &= \left(1 - \tfrac{1}{3}\Lambda r^2\right)\left(r^2 + a^2\right) - 2Mr - n^2 + Q_\mathrm{e}^2 + Q_\mathrm{m}^2 - \Lambda n^2\left(2r^2 + a^2 - n^2\right) \\ A_\vartheta &= a\sin^2\vartheta + 2n\cos\vartheta \\ A_r &= r^2 + a^2 + n^2 \end{aligned}$$

- M = mass, a = Kerr parameter, Λ = cosmological constant, n = NUT parameter, Q_e = electric charge, Q_m = magnetic charge
- this metric contains all standard black hole space–times, Petrov Type D
- Plebański & Demiański, AP 1976; Griffiths & Podolski, IJMP 2006
- horizons given by $\Delta_r = 0$

Conservation laws

There are two Killing vectors ∂_t and ∂_φ
$\Rightarrow$ two conservation laws

$$E := g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi}$$
$$-L := g_{\varphi t}\dot{t} + g_{\varphi\varphi}\dot{\varphi}$$

or

$$E = \frac{\Delta_r}{p^2}(\dot{t} - A_\vartheta\dot{\varphi}) - a\frac{\Delta_\vartheta}{p^2}\sin^2\vartheta(a\dot{t} - A_r\dot{\varphi})$$
$$L = A_\vartheta\frac{\Delta_r}{p^2}(\dot{t} - A_\vartheta\dot{\varphi}) - A_r\frac{\Delta_\vartheta}{p^2}\sin^2\vartheta(a\dot{t} - A_r\dot{\varphi}),$$

this corresponds to

- energy
- angular momentum in z–direction

Outline

String space–times

Plebański–Demiański string space–time $d\varphi \to \beta d\varphi$

$$ds^2 = \frac{\Delta_r}{p^2}\left(dt - A_\vartheta \beta d\varphi\right)^2 - \frac{p^2}{\Delta_r} dr^2 - \frac{\Delta_\vartheta}{p^2}\sin^2\vartheta (a dt - A_r \beta d\varphi)^2 - \frac{p^2}{\Delta_\vartheta} d\vartheta^2$$

describes space–time with string along symmetry axis (space–time with matter)

- same Killing vectors
- conserved energy and angular momentum

$\longrightarrow$ see Betti's talk

Outline

General Relativity in higher dimensions

spherically symmetric space–times in d dimensions: Reissner–Nordström–(anti-)de Sitter

$$ds^2 = g_{tt}dt^2 - g_{rr}dr^2 - r^2 d\Omega_{d-2}^2$$

with

$$g_{tt} = \frac{1}{g_{rr}} = 1 - \frac{M^{d-3}}{r^{d-3}} + \frac{q^{2(d-3)}}{r^{2(d-3)}} - \frac{2\Lambda}{(d-1)(d-2)} r^2$$

Equation of motion

$$\begin{aligned}\left(\frac{dr}{d\varphi}\right)^2 &= \frac{r^4}{L^2}\frac{1}{g_{rr}g_{tt}}\left(E^2 - g_{tt}\left(\epsilon + \frac{L^2}{r^2}\right)\right) \\ &= \frac{r^4}{L^2}\left(E^2 - \left(1 - \frac{M^{d-3}}{r^{d-3}} + \frac{q^{2(d-3)}}{r^{2(d-3)}} - \frac{2\Lambda}{(d-1)(d-2)}r^2\right)\left(\epsilon + \frac{L^2}{r^2}\right)\right)\end{aligned}$$

Substitution $u = \frac{m}{r}$,

various cases can be solved by elliptic and hyperelliptic integrals

$\longrightarrow$ for general case, see Valeria's talk

Outline

PPN space–times

PPN metric

$$\begin{aligned} g_{00} &= 1 - 2U + 2\beta U^2 + \dots \\ g_{0i} &= 0 \\ g_{ij} &= -(1 + 2\gamma U)\delta_{ij} \end{aligned}$$

with Newtonian potential

$$U(t, \boldsymbol{x}) = \int \frac{\rho(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x'$$

- same Killing vectors
- conserved energy and angular momentum
- geodesic equation leads to differential equations which have the same mathematical structure as in Schwarzschild space–time

Outline

Outline

The geodesic equation

geodesic equation

$$0 = \frac{d^2 x^\mu}{ds^2} + \left\{ {\mu \atop \rho\sigma} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}$$

is equivalent to the Hamilton–Jacobi equation

$$2 \frac{\partial S}{\partial s} = g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\mu}$$

separation ansatz

$$S = \tfrac{1}{2}\epsilon s - Et + L\varphi + S_r(r) + S_\vartheta(\vartheta)$$

- insertion into Hamilton–Jacobi
- separation of r and ϑ equations
- separation constant $=$ k $=$ Carter constant (Carter, PR 1968)
- introduction of Mino time τ through $d\tau = \rho^2 ds$ (Mino, PRD 2003)
- substitution $\xi = \cos\vartheta$
- renormalization: all quantities in units of $r_{\rm S}$

The geodesic equation

$$\left(\frac{dr}{d\tau}\right)^2 = \left((r^2+a^2+n^2)E - aL\right)^2 - \Delta_r(\epsilon r^2 + k) \qquad =: R(r)$$

$$\left(\frac{d\xi}{d\tau}\right)^2 = \Delta_\xi(1-\xi^2)\left(k - \epsilon(n-a\xi)^2\right) - (L - A_\xi E)^2 \qquad =: \Theta(\xi)$$

$$\frac{d\varphi}{d\tau} = a\frac{(r^2+a^2+n^2)E - aL}{\Delta_r} + \frac{L - A_\xi E}{\Delta_\xi(1-\xi^2)} \qquad =: f(r) + g(\xi)$$

$$\frac{dt}{d\tau} = A_r\frac{(r^2+a^2+n^2)E - aL}{\Delta_r} + \frac{A_\xi\,(L - A_\xi E)}{\Delta_\xi(1-\xi^2)} \qquad =: h(r) + j(\xi)$$

analytic solution given by hyperelliptic functions (Hackmann & C.L., PRL 2008)

$$r(\tau) = \mp\frac{\sigma_2^{(r)}(\vec{x})}{\sigma_1^{(r)}(\vec{x})} + r_0 \qquad \text{with} \qquad \sigma^{(r)}(\vec{x}) = 0\,, \quad \vec{x} = \begin{pmatrix}\tau_1\\ \tau\end{pmatrix}$$

$$\xi(\tau) = \mp\frac{\sigma_2^{(\xi)}(\vec{y})}{\sigma_1^{(\xi)}(\vec{y})} + \xi_0 \qquad \text{with} \qquad \sigma^{(\xi)}(\vec{y}) = 0\,, \quad \vec{x} = \begin{pmatrix}\tau_1\\ \tau\end{pmatrix}$$

The geodesic equation

integration of φ and t motion

$$\varphi - \varphi_0 = \int_{r_0}^{r(\tau)} f(r) \frac{dr}{\sqrt{R}} + \int_{\xi_0}^{\xi(\tau)} g(\xi) \frac{d\xi}{\sqrt{\Theta(\xi)}}$$
$$t - t_0 = \int_{r_0}^{r(\tau)} h(r) \frac{dr}{\sqrt{R}} + \int_{\xi_0}^{\xi(\tau)} j(\xi) \frac{d\xi}{\sqrt{\Theta(\xi)}}$$

f, g, h, and j are rational functions
$\rightarrow$ partial fraction expansion: hyperelliptic integrals of first, second and third kind

$$\int \frac{x^p\, dx}{\sqrt{P_n(x)}}, \qquad \int \frac{x^q\, dx}{\sqrt{P_n(x)}}, \qquad \int \frac{dx}{(x-c)\sqrt{P_n(x)}}$$

with $p < \left[\frac{n-1}{2}\right]$ and $q \geq \left[\frac{n-1}{2}\right]$, can be integrated explicitly, but gives rather complicated expressions

regularity of geodesic equation for $\vartheta = 0$ or π

The geodesic equation

solution for φ (similar for t)

$$\begin{aligned}\varphi &= \varphi_0 + \mathrm{sign}(r_0')\left(C_1^r f^r(\tau-\tau_0^r) + C_2^r(\tau-\tau_0^r) + I_{34}^r(\tau-\tau_0^r)\right)\\ &\quad -\mathrm{sign}(\vartheta_0')\left(C_1^\vartheta f^\vartheta(\tau-\tau_0^\vartheta) + C_2^\vartheta(\tau-\tau_0^\vartheta) - aI_{32}^\vartheta(\tau-\tau_0^\vartheta) - I_{44}^\vartheta(\tau-\tau_0^\vartheta)\right)\end{aligned}$$

with

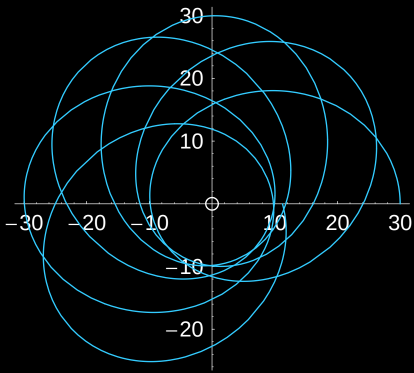
$$\begin{aligned}I_{mn}^x(w) &= \sum_{i=1}^n \frac{C_{m,i}^x}{\sqrt{P_5^x(u_i)}}\left(-\begin{pmatrix} f^x(w)-f^x(w_0)\\ w-w_0\end{pmatrix}^T \cdot \int_{p_i^-}^{p_i^+} d\vec{r}\right.\\ &\quad \left. + \frac{1}{2}\log\frac{\sigma\left((f^x(w),w)^T - 2\int_\infty^{p_i^+} d\vec{z}\right)}{\sigma\left((f^x(w),w)^T - 2\int_\infty^{p_i^-} d\vec{z}\right)} - (w \leftrightarrow w_0)\right)\end{aligned}$$

$d\vec{r} =$ holomorphic differentials of second kind
$d\vec{z} =$ meromorphic differentials of first kind

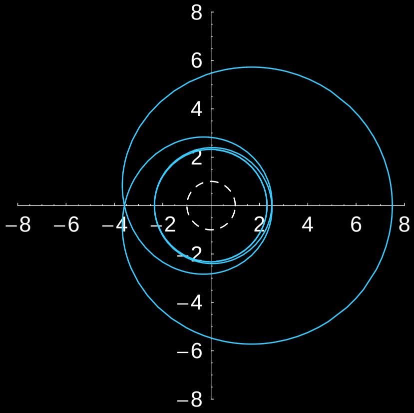
$$p_i^\pm = \begin{pmatrix} u_i \\ \pm\sqrt{P_5(u_i)}\end{pmatrix} \qquad \text{with} \qquad u_i \ \text{pole}$$

Outline

Schwarzschild



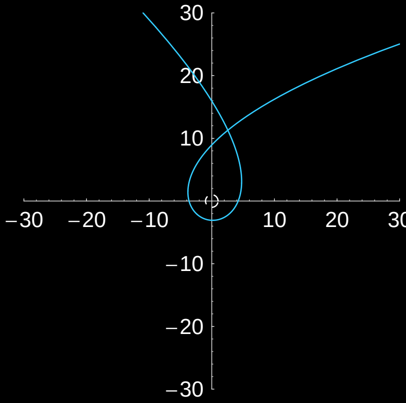
bound orbit



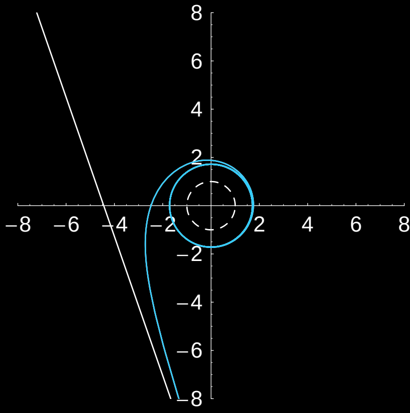
homocline orbit

Hagihara, JJGA 1931

Schwarzschild



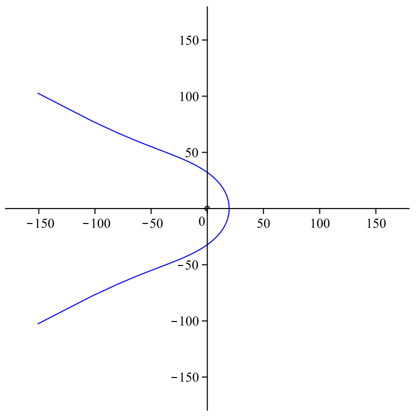
quasi parabolic escape orbit



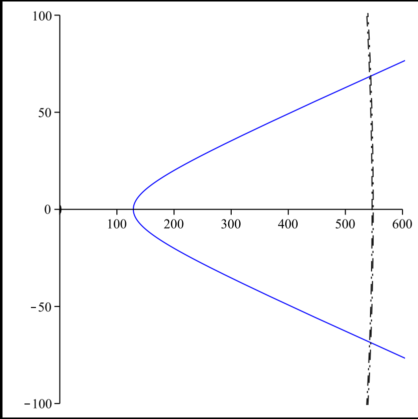
inspiral orbit

Hagihara, JJGA 1931

Schwarzschild–de Sitter



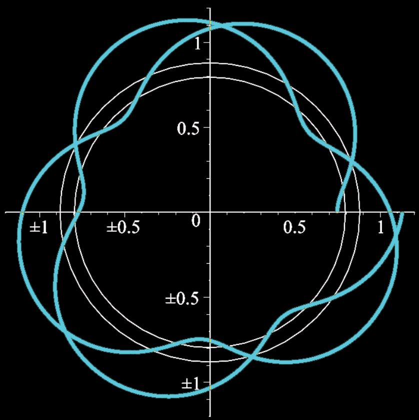
escape orbit



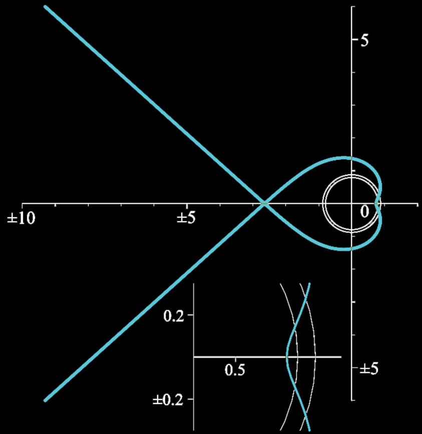
reflection at cosmic wall

Hackmann & C.L., PRL 2008, PRD 2008

Reissner–Nordström in higher dimensions



bound orbit
many universe orbit

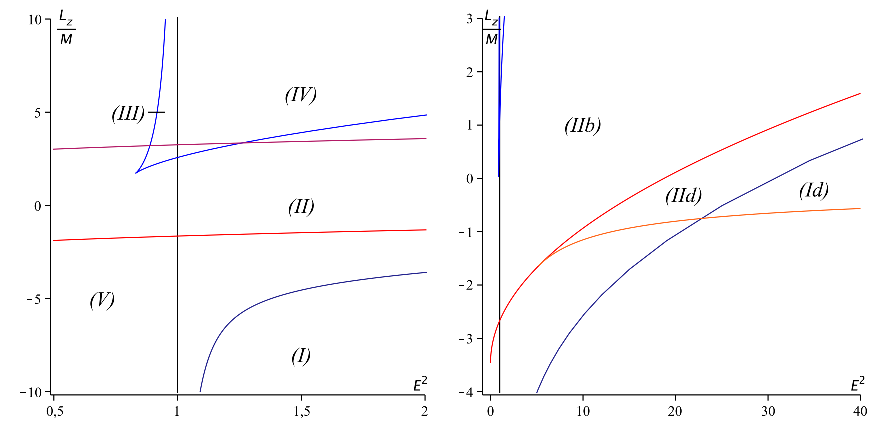


escape orbit
escape in different universe

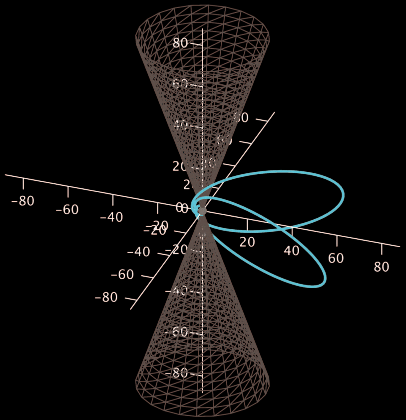
Hackmann, Kagramanova, Kunz, & C.L., PRD 2008

Kerr space–time

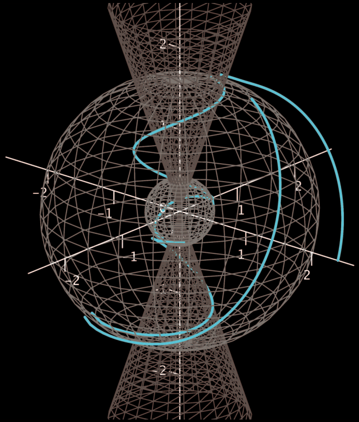
parameter plots for r and ϑ



Kerr–de Sitter



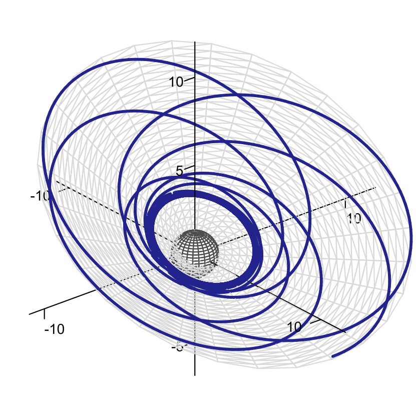
bound orbit



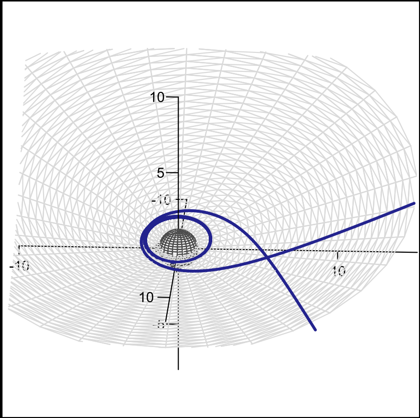
escape orbit

Hackmann, Kagramanova, Kunz, C.L., PRD 2010

Taub–NUT



bound orbit



escape orbit

Hackmann, Kagramanova, Kunz, & C.L., PRD 2010

Outline

Orbits

Types of orbits

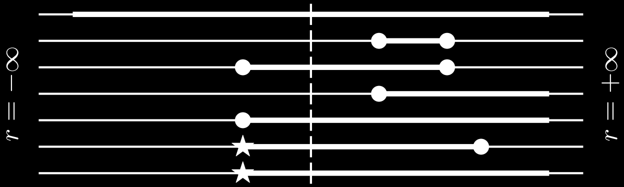
- bound orbit: $r_{\min} \leq r \leq r_{\max}$
- escape orbit: $r_{\min} \leq r \leq \infty$
- terminating bound orbit: $r \leq r_{\max}$, orbit terminates (falls into singularity)
- terminating escape orbit $r \leq \infty$, orbit terminates (falls into singularity)

r can also become negative $\Rightarrow$ more possibilities

Orbits

Types of orbits allowing $r < 0$

- transit orbit: $-\infty < r < +\infty$
- bound orbit: $0 < r_{\rm min} \leq r \leq r_{\rm max}$
- crossover bound orbit: $r_{\rm min} \leq r \leq r_{\rm max}$ with $r_{\rm min} < 0 < r_{\rm max}$
- escape orbit: $r_{\rm min} \leq r \leq \infty$
- crossover escape orbit: $r_{\rm min} \leq r \leq \infty$ with $r_{\rm min} < 0$ or $-\infty < r < r_{\rm max}$ with $0 < r_{\rm max}$
- terminating – bound – crossover orbit: bound orbit $|r| < r_{\rm max}$ terminates
- terminating – escape – crossover orbit: escape orbit terminates



Orbits

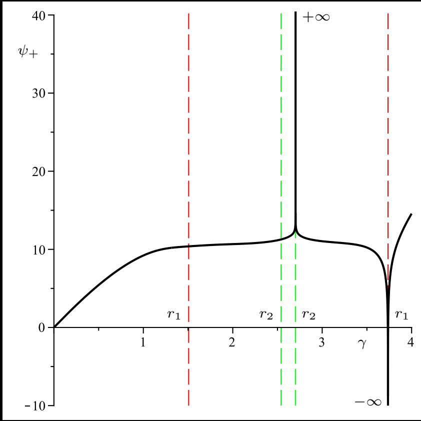
Further discussion of effects

- geodesic incompleteness
- geodesics in analytic continuation of space-time
- closed time–like curves (CTC)
- crossing horizons
- homocline orbits
- many crosses of z–axes
- fast, slow rotation

Taub–NUT space–time: incompleteness

- Taub–NUT space–time possess no curvature singularity
- but is geodesic incomplete ... during second transition through a horizon proper time terminates

(Hackmann, Kagramanova, Kunz, C.L. 2010)



Outline

Outline

Observables: for bound orbits

For bound orbits

- two oscillatory coordinates: r and ϑ (generalized Lissajous figures)
- two (secularly) increasing coordinates: t and φ

Periods

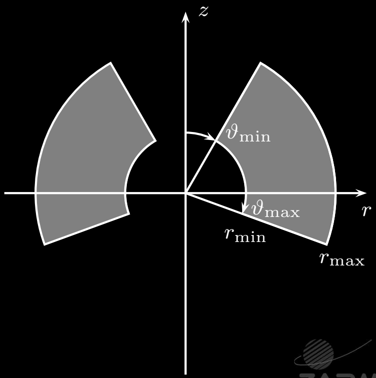
- radial period

$$\omega_r = 2 \int_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{R}}$$

  is time needed to go from $r_{\min}$ to $r_{\min}$
- polar angle period

$$\omega_\vartheta = 2 \int_{\vartheta_{\min}}^{\vartheta_{\max}} \frac{dr}{\sqrt{\Theta}}$$

  is time needed to go from $\vartheta_{\min}$ to $\vartheta_{\max}$



Observables: for bound orbits

Secular increases

- secular time increase

$$\Gamma = \left\langle \frac{dt}{d\tau} \right\rangle = \frac{2}{\omega_r} \int_{r_{\min}}^{r_{\max}} h(r) \frac{dr}{\sqrt{R}} + \frac{2}{\omega_\vartheta} \int_{\vartheta_{\min}}^{\vartheta_{\max}} j(\vartheta) \frac{dr}{\sqrt{\Theta}}$$

- secular azimuthal increase

$$Y = \left\langle \frac{d\varphi}{d\tau} \right\rangle = \frac{2}{\omega_r} \int_{r_{\min}}^{r_{\max}} f(r) \frac{dr}{\sqrt{R}} + \frac{2}{\omega_\vartheta} \int_{\vartheta_{\min}}^{\vartheta_{\max}} g(\vartheta) \frac{dr}{\sqrt{\Theta}}$$

orbital frequencies (Drasco & Hughes, PRD 2004; Schmidt, CQG 2004)

$$\Omega_r = \frac{2\pi}{\Gamma\omega_r}\,, \qquad \Omega_\vartheta = \frac{2\pi}{\Gamma\omega_\vartheta}\,, \qquad \Omega_\varphi = \frac{Y}{\Gamma}$$

- angular velocity of r–oscillations
- angular velocity of ϑ–oscillations
- secular angular velocity

Observables: for bound orbits

observables: self referential comparison, invariant

The observables

- periastron shift

$$\Delta_{\rm periastron} := \Omega_\varphi - \Omega_r = \left(Y - \frac{2\pi}{\omega_r}\right)\frac{1}{\Gamma}$$

- Lense–Thirring effect

$$\Delta_{\rm Lense-Thirring} := \Omega_\varphi - \Omega_\vartheta = \left(Y - \frac{2\pi}{\omega_\vartheta}\right)\frac{1}{\Gamma}$$

- $\Delta_{\rm periastron}$ compares the φ–advance for $r_{\rm min}$ with 2π
  $\rightarrow$ in weak field motion of $r_{\rm min}$ within orbital plane or orbital cone
- $\Delta_{\rm Lense-Thirring}$ compares the φ–advance for $\vartheta_{\rm min}$ with 2π
  $\rightarrow$ in weak field precession of orbital plane or orbital cone

all observables for bound orbits should be functions of Ω_r, Ω_ϑ, and Ω_φ which can be evaluated explicitly by complete hyperelliptic integrals

Observables: for bound orbits

- Schwarzschild, Schwarzschild–de Sitter, Reissner–Nordström, Taub–NUT:

$$\Delta_{\text{perihelion}} \neq 0\,, \qquad \Delta_{\text{Lense-Thirring}} = 0$$

- Kerr, Kerr–de Sitter, Kerr–Newman, Kerr–NUT:

$$\Delta_{\text{perihelion}} \neq 0\,, \qquad \Delta_{\text{Lense-Thirring}} \neq 0$$

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$$\Delta_{\mathrm{perihelion}} \neq 0\,, \qquad \Delta_{\mathrm{Lense-Thirring}} \neq 0$$

- task: expansion of $\Delta_{\mathrm{perihelion}} = \Delta_{\mathrm{perihelion}}(M, a, n, \Lambda, Q_{\mathrm{e}}, Q_{\mathrm{m}})$ and $\Delta_{\mathrm{Lense-Thirring}} = \Delta_{\mathrm{Lense-Thirring}}(M, a, n, \Lambda, Q_{\mathrm{e}}, Q_{\mathrm{m}})$ (Hackmann, Kagramanova, Kunz, C.L., in preparation)

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- In general, for complicated potentials there are several periods $\Rightarrow$ many perihelion shifts or Lense–Thirring effects ($\rightarrow$ Valeria's talk)

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- In general, for complicated potentials there are several periods $\Rightarrow$ many perihelion shifts or Lense–Thirring effects ($\rightarrow$ Valeria's talk)
- observables on higher dimensions: many φ_p, ϑ_p, $p = 1, \ldots, n-3$ periods possible to define Γ and Y_p, and

$$\begin{aligned} \Delta_{\text{periastron},p} &:= \Omega_{\varphi_p} - \Omega_p \\ \Delta_{\text{Lense-Thirring},p,p'} &:= \Omega_{\varphi_p} - \Omega_{\vartheta_{p'}} \end{aligned}$$

does this make sense?

Post–Schwarzschild of perihelion shift

Perihelion shift in Schwarzschild–de Sitter (for bound orbit, Kraniotis & Whitehouse, CQG 2003)

$$\delta\varphi_{\text{perihelion}} = 2\pi - \omega_{22} = 2\pi - \oint \frac{x dx}{\sqrt{P_5(x)}}$$

with

$$\begin{aligned}
\oint_{a_2} \frac{x dx}{\sqrt{P_5(x)}} &= \oint_{a_2} \frac{1}{\sqrt{P_3(x)}} - \frac{2}{3}\Lambda m^2 \oint_{a_2} \frac{x^2+\lambda}{x^2 P_3(x)\sqrt{P_3(x)}} dx + \mathcal{O}(\Lambda^2) \\
& x \\
&= \omega_1 + \Lambda \frac{m^2}{96}\left(\sum_{j=1}^{3} \frac{\eta_1 + \omega_1 z_j}{(\wp''(\rho_j))^2}\left(1 + \frac{\lambda}{\left(4z_j + \frac{1}{3}\right)^2}\right)\right. \\
&\quad \left. + \lambda\left(\frac{2\eta_1 - \frac{1}{6}\omega_1}{16(\wp'(u_0))^2} + \frac{6}{16}\frac{\wp''(u_0)}{(\wp'(u_0))^5}\left(\zeta(u_0) - \eta_1 u_0\right)\right)\right) + \mathcal{O}(\Lambda^2)
\end{aligned}$$

- Needs introduction of $r_{\min}$ and $r_{\max}$ or a and e for interpretation
- Needs relativistic approximation for interpretation

Perihelion shift

$$\omega_1 = \int_{r_{\min}}^{r_{\max}} \frac{d\varphi}{dr} dr = \int_{e_2}^{e_3} \frac{d\varphi}{dx} dx = \int_{e_2}^{e_3} \frac{dx}{\sqrt{\left(\frac{dx}{d\varphi}\right)^2}} = \int_{e_2}^{e_3} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}$$

Perihelion shift

$$\delta\varphi = 2\omega_1 - 2\pi = \frac{4}{\sqrt{-e_2 - 2e_3}} \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - \frac{e_2 - e_3}{-e_2 - 2e_3} \sin^2 x}} - 2\pi \,.$$

One can identify

$$e_2 = \frac{2M}{r_{\min}} - \frac{1}{3} \,, \qquad e_3 = \frac{2M}{r_{\max}} - \frac{1}{3} \,.$$

- Can be used for approximation
- Can be used for representation in terms of semi–major axis and eccentricity

Observables: for bound orbits

Further observables

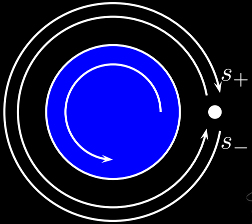
- progression of nodes: determination of φ and t or τ for which $\vartheta = \pi/2$: then

$$\Delta_i \varphi = \varphi(\tau_{i+1}) - \varphi(\tau_i) \qquad \text{with} \qquad \vartheta(\tau_i) = \frac{\pi}{2}\,, \quad i = 1, 2, \ldots$$

  one has to determine τ_i and then to integrate $d\varphi/d\tau$ from τ_i to τ_{i+1} (Gebhardt, Hackmann & C.L., in preparation)

- clock effect $s_+ - s_- \sim 4\pi \dfrac{J}{M} \sim 10^{-7}$ s — and generalizations of it ...

- analytic expressions still have to be calculated
- application to Galileo?



Outline

Observables: for flyby orbits

flyby orbit: $r \longrightarrow \infty$ for $s \longrightarrow \pm\infty$
Then $\vartheta^{\pm} = \vartheta(\pm\infty)$ and $\varphi^{\pm} = \varphi(\pm\infty)$

Deflection angles

- azimuthal deflection angle

$$\Delta\varphi = \varphi^{+} - \varphi^{-}$$

- polar deflection angle

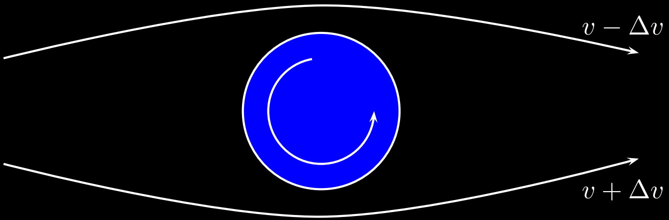
$$\Delta\vartheta = \vartheta^{+} - \vartheta^{-}$$

- analytic expressions still have to be calculated
- application to rotating black holes
- no impact on flyby anomaly (Hackmann & C.L. 2010)

In addition for light:

- gravitational time delay

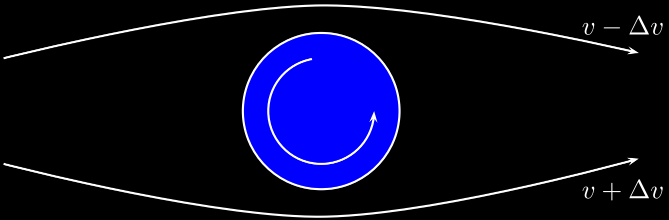
Further observables



- flyby at rotating body: different direction, different velocity
- no impact on flyby anomaly (Hackmann & C.L. 2010)
- deflection of light
- timing formula

will depend on impact parameter as well as on polar angle

Further observables



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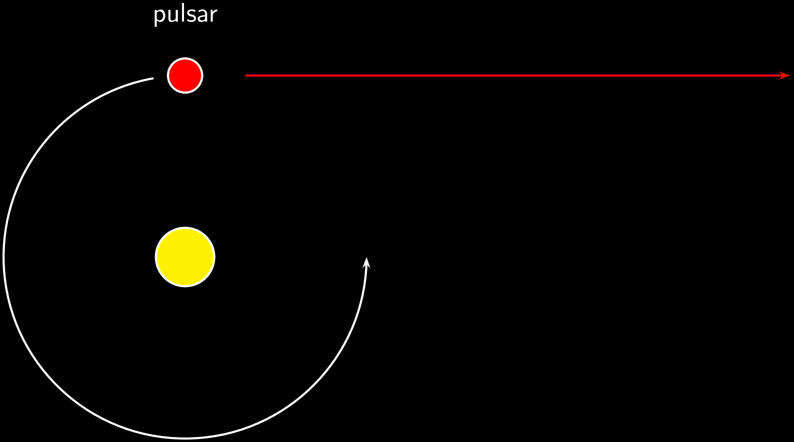
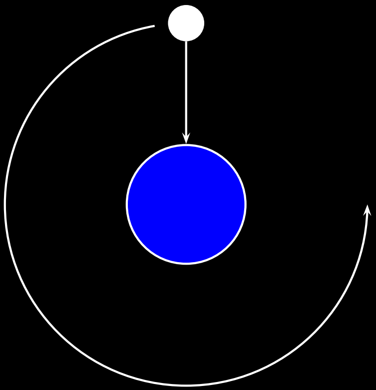
Timing formula



Question: frequency of pulses arriving at Earth — function of orbit (position, velocity, structure of gravitational field)

Timing formula

satellite



Question: frequency of satellites (time of satellites, Galileo) arriving at Earth — function of orbit

- signals arriving at fixed position on Earth
- signals arriving at surface of Earth

Outline

Outline

Discussion: General meaning

Theorem: Separability

The Hamilton–Jacobi equation for the geodesic equation is separable if and only if space–time is of Petrov type D without acceleration (Demianski & Francaviglia, JPA 1981)

Theorem: Petrov type D

The general type D Petrov space–times are exhausted by the electro–vac Plebański–Demiański solutions (Plebański & Demiański, AP 1976)

Theorem: Integration

The geodesic equation in a general electro–vac Plebański–Demiański space–time without acceleration can be integrated using the method of hyperelliptic integrals (Hackmann, Kagramanova, Kunz, C.L., EPL 2009)

All what is separable can be solved analytically

Summary and outlook

Summary

- Complete analytic solution of geodesic equation in Plebański–Demiański space–times
- Analytic solution for all electro–vac space-times for which Hamilton–Jacobi separates
- Complete set of fundamental observables for bound orbits

mathematics is essentially under control $\longrightarrow$ discussion of solutions and observables

Summary and outlook

Outlook 1: discussion of obtained solutions

- analytic description of progression of nodes
- further observables (deflection angle, clock effect, timing, time delay, ...)
- post–Newton, post–Schwarzschild, post–Kerr, ... expansions of solutions
- post–Newton, post–Schwarzschild, post–Kerr, ... expansions of observables

Outlook 2: new solutions

- motion in axially symmetric mass multipole fields (e.g. Quevedo, FP 1990)
- we are now able to analytically solve

$$\left(\frac{dr}{d\tau}\right)^2 = P_n(r) \qquad \text{for all } n$$

(Enolskii, Hackmann, Kagramanova, Kunz, C.L. in preparation, ⟶ see Valeria's talk)

Outline

Spinning objects

spherically symmetric metric

$$ds^2 = \alpha dt^2 - \alpha^{-1} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 \,, \qquad \alpha = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \frac{Q^2}{r^2}$$

additional constant of motion

$$C \equiv \xi^\mu p_\mu + D_\nu \xi_\mu S^{\mu\nu}$$

spin motion can be solved

$$\frac{dS^{\mu\nu}}{ds} - \frac{1}{r}\frac{dr}{ds} S^{r\varphi} = 0 \quad \Rightarrow \quad S^{r\varphi} = \frac{S}{r}$$

equation of motion for radial coordinate (with $J = L + S$)

$$\left(\frac{dr}{ds}\right)^2 = E^2 - \frac{(\partial_r g_{tt})\, SJ}{r} + g_{\varphi\varphi} g_{tt} \left(\frac{J^2 + 2JS}{r^4}\right) - g_{tt}$$

Spinning objects

for Reissner–Nordström–de Sitter

$$\left(\frac{dr}{ds}\right)^2 = E^2 - 1 + \frac{\Lambda r^2}{3} - \Lambda S^2 - \frac{2\Lambda L S}{3} + \frac{\Lambda L^2}{3} + \frac{2}{r} + \frac{S^2 - L^2 - Q^2}{r^2} + \frac{2LS + 2L^2}{r^3} - Q^2 \frac{(L+S)^2}{r^4}$$

Outline

Effective one–body problem

- Newton: 2–body problem can be reduced to a one–body problem
- Einstein: not possible in closed form
- series expansion method for successive reduction to an "effective" one–body problem within post–Newtonian expansion
- effective dynamics of two black holes described by

$$ds^2 c = -g_{tt}(r,\nu)dt^2 + g_{rr}(r,\nu)dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

with $u = 2(M_1 + M_2)/r$, $\nu = M_1 M_2/(M_1 + M_2)^2$ and

$$\begin{aligned} g_{tt}(r,\nu) &= 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + \mathscr{O}(u^5) \\ (g_{tt}(r,\nu) g_{rr}(r,\nu))^{-1} &= 1 + 6\nu u^2 + 2(26 - 3\nu)\nu u^3 + \mathscr{O}(u^4) \end{aligned}$$

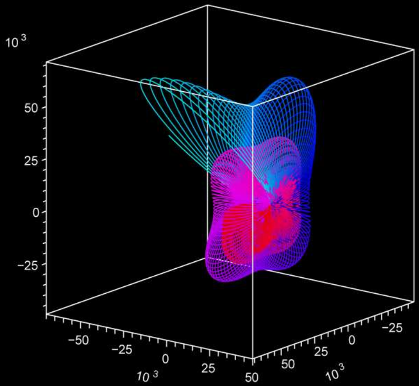
- effective one–body equation of motion

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{L^2}\frac{1}{g_{rr}g_{tt}}\left(E^2 - g_{tt}\left(\epsilon + \frac{L^2}{r^2}\right)\right)$$

Effective one–body problem

- physical questions
  - orbits
  - last stable orbit
  - last spherical stable orbit
  - last circular stable orbit
- has to be complemented by radiation reaction effects
  - variation of orbital parameters, variation of observables
  - gravitational radiation
  - inspiraling orbit
- can be supplemented by spin $\longrightarrow$ axially symmetric case

The End



Thank you

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