International Centre for Mathematical Sciences
Workshop October 11–15, 2010
The higher-genus sigma function and applications

Subvarieties of moduli spaces of curves

E. Previato (Boston University) ep@bu.edu

A mix of Computer-Algebra applications

0. Setting and problems

1. Invariant theory (Y. Demirbas)

2. PDEs (Y. Kodama and S. Matsutani)

3. Thetanulls (T. Shaska, S. Wijesiri)

4. AgM (E.S.A. Farrington)

0. Setting and problems

Motivation – A crash course on GIT

(smooth) Curves = Compact Riemann Surfaces

Automorphisms: Genus zero, one are exceptional (infinite group).

Moduli space $\mathcal{M}_g$ for genus 0 is one point, for genus 1 is described by one parameter, the j-invariant.

Weierstrass form:

$$y^2 = 4x^3 + b_2x^2 + b_4x + b_6,$$

$$j := (b_2^2 - 24b_4)^3/\Delta,$$

$$\Delta := -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6, \; b_8 := (b_2b_6 - b_4^2)/4,$$

Legendre normalization:

$$y^2 = x(x-1)(x-\lambda),$$

$$j = 2^8(\lambda^2 - \lambda + 1)^3/\lambda^2(\lambda-1)^2,$$

but the map from the λ line to $\mathcal{M}_1$ is $6:1$, with fibre given by $\{\lambda, (1-\lambda), \frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\}$, with the exception of the fibres over $j = 0, 1728$ which consist of 2, 3 points respectively.

The problem of classifying sets of k points in $\mathbb{P}^1$ is equivalent to that of classifying binary forms of degree k:

$$f = a_0X_0^k + a_1X_0^{k-1}X_1 + \ldots + a_kX_1^k,$$

under the natural action of $SL(2)$ on $\mathbb{P}^1$. This action induces naturally an action on the projective space of the coefficients: $(a_0, ..., a_k) \in \mathbb{P}^k$.

Fact I. [Newstead, Prop. 4.13] A binary form of degree k is stable (semi-stable) if and only if no point of $\mathbb{P}^1$ occurs as a point of multiplicity $\geq \frac{k}{2}$ ($> \frac{k}{2}$) for the given form.

Fact II. In the case of binary quartics that possess a simple root, after normalizing them: $X_0^3X_1 + aX_0X_1^3 + bX_1^4$ by putting one root 'at infinity' [1,0], the two invariants: $I = -\frac{a}{4}, J = -\frac{b}{16}$ generate the ring of invariant polynomials, $\frac{J^2}{\Delta}$ generates the ring of invariants inside the ring of regular functions of the affine variety $\mathbb{P}^4 \backslash (\Delta = 0)$, where $\Delta = I^3 - 27J^2$ is the discriminant of the other three roots, so the geometric quotient of $\mathbb{P}^4 \backslash (\Delta = 0)$ may be identified with the affine line. In terms of the cross-ratio λ of the four points, three of which are normalized to be [1,0], [0,1] and [1,1], or rather, of the six different values $\{\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1}, \frac{1}{1-\lambda}\}$ that are obtained permuting the points in all 24 possible ways, the invariant becomes:

$$\left(\frac{(2\lambda-1)(\lambda-2)(\lambda+1)}{\lambda(\lambda-1)}\right)^2 = \frac{3^6 J^2}{\Delta}.$$

This is a coarse moduli space; it is not fine as can be seen from the quotient morphism $\frac{J^2}{\Delta}: \mathbb{P}^4\backslash(\Delta = 0) \to \mathbb{A}^1$; indeed, in a neighborhood of 0 in $\mathbb{A}^1$ (or, of $-\frac{1}{27}$), the identity factors through $\frac{J^2}{\Delta}$, a coordinate x on $\mathbb{A}^1$ could be written as the quotient $\frac{f^2}{g}$, with f, g polynomials and $g \neq 0$. In this GIT context, the issue is that the stabilizers at those points are larger than at all other points.

The moduli space of elliptic curves is defined by the further choice of a point on the curve of genus 1, but as we saw it is also parametrized by $\mathbb{A}^1$. Taking ∞ as the origin, the curve in Legendre form is: $y^2 = x(x-1)(x-\lambda)$, with λ the same as above. There are six possible values of λ (as above) over one elliptic curve, whose j invariant is

$$j = \frac{2^8(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$

unless $j = 0$ (which has only two corresponding λ's, corresponding to the equianharmonic set of four points and to elliptic curves with 6 automorphisms) or $j = 1728$ ($\lambda = -1, 2, \frac{1}{2}$, corresponding to harmonic quadruples, for which the curve has 4 automorphisms).

Analytic counterpart: the moduli space $\mathcal{M}_g$ is described via the Siegel upper-half space $\mathfrak{H}_g$; for $g = 1$, the τ's in the (standard) fundamental domain corresponding to curves with automorphisms are $\imath$ and $e^{2\pi\imath/3}$.

In higher genus, analogous statements are not completely known. What unifies our mix of examples is the issue of defining equations for subvarieties of $\mathcal{M}_g$. One central question is the determination of the cohomology ring $\oplus_{k\geq 0}H^k(\mathcal{M}_g)$, significant in physical theories.

One example (surveyed in [FontanariP]) would serve to reduce the calculation of homology groups to the boundary and yield an induction procedure. We recall some classical notation.

The Riemann theta function is the fundamental (e.g., as solution of the heat equation) analytic function on $\mathbb{C}^g \times \mathfrak{H}_g$, defined as:

$$\vartheta(z,\Omega) = \sum_{n\in\mathbb{Z}^g} \exp(2\pi\imath {}^t n\Omega n + \pi\imath {}^t n z).$$

The Abel map $\mathcal{A}$, embeds the curve in the Jacobian and sends the s-fold symmetric product of the curve, S^sX, to $W^s \subset \mathrm{Jac}X$; the model depends on a choice of (normalized) basis of the first homology $H_1(X,\mathbb{Z})$, which gives a 'dual' basis of holomorphic differentials $\omega_1, ..., \omega_g$ on X, the period matrix Ω gives a model for the Jacobian, $\mathrm{Jac}X = \mathbb{C}^g/\Lambda$, $\Lambda := \mathbb{I}\vec{u}+\Omega\vec{v}$,

$\vec{u}, \vec{v} \in \mathbb{Z}^g$; The Jacobian together with a principal polarization can be identified with $\Omega \in \mathfrak{H}_g$. The Abel map is then:

$$\mathcal{A} : P \mapsto \int_{P_0}^{P} (\omega_1, ..., \omega_g),\; P_1 + ... + P_s \mapsto \sum_{i=1}^{s} \int_{P_0}^{P_i} (\omega_1, ..., \omega_g)$$

Riemann's Theorem: *If $\vartheta(e) \neq 0$, then $e \equiv \mathcal{A}(D) + \mathcal{A}(K_X)$ for a unique $D \in S^g X$ and $i(D)(= H^1(X, D)) = 0$, K_X the canonical divisor. If $\vartheta(e) = 0$ and s is least such that $1 \leq s \leq g-1 \Rightarrow \vartheta(W^s - W^s - e) \neq 0$, then for some $D \in S^{g-1}X, i(D) = s$, $e \equiv \mathcal{A}(D) + \mathcal{A}(K_X)$, and all partial derivatives of ϑ of order $< s$ (but not s) vanish at e.*

This theorem tells us that the Jacobian is principally polarized, namely the zero-divisor Θ_0 of ϑ has a one-dimensional space of sections, spanned in fact by ϑ; Lefschetz' theorem says that the linear system $|3\Theta_0|$ embeds the Jacobian into projective space; theta functions with characteristics $[n/k, m/k]$, $n, m \in \mathbb{Z}^g$, provide a basis of global sections for $|k\Theta_0|$. More generally, for two complex g-vectors α, β, the theta function with characteristics $[\alpha, \beta]$ is defined as:

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z) = \sum_{n \in \mathbb{Z}^g} \exp(\pi \imath^t (n+\alpha) \Omega (n+\alpha) + 2\pi \imath^t (n+\alpha)(z+\beta)).$$

A basis for $|2\Theta_0|$ is given by $\vartheta^2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z)$ with $\alpha\Omega + \beta$ a point of order two on the Jacobian; there are $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd such points, where the even are those such that $4\alpha \cdot \beta \equiv 0 \mod 2$, equivalently the dimension of space of sections of the corresponding theta divisor is even; the image of the Jacobian in $\mathbb{P}^{2^{2g}-1}$ under the $|2\Theta_0|$ map is called the Kummer variety; the "thetanulls" are the values of these functions at $z = 0$: they give a local immersion of Jacobians into the moduli space of ppav's.

We denote by $\mathcal{H}_g$ the subvariety of points that correspond to hyperelliptic curves. It was known to Riemann that one vanishing (even) thetanull characterizes hyperelliptic curves among curves of genus 3, and to Weber that two such vanishings characterize hyperelliptic curves in genus 4.

Conjecture *("$p-2$ conjecture", "which is most probably false"* [Accola]) The defining equations of $\mathcal{H}_p$ are given by the vanishing of $p-2$ even characteristics.

Accola also considers the stratification Θ^r of Teichmüller space, given by those surfaces where a theta function with period-2 characteristics vanishes to order $r+1$ at $\mathbf{0} \in \mathbb{C}^p$. He proves a "modified" $p-2$ conjecture in genus 5 [Part III, §4]: a point belongs to the hyperelliptic locus if and only if the theta function vanishes to order two at three half-periods, and to order one at their sum. Moreover, he proves the $p-1$ conjecture in genus 3, but "must allow (...) one-quarter integer theta characteristics".

Based on this, he later modified the conjecture: *It would be very appealing to find $p-2$ hypersurfaces in Teichmüller space (or whatever covering of the moduli space of smooth curves of genus p one prefers) and have the intersection of the hypersurfaces be precisely the hyperelliptic locus.*

Note: For genus one there are no vanishing thetanulls. The next-studied loci are given by vanishing derivatives of theta with odd characteristics, but again:

$$\theta' \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (0,\tau) = -\pi \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0,\tau) \theta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (0,\tau) \theta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0,\tau)$$

(Jacobi's derivative formula), is not zero for any τ.

PROBLEMS

- Covers of tori

• What groups may act on a curve of genus g, and what groups occur as automorphism groups (for a given genus).

• Equations for loci of covers of tori and curves with automorphisms; structure of loci (singularities, components)

Group theory: Two main methods
Method I: Fuchsian groups
Method II: Hurwitz spaces

1. Invariant theory

The locus of curves with automorphisms is a proper subvariety, singular locus of $\mathcal{M}_g$ if $g > 3$, whereas:

The components of the singular locus of $\mathcal{M}_3$ are: $S(3,0;1,1,1,1,2)$; $S(7,0;1,1,5)$; $S(2,1;1,1,1,1)$.

[Cornalba] determines the irreducible components of the singular loci: to state Cornalba's method and result, we recall his notation for $S(p,g';a_1,...,a_n)$, where p is a prime number and a_i are positive integers with $\sum a_i = p$: These are the points of $\mathcal{M}_g$ corresponding to curves X that are $p:1$ covers of a curve X' of genus g', totally ramified at n points (including the case $n=0$), so that $2-2g = p(2-2g')+n(p-1)$; the curve is (the normalization of one) defined by a section of a line bundle $\mathcal{O}(\sum a_i q_i)^{\frac{1}{p}}$. Equivalently, these are curves with an automorphism of order p, since the data of $S(p,g';a_1,...,a_n)$ determine the curve up to isomorphism. First, recall that for genus 2, the singular locus of the moduli space is one point, corresponding to the $\mathbb{Z}_{10}$ curve $y^2 = x(x^5-1)$, which is $S(5,0;1,1,3)$ in Cornalba notation, not to the one with maximum number of automorphisms! This curve, among the ones with extra automorphisms, has the peculiarity of being isolated, whereas the others occur in families, as can be seen from the table: as observed by Igusa, the D_{12} and the D_8 families are both specializations of the V_4 family; the $\mathbb{Z}_3 \rtimes D_8$ curve is a specialization of either one of the 1-dimensional families; and the $GL_2(3)$ curve is a specialization of the D_8 family.

[Igusa] determined the singular point of $\mathcal{M}_2$ by computing the Zariski tangent space in the local ring given by the I_{2k} invariants. When $g \geq 4$, by a standard argument of local deformation and the theorem on the purity of the branch locus, one knows that $\mathcal{S}ing(\mathcal{M}_g)$ is indeed the set of curves with extra automorphisms; in particular, is is a union of its largest components, corresponding to automorphisms of prime order, and Cornalba determines the inclusions among them. The argument to show that they are in fact components is similar to the one used for Hurwitz spaces, namely: for $g \geq 2$, a parameter count shows that $S(p, g'; a_1, ..., a_n)$ has dimension $3g' - 3 + n$. The moduli spaces $\mathcal{M}_g$ (case $n = 0$), as well as those parametrizing couples: (genus-g curves X', p-torsion point in JacX') are irreducible; and if $X_t \in S(p, X', D_t, L_t), 0 \leq t \leq 1$, is a family of p-fold covers of X' defined as above by a line bundle L_t whose p-th power is $\mathcal{O}(D_t)$, $D_t = a_1 q_1(t) + a_2 q_2 + ... + a_n q_n$, and $q_1(t)$ moves in a closed loop of homology class ξ, then L_1 equals $L_0 \otimes M$, where M is the p-torsion point in JacX' corresponding to $\frac{\xi}{p}$.

Genus 2.

Problem: description of the sublocus $\mathcal{L}_d \subset \mathcal{M}_2$, namely the curves X which admit a cover $\pi : X \to E$ of degree d, where E is an elliptic curve, or $\mathcal{L}_d^m$ when the cover is minimal

The two questions already give different subloci

Group	Curve	P Matrix
V_4	$y^2 = (x^2 - a^2)(x^2 - b^2)(x^2 - 1)$	$\begin{bmatrix} \tau & \frac{1}{2} \\ \frac{1}{2} & \tau' \end{bmatrix}$
D_8	$y^2 = x(x^2 - a^2)(x^2 - a^{-2})$	$\begin{bmatrix} \tau & \frac{1}{2} \\ \frac{1}{2} & \tau \end{bmatrix}$
D_{12}	$y^2 = (x^3 - a^3)(x^3 - a^{-3})$	$\begin{bmatrix} 2\tau & \tau \\ \tau & 2\tau \end{bmatrix}$
$\mathbb{Z}_3 \rtimes D_8$	$y^2 = x^6 - 1$	$\begin{bmatrix} \frac{2i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{2i}{\sqrt{3}} \end{bmatrix}$
$GL_2(3)$	$y^2 = x(x^4 - 1)$	$\begin{bmatrix} \frac{-1+i\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1+i}{2} \end{bmatrix}$
$\mathbb{Z}_{10}$	$y^2 = x(x^5 - 1)$	$\begin{bmatrix} 1 - \zeta^4 & -\zeta \\ -\zeta^2 - \zeta^4 & 1-\zeta \end{bmatrix}$

where $\zeta = e^{\frac{2\pi i}{5}}$ is a primitive fifth root of unity. The corresponding curve does not cover an elliptic curve.

[Demirbas] uses methods of [Igusa] to list the hyperelliptic curves with automorphisms and the groups, in genus 3 and 4, over an algebraically closed field of characteristic 2. Next project: find the singular loci of $\mathcal{H}_3, \mathcal{H}_4$, not complete in [Lønsted]; Igusa's method would require at a minimum the knowledge of the ring of invariants and covariants of 8, 10 points on the line, respectively. According to [Newstead], it is known only for $d = 4, 5, 6, 8$ points; but in the case of 8, we need covariants besides and [Shioda] does not succeed in giving complete relations. Alternative strategy: [Cornalba]'s method via Galois extensions of curves [recent work by K. Altmann]. Both are computer-algebra based works-in-progress.

Note: The ring of invariants and covariants for pencils of binary cubics were computed recently (over $\mathbb{C}$), using Klein's coordinates on the Grassmannian of points in $\mathbb{P}^3$, and have applications to instanton theory [Newstead].

2. PDEs

As reported in [GesztesyHoldenMichorTeschl], in the early 1950s close-to-periodic and soliton solutions were observed in lattice models with non-linear interaction.

In [Toda], a model that supports exact periodic and soliton solutions was introduced, a nonlinear lattice with exponential interaction:

$$x_{tt}(n,t) = e^{(x(n-1,t)-x(n,t))} - e^{(x(n,t)-x(n+1,t))},$$
$$(n,t) \in \mathbb{Z} \times \mathbb{R}.$$

This turned out to be a completely-integrable, nonlinear partial differential-difference equation.

Original Solution (Genus One Case)

Let X be an elliptic curve given by

$$X: \frac{1}{4}\hat{y}^2 = y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$
$$= (x-e_1)(x-e_2)(x-e_3),$$

with e's in $\mathbb{C}$ and $\lambda_2 = -(e_1+e_2+e_3) = 0$.
Note: We do not pursue the issue of real-valuedness of the solutions [Kodama].

The Weierstrass elliptic σ function associated with the curve X is connected with the Weierstrass $\wp$ and ζ functions by

$$\wp(u) = -\frac{d^2}{du^2}\log\sigma(u), \quad \zeta(u) = \frac{d}{du}\log\sigma(u),$$

the coordinate u in the universal cover of the Jacobian Jac(X) =J is given by

$$u = \int_{\infty}^{(x,y)} \frac{dx}{2y},$$

with $x(u) = \wp(u)$, $\hat{y}(u) = \wp'(u)$ and ∞ the point at infinity of X.
The Jacobian is given by Jac$(X) = \mathbb{C}/(\mathbb{Z}\omega' + \mathbb{Z}\omega'')$ using the double period (ω', ω'').

The key to obtain a $\wp$ function solution of the Toda lattice is the addition formula,

$$\wp(u) - \wp(v) = \frac{\sigma(v+u)\sigma(u-v)}{[\sigma(v)\sigma(u)]^2}.$$

By differentiating the logarithm of this formula with respect to u twice, we have

$$-\frac{d^2}{du^2}\log[\wp(u) - \wp(v)] = \wp(u+v) - 2\wp(u) + \wp(u-v).$$

For a constant number u_0, by letting

$$u = nu_0 + t + t_0, \quad v = u_0,$$

we compute

$$\begin{gathered}-\tfrac{d^2}{dt^2}\log[\wp(nu_0 + t + t_0) - \wp(u_0)] \\ = [\wp((n+1)u_0 + t + t_0) - \wp(u_0)] \\ -2[\wp(nu_0 + t) - \wp(u_0)] \\ +[\wp((n-1)u_0 + t) - \wp(u_0)].\end{gathered}$$

By letting

$$\begin{aligned} V_n(t) &:= -\wp(nu_0 + t + t_0), \quad V_c := -\wp(u_0) \\ q_n &:= -\log[V_n(t) - V_c], \end{aligned}$$

we have

$$-\frac{d^2}{dt^2} q_n = e^{-q_{n+1}} - 2e^{-q_n} + e^{-q_{n-1}} \qquad (n = 1, 2, \ldots).$$

For $q_n = Q_n - Q_{n-1}$, then ($Q_n = \sum_{i=1}^n q_n + Q_0$), Q_n obeys the Toda lattice equation:

$$\frac{d^2}{dt^2} Q_n = e^{Q_n - Q_{n+1}} - e^{Q_{n-1} - Q_n} \qquad (n = 1, 2, \ldots).$$

Here we assume $Q_0 \equiv 0$ which corresponds to the base point of the oscillation.

By letting $t_0 = -\omega''$, we have the relation:

$$\wp(t+nu_0-\omega'') = (e_1-e_3)\,\mathrm{dn}((e_1-e_3)^{1/2}(u+nu_0))^2+e_1,$$

which provides the connection with Toda's original solution for $e_1, e_2, e_3 \in \mathbb{R}$.

In detail, for the hyperelliptic case, we exhibit θ, and the σ function as a generalization of the Weierstrass elliptic σ function.

Let X be a genus-g hyperelliptic curve defined by

$$X \;:\; y^2 = f(x) := x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_0$$

together with a smooth point ∞ at infinity. Let the affine ring related to X be $R_g := \mathbb{C}[x, y]/(y^2 - f(x))$. Here λ's are complex numbers. We fix a basis of holomorphic one-forms $\nu_i^I = \frac{x^{i-1}dx}{2y}$ $(i = 1, \ldots, g)$. We also fix a homology basis for the curve X so that

$$H_1(X, \mathbb{Z}) = \oplus_{j=1}^{g} \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g} \mathbb{Z}\beta_j,$$

where the intersections are given by $[\alpha_i, \alpha_j] = 0$, $[\beta_i, \beta_j] = 0$ and $[\alpha_i, \beta_j] = -[\beta_i, \alpha_j]\delta_{ij}$. We take the half-period matrix $\omega = \begin{bmatrix} \omega' \\ \omega'' \end{bmatrix}$ of X_g with respect to the given basis where

$$\omega' = \frac{1}{2}\left[\oint_{\alpha_j} \nu_i^I\right], \quad \omega'' = \frac{1}{2}\left[\oint_{\beta_j} \nu_i^I\right],$$

Let Λ be the lattice in $\mathbb{C}^g$ generated by the column vectors in $2\omega'$ and $2\omega''$. The Jacobian variety of X is denoted by J and is identified with $\mathbb{C}^g/\Lambda$. For a non-negative integer k, we define the Abel map from the k-th symmetric product $Sym^k X$ of the curve X to J, $w : Sym^k X \to \mathrm{J}$ by,

$$w((x_1, y_1), \ldots, (x_k, y_k)) = \sum_{i=1}^{k} \int_{\infty}^{(x_i, y_i)} \begin{bmatrix} \nu_1^I \\ \vdots \\ \nu_g^I \end{bmatrix} \mod \Lambda.$$

The image of w is denoted by $\mathcal{W}_k = w(Sym^k X_g)$. The mapping w is surjective when $k = g$ by Abel's theorem, and is injective if we restrict it to the pre-image of the complement of a specific connected Zariski closed subset of dimension at most $g - 2$ in J, by Jacobi's theorem.

We define differentials of the second kind,

$$\nu_j^{II} = \frac{1}{2y} \sum_{k=j}^{2g-j} (k + 1 - j)\lambda_{k+1+j} x^k dx, \quad (j = 1, \ldots, g)$$

and complete hyperelliptic integrals of the second kind

$$\eta' = \frac{1}{2}\left[\oint_{\alpha_j} \nu_i^{II}\right], \quad \eta'' = \frac{1}{2}\left[\oint_{\beta_j} \nu_i^{II}\right].$$

For these bases the half-periods $\omega', \omega'', \eta', \eta''$ satisfy the generalized Legendre relation

$$\mathfrak{M} \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix} \mathfrak{M}^T = \frac{\imath\pi}{2} \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}.$$

where $\mathfrak{M} = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}$.

Let $\mathbb{T} = {\omega'}^{-1}\omega''$. The theta function on $\mathbb{C}^g$ with modulus $\mathbb{T}$ and characteristics $\mathbb{T}a + b$ for $a, b \in \mathbb{C}^g$ is given by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z; \mathbb{T}) = \sum_{n \in \mathbb{Z}^g} e^{\left[2\pi\imath\left(\frac{1}{2}\ {}^t\,(n+a)\mathbb{T}(n+a) +\ {}^t\,(n+a)(z+b)\right)\right]}.$$

The σ-function is an analytic function on $\mathbb{C}^g$, is associated to the theta function, and has modular invariance of a given weight with respect to $\mathfrak{M}$:

$$\sigma(u) = \gamma_0 \exp\left(-\frac{1}{2}{}^t u\eta'{\omega'}^{-1}u\right) \theta \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} \left(\frac{1}{2}{\omega'}^{-1}u\, ; \mathbb{T}\right),$$

where δ' and δ'' are half-integer characteristics giving the vector of Riemann constants with basepoint at ∞

and γ_0 is a certain non-zero constant. The σ-function vanishes simply on $\kappa^{-1}(\mathcal{W}_{g-1})$, where the map κ is the natural projection, $\kappa : \mathbb{C}^g \longrightarrow \mathrm{J}$.

The Kleinian $\wp$ and ζ functions are defined by

$$\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u), \quad \zeta_i = \frac{\partial}{\partial u_i} \log \sigma(u).$$

Let $\{\phi_i(x,y)\}$ be an ordered set of $\mathbb{C} \cup \{\infty\}$-valued functions over X defined by

$$\phi_i(x,y) = \begin{cases} x^i & \text{for } i \le g, \\ x^{\lfloor (i-g)/2 \rfloor + g} & \text{for } \ i > g \text{ and } i-g \ \text{ even}, \\ x^{\lfloor (i-g)/2 \rfloor} y & \text{for } \ i > g \text{ and } i-g \text{ odd}. \end{cases}$$

We note that $\{\phi_i(x,y)\}$ is a basis of R_g as a $\mathbb{C}$ vector space.

Following [Ônishi], we introduce a multi-index $\natural^n$. For n with $1 \le n < g$, we let $\natural^n$ be the set of positive integers i such that $n+1 \le i \le g$ with $i \equiv n+1 \bmod 2$. Namely,

$$\natural^n = \begin{cases} n+1, n+3, \dots, g-1 & \text{for } g-n \equiv 0 \bmod 2 \\ n, n+2, \dots, g & \text{for } g-n \equiv 1 \bmod 2 \end{cases}$$

and partial derivative over the multi-index $\natural^n$

$$\sigma_{\natural^n} = \left(\prod_{i \in \natural^n} \frac{\partial}{\partial u_i} \right) \sigma(u).$$

For $n \geq g$, we define $\natural^n$ as empty and $\sigma_{\natural^n}$ as σ itself. The first few examples are given in Table 1, where we let $\sharp$ denote $\natural^1$ and $\flat$ denote $\natural^2$.

Table 1

g	$\sigma_\sharp \equiv \sigma_{\natural^1}$	$\sigma_\flat \equiv \sigma_{\natural^2}$	$\sigma_{\natural^3}$	$\sigma_{\natural^4}$	$\sigma_{\natural^5}$	$\sigma_{\natural^6}$	$\sigma_{\natural^7}$	$\sigma_{\natural^8}$
1	σ	σ	σ	σ	σ	σ	σ	σ
2	σ_2	σ	σ	σ	σ	σ	σ	σ
3	σ_2	σ_3	σ	σ	σ	σ	σ	σ
4	σ_{24}	σ_3	σ_4	σ	σ	σ	σ	σ
5	σ_{24}	σ_{35}	σ_4	σ_5	σ	σ	σ	σ
6	σ_{246}	σ_{35}	σ_{46}	σ_5	σ_6	σ	σ	σ
7	σ_{246}	σ_{357}	σ_{46}	σ_{57}	σ_6	σ_7	σ	σ
8	σ_{2468}	σ_{357}	σ_{468}	σ_{57}	σ_{68}	σ_7	σ_8	σ
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

For $u \in \mathbb{C}^g$, we denote by u' and u'' the unique vectors in $\mathbb{R}^g$ such that

$$u = 2\,{}^t\omega' u' + 2\,{}^t\omega'' u''.$$

We define

$$L(u,v) = {}^t u(2\,{}^t\eta' v' + 2\,{}^t\eta'' v''),$$
$$\chi(\ell) = \exp\left[2\pi\imath\left({}^t\ell'\delta'' - {}^t\ell''\delta' + \frac{1}{2}{}^t\ell'\ell''\right)\right] \quad (\in \{1, -1\})$$

for u, $v \in \mathbb{C}^g$ and for ℓ $(= 2\,{}^t\omega'\ell' + 2\,{}^t\omega''\ell'') \in \Lambda$. Then $\sigma_{\natural^n}(u)$ for $u \in \kappa^{-1}(\mathcal{W}_1)$ satisfies the periodicity relation:

$$\sigma_{\natural^n}(u+\ell) = \chi(\ell)\sigma_{\natural^n}(u)\exp L(u+\frac{1}{2}\ell, \ell) \text{ for } u \in \kappa^{-1}(\mathcal{W}_1).$$

For $n \leqq g$, $\sigma_{\natural^n}(-u) = (-1)^{ng+\frac{1}{2}n(n-1)}\sigma_{\natural^n}(u)$ for $u \in \kappa^{-1}(\mathcal{W}_n)$, especially,

$$\begin{cases} \sigma_\flat(-u) = -\sigma_\flat(u) & \text{for } u \in \kappa^{-1}(\mathcal{W}_2) \\ \sigma_\sharp(-u) = (-1)^g\sigma_\sharp(u) & \text{for } u \in \kappa^{-1}(\mathcal{W}_1) \end{cases}$$

Key for the result are addition formulas of the hyperelliptic σ functions [EilbeckEnolskiiMatsutaniÔnishiP] which generalize the genus-1 case.

Generalized Frobenius-Stickelberger determinant.

For a positive integer $n \geq 1$ and $(x_1, y_1), \cdots, (x_n, y_n)$ in X_g, we define the Frobenius-Stickelberger determinant, $\Psi_n((x_1, y_1), \cdots, (x_n, y_n)) :=$

$$\begin{vmatrix} 1 & \phi_1(x_1, y_1) & \cdots & \phi_{n-2}(x_1, y_1) & \phi_{n-1}(x_1, y_1) \\ 1 & \phi_1(x_2, y_2) & \cdots & \phi_{n-2}(x_2, y_2) & \phi_{n-1}(x_2, y_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_1(x_{n-1}, y_{n-1}) & \cdots & \cdot & \phi_{n-1}(x_{n-1}, y_{n-1}) \\ 1 & \phi_1(x_n, y_n) & \cdots & \phi_{n-2}(x_n, y_n) & \phi_{n-1}(x_n, y_n) \end{vmatrix}$$

We prove the following theorem:

For a positive integer $n > 1$, let $(x_1, y_1), \ldots, (x_n, y_n)$ in X, and $u^{(1)}, \ldots, u^{(n)}$ in $\kappa^{-1}(\mathcal{W}_1)$ be points such that $u^{(i)} = w((x_i, y_i))$. Then the following relation holds:

$$\frac{\sigma_{\natural^n}(\sum_{i=1}^n u^{(i)}) \prod_{i<j} \sigma_\flat(u^{(i)} - u^{(j)})}{\prod_{i=1}^n \sigma_\sharp(u^{(i)})^n}$$
$$= \epsilon_n \Psi_n((x_1, y_1), \cdots, (x_n, y_n)),$$

where $\epsilon_n = (-1)^{g+n(n+1)/2}$ for $n \leq g$
and $\epsilon_n = (-1)^{(2n-g)(g-1)/2}$ for $n \geq g+1$.

The Picard group of the curve X has the following addition structure:

For given $P_1, \cdots, P_n \in X$, we define

$$\mu_n(P; P_1, \ldots, P_n) = \lim_{Q_i \to P_i} \frac{\Psi_{n+1}(P, Q_1, \ldots, Q_n)}{\Psi_n(Q_1, \ldots, Q_n)}$$

for distinct Q_i's.

For given $P_1, \ldots, P_n \in X$, we have $Q_i, \ldots, Q_\ell$ with $\ell = g$ for $n \geq g$ and $\ell = n$ otherwise, such that

$$P_1 + P_2 + \ldots + P_n + Q_1 + Q_2 + \ldots + Q_\ell - (n+\ell)\infty \sim 0$$

as a non trivial zero of $\mu_n(P; P_1, \ldots, P_n)$. For each $Q_i = (x_i, y_i)$, by letting $-Q_i = (x_i, -y_i)$, we have the addition property,

$$P_1 + P_2 + \ldots + P_n - n\infty \sim (-Q_1) + \ldots + (-Q_\ell) - (-\ell)\infty.$$

The hyperelliptic involution $\iota : (x, y) \mapsto (x, -y)$ induces the $[-1]$-action on J, $u \mapsto -u$.

In this sense, since we fix the base point ∞, we use the notations, $n(x, y)$ for a point $(x, y) \equiv P \in X_g$, addition $(x, y) + (x', y')$, and equality "="; for example $(x, y) + (x', y') \neq (x + x', y + y')$ and $2(x, y) \neq (2x, 2y)$ in general. We note that for every $P \in X$, $2P = -\sum_{i=1}^{2g-1} Q_i$ for some points Q_i's.

We give the addition formula for the hyperelliptic σ functions:

Assume that (m,n) is a pair of positive integers. Let (x_i,y_i) $(i=1,\dots,m)$, (x'_j,y'_j) $(j=1,\dots,n)$ in X and $u\in\kappa^{-1}(\mathcal{W}_m)$, $v\in\kappa^{-1}(\mathcal{W}_n)$ be points such that $u=w((x_1,y_1),\cdots,(x_m,y_m))$ and $v=w((x'_1,y'_1),\cdots,(x'_n,y'_n))$. Then the following relation holds:

$$\frac{\sigma_{\natural^{m+n}}(u+v)\sigma_{\natural^{m+n}}(u-v)}{\sigma_{\natural^m}(u)^2\sigma_{\natural^n}(v)^2}$$

$$=\delta(g,m,n)\frac{\prod_{i=0}^{1}\Psi_{m+n}((\mathbf{x},\mathbf{y}),(\mathbf{x}',(-1)^i\mathbf{y}'))}{[\Psi_m((\mathbf{x},\mathbf{y}))\Psi_n((\mathbf{x}',\mathbf{y}'))]^2}$$

$$\times\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{\Psi_2((x_i,y_i),(x'_j,y'_j))}$$

where $\delta(g,m,n)=(-1)^{gn+\frac{1}{2}n(n-1)+mn}$.

For $m=g$ and $n=2$ we derive the Corollary [KodamaMatsutaniP]:

Let $(x_i,y_i)\in X_g$ $(i=1,\dots,g)$, $(x'_j,y'_j)\in X_g$ $(j=1,2)$, $u\in\mathbb{C}^g$, $v:=v^{[1]}+v^{[2]}\in\kappa^{-1}(\mathcal{W}_2)$, and $v^{[j]}\in\kappa^{-1}(\mathcal{W}_1)$ $(j=1,2)$ be points such that $u=w((x_1,y_1),\dots,(x_g,y_g))$ and $v^{[j]}=w((x'_j,y'_j))$, $(j=1,2)$. Then the following relation holds:

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma_\flat(v)^2}=-\Xi(u,v),$$

where $\Xi(u,v)$ is equal to

$$F(x_1')F(x_2')\left(\sum_{i=1}^{g}\frac{y_i}{(x_i-x_1')(x_i-x_2')F'(x_i)}\right)^2 -F(x_1')F(x_2')\left(\sum_{i=1}^{2}\frac{(-1)^i y_i'}{(x_1'-x_2')F(x_i')}\right)^2,$$

and $F(x) := (x-x_1)(x-x_2)\cdots(x-x_g)$ and $F'(x) := \partial F(x)/\partial x$.

The proof is a Vandermonde:

$$\Delta(x_1,x_2,\ldots,x_\ell) = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{\ell-1} \\ 1 & x_2 & \cdots & x_2^{\ell-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_\ell & \cdots & x_\ell^{\ell-1} \end{vmatrix} = \prod_{i,j=1,i<j}^{\ell}(x_j-x_i),$$

we have $\Psi_{g+2}((x_1,y_1),\ldots(x_g,y_g),(x_1',\pm y_1'),(x_2',\pm y_2'))$

$$
= \begin{vmatrix}
1 & x_1 & \cdots & x_1^g & y_1 \\
1 & x_2 & \cdots & x_2^g & y_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_g & \cdots & x_g^g & y_g \\
1 & {x'}_1 & \cdots & {x'}_1^g & \pm {y'}_1 \\
1 & {x'}_2 & \cdots & {x'}_2^g & \pm {y'}_2
\end{vmatrix}
$$

$$
= \sum_{i=1}^{g} \Delta(x_1, \ldots, \check{x}_i, \cdots, x_g, x'_1, x'_2) y_i
$$

$$
\pm \sum_{i=1}^{2} (-1)^i \Delta(x_1, \cdots, x_g, x'_{3-i}) y'_i.
$$

[Baker] proved:

Let $(x_i, y_i) \in X$ $(i = 1, \ldots, g)$ and $u \in \mathbb{C}^g$ such that $\kappa(u) = w((x_1, y_1), \ldots, (x_g, y_g))$. The following relation holds for generic x'_i $(i = 1, 2)$,

$$
\sum_{i=1}^{g} \sum_{j=1}^{g} \wp_{ij}(u) {x'_1}^{i-1} {x'_2}^{j-1}
$$

$$
= F(x'_1) F(x'_2) \left(\sum_{i=1}^{g} \frac{y_i}{(x'_1 - x_i)(x'_2 - x_i) F'(x_i)} \right)^2
$$

$$
- \frac{f(x'_1) F(x'_2)}{(x'_1 - x'_2)^2 F(x'_1)} - \frac{f(x'_2) F(x'_1)}{(x'_1 - x'_2)^2 F(x'_2)}
$$

$$
+ \frac{f(x'_1, x'_2)}{(x'_1 - x'_2)^2}.
$$

where

$$f(x_1, x_2) = \sum_{i=0}^{g} x_1^i x_2^i (\lambda_{2i+1}(x_1 + x_2) + 2\lambda_{2i}).$$

We can conclude:

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2 \sigma_\flat(v)^2} =$$

$$\frac{f(x'_1, x'_2) - 2y'_1 y'_2}{(x'_1 - x'_2)^2} - \sum_{i=1}^{g} \sum_{j=1}^{g} \wp_{ij}(u) {x'_1}^{i-1} {x'_2}^{j-1},$$

which correspond to a formula in [Fay], underlying the "trisecant identity".

Note: When $v^{[1]} = v^{[2]}$,

$$\frac{\sigma(u + 2v^{[1]})\sigma(u - 2v^{[1]})}{\sigma(u)^2 \sigma_\flat(2v^{[1]})^2}$$

$$= f_{1,2}(x'_1) - \sum_{i=1}^{g} \sum_{j=1}^{g} \wp_{ij}(u) {x'_1}^{i+j-2}$$

$$= - \lim_{x'_2 \to x'_1} \Xi(u, v),$$

where

$$f_{1,2}(x) := \frac{\partial_x^2 f(x)}{2f(x)} - f_{1,2}^I(x),$$

$$f_{1,2}^{I}(x) := \sum_{i=0}^{g}(i^2\lambda_{2i+1}x^{2i-1} + i(i-1)\lambda_{2i}x^{2i}).$$

When $v^{[2]} = 0$ or $(x_2', y_2') = \infty$,

$$\frac{\sigma(u+v^{[1]})\sigma(u-v^{[1]})}{\sigma(u)^2\sigma_\flat(v^{[1]})^2} =$$

$${x_1'}^g - \sum_{i=1}^{g} \wp_{gj}(u){x_1'}^{i+j-2} =$$

$$F(x_1') = -(x_1' - x_1)(x_1' - x_2)\cdots(x_1' - x_g).$$

'Algebraic' differential operators on the universal cover of the Jacobian:

For $j = 1$ or 2, define:

$$D_j := \sum_{i=1}^{g} {x'_j}^{i-1} \frac{\partial}{\partial u_i}$$

$$= \frac{1}{\Delta(x_1, x_2, \ldots, x_g)} \begin{vmatrix} 1 & x_1 & \cdots & x_1^{g-1} & 2y_1\partial_{x_1} \\ 1 & x_2 & \cdots & x_2^{g-1} & 2y_2\partial_{x_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_g & \cdots & x_g^{g-1} & 2y_g\partial_{x_g} \\ 1 & x'_j & \cdots & {x'_j}^{g-1} & 0 \end{vmatrix}$$

$$= \sum_{i=1}^{g} \frac{y_i F(x'_j)}{F'(x_i)(x'_j - x_i)} \frac{\partial}{\partial x_i},$$

then for $j, j' = 1$ or 2,

$$[D_j, D_{j'}] \equiv D_j D_{j'} - D_{j'} D_j = 0.$$

Also: For $v^{(j)} = w(x'_j, y'_j)$ with $j = 1, 2$,

$$\frac{\partial}{\partial x'_j} = \frac{1}{2y'_j} \sum_{i=1}^{g} {x'_1}^{i-1} \frac{\partial}{\partial {v^{(j)}}_i}.$$

For $h \in \Gamma(\mathbb{C}^g, \mathcal{O}(\mathbb{C}^g))$ and $j = 1, 2$,

$$D_{j'} h(u + v^{(j)}) := 2y'_j \frac{\partial}{\partial x'_j} h(u + v^{(j)}) = D_j h(u + v^{(j)}).$$

One checks:

$$D_1 \log \sigma(u+v) = \frac{1}{2}\left(D_1 \log \Xi(u,v) + D_{1'} \log \Xi(u,v)\right) + D_1 \log \sigma(u) + D_{1'} \log \sigma_\flat(v),$$

$$\begin{aligned} D_1 D_2 \log \sigma(u+v) &= \frac{1}{2} D_1 D_2 \log \Xi(u,v) \\ &+ \frac{1}{2} D_1 D_{2'} \log \Xi(u,v) + D_1 D_2 \log \sigma(u) \\ &= \frac{1}{2} D_1 D_2 \log \Xi(u,v) \\ &+ \frac{1}{2} D_2 D_{1'} \log \Xi(u,v) + D_1 D_2 \log \sigma(u) \end{aligned}$$

Now we can give the σ function solution of the Toda lattice equation:

Let $(x_i, y_i) \in X$ $(i = 1, \ldots, g)$, $(x_1', y_1') \in X$ $u \in \mathbb{C}^g$, and $v^{[1]} \in \kappa^{-1}(\mathcal{W}_1)$ be points such that $u = w((x_1, y_1), \cdots, (x_g, y_g))$ and $v^{[1]} = w((x_1', y_1'))$.

Define $c := 2v^{[1]}$, $\tilde{D}_1 = \sigma_\flat(c) D_1$,

$$\mathcal{V}(u) := \mathcal{V}(u, v^{[1]}) := \sum_{i=1}^{g} \sum_{j=1}^{g} \wp_{ij}(u) {x_1'}^{i+j-2},$$

$$\mathcal{V}_c(v^{[1]}) := f_{1,2}(x_1'),$$

and $t := (t_{11}, t_{12}, \dots, t_{1g}) \in \mathbb{C}^g$ with

$$t_{1j} := (x_1')^{1-j} \sum_{i=1}^{g} \int_{\infty}^{(x_i,y_i)} \nu_j^I, \qquad (j = 1, 2, \dots, g).$$

Then with the coordinate change $u = nc + t^\perp + t$

$$\begin{aligned} -D_1^2 \log \big(\mathcal{V}(t + nc + t^\perp) - \mathcal{V}_c(c)\big) \\ = \mathcal{V}(t + (n+1)c + t^\perp) - 2\mathcal{V}(t + nc + t^\perp) \\ + \mathcal{V}(t + (n-1)c + t^\perp). \end{aligned}$$

And Hirota's bilinear equation,

$$\begin{aligned} \sigma(t + nc + t^\perp)\tilde{D}_1^2 \sigma(t + nc + t^\perp) \\ -\tilde{D}_1 \sigma(t + nc + t^\perp)\tilde{D}_1 \sigma(t + nc + t^\perp) \\ -\mathcal{V}_c(c)\sigma(t + nc + t^\perp)^2 \\ -\sigma(t + (n+1)c + t^\perp)\sigma(t + (n-1)c + t^\perp) = 0. \end{aligned}$$

By letting $\mathcal{V}_n(t + t^\perp) := \mathcal{V}(t + nc + t^\perp)$ and $q_n(t) := -\log \big(\mathcal{V}_n(t + t^\perp) - \mathcal{V}_c(c)\big)$,

$$D_1^2 q_n(t) = e^{-q_{n+1}} - 2e^{-q_n} + e^{-q_{n-1}}.$$

Aside. In [ChoNakayashiki], the related problem of finding the $\mathcal{D}$-module structure of the space of abelian functions of a ppav (J, Θ) is posed, in the case that Θ is non-singular,

$$\mathcal{D} := \mathbb{C}[\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_g}].$$

The starting point is the observation that the classical Frobenius-Stickelberger formula gives the $\mathcal{D}$-module structure of the ring of elliptic functions generated by 1 and $\wp$, specifically in the $\mathbb{C}$-basis $1, \wp, \wp', \wp'', ...$ More generally [MatsutaniP], let X be a Burchnall-Chaundy curve whose affine equation is given by $f(x, y) = H(y) - h(x)$, where $H(y) \in \mathbb{C}[y]$, $h(x) \in \mathbb{C}[x]$, $\deg_y H(y) = r$ and $\deg_x h(x) = s$, $(r < s)$. Then

1) we have a derivation of R, the ring of functions regular on $X \backslash \infty$, given by,

$$D := H'(y)\frac{d}{dx} : R \to R,$$

where $H'(y) = \partial H(y)/\partial y$,

2) $du_1 := \dfrac{dx}{H'(y)}$ is a holomorphic one form, $\langle D, du_1 \rangle = 1$,

3) For smallest positive integers p and q satisfying

the relation $ps - qr = 1$, $du_g := \dfrac{y^{r-p-1}x^{q-1}dx}{H'(y)}$ is a holomorphic one form, $\langle \dfrac{\partial}{\partial z_\infty}, du_g \rangle = 1$.

This gives $R[D]$ the structure of a differential ring, and we can derive some algebraic relations:

For an integer $n > 1$, we define

$$\psi_\infty^{(n+1)} := \det \begin{bmatrix} D\phi_1 & D\phi_2 & \cdots & D\phi_n \\ D^2\phi_1 & D^2\phi_2 & \cdots & D^2\phi_n \\ \vdots & \vdots & \ddots & \vdots \\ D^n\phi_1 & D^n\phi_2 & \cdots & D^n\phi_n \end{bmatrix}.$$

It follows easily from the assumptions that $\psi_\infty^{(n)}$ is an element of R.

If $h(x) \in \mathbb{Q}[x]$, for an integer $m > s$,

$$D_x^m y \in \mathbb{Q}[y, D_x y, D_x^2 y, \dots, D_x^s y, 1/H'(y)].$$

If the polynomials h, H have rational coefficients, for every positive integer n, we have

$$\psi_\infty^{(n+1)} \in \mathbb{Q}[y, D_x y, D_x^2 y, \dots, D_x^s y].$$

entails that some of $\{\psi_\infty^{(n_i)}\}_{i=1,\cdots,t}$ ($t > s$, $n_i \neq n_j$ if $i \neq j$) are not algebraically independent and might satisfy a relation. In fact, in the case of genera $g = 1$

$(r=2, s=3)$ [Weber, p.196-200] and $g=2$ $(r=2, s=5)$ [BradenEnolskiiHone], [Cantor], [Matsutani], $\psi_\infty^{(n)}$'s obey the following relations: $g=1$ case:

$$\psi_\infty^{(m+n)}\psi_\infty^{(m-n)} = \det \begin{bmatrix} \psi_\infty^{(m-1)}\psi_\infty^{(n)} & \psi_\infty^{(m)}\psi_\infty^{(n+1)} \\ \psi_\infty^{(m)}\psi_\infty^{(n-1)} & \psi_\infty^{(m+1)}\psi_\infty^{(n)} \end{bmatrix},$$

and $g=2$ case: $\psi_\infty^{(2)2}\psi_\infty^{(m)}\psi_\infty^{(n)}\psi_\infty^{(m+n)}\psi_\infty^{(m-n)} =$

$$\det \begin{bmatrix} \psi_\infty^{(m-2)}\psi_\infty^{(n)} & \psi_\infty^{(m-1)}\psi_\infty^{(n+1)} & \psi_\infty^{(m)}\psi_\infty^{(n+2)} \\ \psi_\infty^{(m-1)}\psi_\infty^{(n-1)} & \psi_\infty^{(m)}\psi_\infty^{(n)} & \psi_\infty^{(m+1)}\psi_\infty^{(n+1)} \\ \psi_\infty^{(m)}\psi_\infty^{(n-2)} & \psi_\infty^{(m+1)}\psi_\infty^{(n-1)} & \psi_\infty^{(m+2)}\psi_\infty^{(n)} \end{bmatrix}.$$

This suggests the possibility of recursion relations and difference equations.

Division polynomials $F_n(x,y)$ arise in expressing the coordinates of nP in terms of those of P, a point of an elliptic curve in Weierstrass form.

Kiepert (1873) and Brioschi (1864) published algebraic equations for the n-division points of an elliptic curve, in terms of the Weierstrass $\wp$-function and its derivatives with respect to a uniformizing parameter, or another elliptic function, respectively.

For an elliptic curve $X: y^2 = 4x^3 - g_2x - g_3$ and an integer $n \geq 2$, an elliptic function ψ_n can be defined

[WW, Ch. XX, Misc. Ex. 24] by

$$\psi_n(u) = \frac{\sigma(nu)}{\sigma(u)^{n^2}},$$

where $\sigma(u)$ is the Weierstrass σ-function. The σ-function has a simple zero at each point of the period lattice [WW, Ch. XX 20·42], and the following can be proved similarly:

The $n^2 - 1$ zeros of ψ_n correspond to the group of points of period n minus the origin on the torus corresponding to X.

[Kiepert] showed that the ψ-functions have the following representation, $\psi_n(u) =$

$$\frac{(-1)^{n-1}}{(1!2!\cdots(n-1)!)^2}\begin{vmatrix} \wp'(u) & \wp''(u) & \cdots & \wp^{(n-1)}(u) \\ \wp''(u) & \wp'''(u) & \cdots & \wp^{(n)}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \wp^{(n-1)}(u) & \wp^{(n)}(u) & \cdots & \wp^{(2n-3)}(u) \end{vmatrix} =$$

$$\frac{(-1)^{n-1}(1/2)^{\lfloor (n-1)/2 \rfloor}}{1!2!\cdots(n-1)!}\times$$

$$\begin{vmatrix} x' & y' & \cdots & (x^{\lfloor (n-3)/2 \rfloor_+}y)' & (x^{\lfloor n/2 \rfloor})' \\ x'' & y'' & \cdots & (x^{\lfloor (n-3)/2 \rfloor_+}y)'' & (x^{\lfloor n/2 \rfloor})'' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{(n-1)} & y^{(n-1)} & \cdots & (x^{\lfloor (n-3)/2 \rfloor_+}y)^{(n-1)} & (x^{\lfloor n/2 \rfloor})^{(n-1)} \end{vmatrix},$$

where $\wp$ is the Weierstrass $\wp$-function, $\lfloor r \rfloor_+$ is equal to 0 for a real number $r < 0$ and to the floor function $\lfloor r \rfloor$ for $r \geq 0$, and the derivatives are taken with respect to u.

[Brioschi] gave another expression for the ψ-functions for $n > 2$, $\psi_n(u) =$

$$\epsilon y^{n(n-1)/2} \left(\prod_{k=\lfloor (n+2)/2 \rfloor}^{n-1} \frac{1}{k!} \right)$$

$$\times \begin{vmatrix} y_{\ell(n)x} & \cdots & y_{(L(n)+\ell(n))x} \\ y_{(\ell(n)+1)x} & \cdots & y_{(L(n)+\ell(n)+1)x} \\ \vdots & \ddots & \vdots \\ y_{(L(n)+\ell(n))x}(u) & \cdots & y_{(2L(n)+\ell(n))x}(u) \end{vmatrix},$$

where ϵ is a $\pm$ sign, $\ell(n) = n - 2\lfloor (n-3)/2 \rfloor - 1$, $L(n) = \lfloor (n-3)/2 \rfloor_+$, and ${}_{nx}$ is n-th derivative in x.

When $n = 2$, indeed $\sigma(2u)/\sigma(u)^4 = -y$ [WW, Ch. XX, Misc. Ex. 24]; the empty product should be interpreted as 1.

The hyperelliptic version of the ψ_n function for genus g over $w(X_g) = \kappa^{-1}\mathcal{W}_1$ is defined by

$$\psi_n(u) = \frac{\sigma(nu)}{\sigma_\sharp(u)^{n^2}}.$$

A zero u of ψ_n is a point such that $nu \in \kappa^{-1}\mathcal{W}_{g-1}$.

The translation formula for σ shows that ψ_n is defined on the curve, hence belongs to R_g, so it can be viewed as a generalization of a division polynomial.

By taking a certain limit of the σ expressions (when the points of the divisor coincide) we can give ψ_n in terms of ϕ_i's in R_g.

In [MatsutaniP] we gave a Kiepert-type formula. Let $n \geq 1$ be a positive integer. For

$$\psi_n(u) = \varepsilon_{n,g} \left| \begin{array}{cccc} \frac{\partial \phi_1}{\partial u_1} & \frac{\partial \phi_2}{\partial u_1} & \cdots & \frac{\partial \phi_{n-1}}{\partial u_1} \\ \frac{\partial^2 \phi_1}{\partial u_1^2} & \frac{\partial^2 \phi_2}{\partial u_1^2} & \cdots & \frac{\partial^2 \phi_{n-1}}{\partial u_1^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{n-1} \phi_1}{\partial u_1^{n-1}} & \frac{\partial^{n} \phi_2}{\partial u_1^{n-1}} & \cdots & \frac{\partial^{n-1} \phi_{n-1}}{\partial u_1^{n-1}} \end{array} \right|,$$

with $\psi_1 = 1$ and $\varepsilon_{n,g}$ is a plus/minus sign, the vanishing of ψ_n on $P \in X$ in is a necessary and sufficient condition for $w(n \cdot P)$ to belong to $\mathcal{W}_{g-1}$.

Moreover, let $n(\geq g)$, $k(< g)$ and $\ell := g - k - 1$ be non-negative integers. The vanishing of $\psi_{n+\ell}$, …, ψ_{n+1}, ψ_n, ψ_{n-1}, …, $\psi_{n-\ell}$, at a point P of X is a necessary and sufficient condition for $w(n \cdot P)$ to belong to $\mathcal{W}_k$.

The Brioschi-type expression of the ψ_n-function [Cantor, Matsutani[illegible]] is similarly,

$$\psi_n(u) = \begin{cases} \varepsilon'_{n,g}(2y)^{n(n-1)/2} \cdot T^{(g+2)}_{(n-g-1)/2}(y, \frac{d}{dx}) & n \not\equiv g \ (2) \\ \varepsilon'_{n,g}(2y)^{n(n-1)/2} \cdot T^{(g+1)}_{(n-g)/2}(y, \frac{d}{dx}) & n \equiv g \ (2) \end{cases}$$

Here $\varepsilon'_{n,g}$ is a plus/minus sign and $T_n^{(m)}$ is a Toeplitz determinant [Matsutani1],

$$T_n^{(m)}\left(g(s), \frac{d}{ds}\right) =$$

$$\left|\begin{array}{ccccc} g^{[m+n-1]} & g^{[m+n-2]} & \cdots & g^{[m+1]} & g^{[m]} \\ g^{[m+n]} & g^{[m+n-1]} & \cdots & g^{[m+2]} & g^{[m+1]} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{[m+2n-3]} & g^{[m+2n-4]} & \cdots & g^{[m+n-1]} & g^{[m+n-2]} \\ g^{[m+2n-2]} & g^{[m+2n-3]} & \cdots & g^{[m+n]} & g^{[m+n-1]} \end{array}\right|,$$

$T_{1-n}^{(m)}\left(g(s), \frac{d}{ds}\right) \equiv 0$ where m and n are positive integers, $g(s)$ is a function of an argument s and

$$g^{[n]}(s) := \frac{1}{n!}\frac{d^n}{ds^n} g(s).$$

Noting for $y^2 = f(x)$ that $y^{2n-1} d^n y/dx^n$ is a polynomial in x coprime to $f(x)$ in general, the function $y^{n(2m+2n-3)} T_n^{(m)}(y, \frac{d}{dx})$ is an element of $\mathbb{C}[x]$ and coprime (in the sense of having no zeros in common) to $y^2 = f(x)$. In conclusion, $\psi_n(u)$ can be expressed by

$$\psi_n = \begin{cases} (2y)^{g(g+1)/2}\alpha_n(x) & \text{for } \ n-g = \text{odd} \\ (2y)^{g(g-1)/2}\alpha_n(x) & \text{for } \ n-g = \text{even} \end{cases},$$

where $\alpha_n(x)$ is a polynomial in x and coprime to y. As shown in [Cantor], the degree of $\alpha_n(x)$ is

$$\deg(\alpha_n) = \begin{cases} \dfrac{g(n+g)(n-g) - g(2g+1)}{2} & n-g = \text{odd} \\ \dfrac{g(n+g)(n-g)}{2} & n-g = \text{even} . \end{cases}$$

We define

$$\Phi_n := \{P \in X \mid P \text{ is a zero of } \alpha_n\},$$

and for $n > g$,

$$\Xi_n := \Phi_{n-g+1} \cap \cdots \cap \Phi_{n-1} \cap \Phi_n \cap \Phi_{n+1} \cap \cdots \cap \Phi_{n+g-1}.$$

One should note here that there is no guarantee that Ξ_n is not empty.

[KacvanMoerbeke] gave a solution of the periodic Toda system:

The Hamiltonian of the Toda lattice equation is

$$H = \frac{1}{2}\sum_{k=1}^{N} P_k^2 + \sum_{k=1}^{N} \exp(Q_k - Q_{k+1}).$$

where $P_k = P_{k+N}$ and $Q_k = Q_{k+N}$. For Flaschka's coordinates, $a_k = \exp(Q_k - Q_{k+1})$ and $b_k = -P_k$, the equation of motion of H is reduced to

$$\begin{cases} \dfrac{d}{dt} a_k = a_k(b_{k+1} - b_k), \\ \dfrac{d}{dt} b_k = a_k - a_{k-1}, \qquad (k = 1, 2, \dots, N). \end{cases}$$

The Hamiltonian system admits the time inversion $t \mapsto -t$, which is identified with the hyperelliptic involution of X.

For brevity, we introduce the notation

$$\sigma^{(n)}(t; t^{\perp}) := \sigma(t + nc + t^{\perp}), \quad \sigma^{(c)} := \sigma_\flat(c),$$

$$\zeta^{(n)}(t; t^{\perp}) := \sum_{i=1}^{g} {x'_1}^{i-1} \zeta_i(t + nc + t^{\perp}),$$

$$\zeta^{(c)} := \frac{1}{2} D_{1'} \log \sigma_\flat(c),$$

$$\wp^{(n)}(t; t^{\perp}) := \sum_{i,j=1}^{g} {x'_1}^{i+j-2} \wp_{ij}(t + nc + t^{\perp}),$$

$$\wp^{(c)}(t^{\perp}) := f_{1,2}(x'_1).$$

The periodic solution of the Toda lattice are expressed by

$$a_n = \wp^{(n)}(t;t^{\perp}) - \wp^{(c)}(t^{\perp}) = \frac{\sigma^{(n+1)}(t;t^{\perp})\sigma^{(n-1)}(t;t^{\perp})}{\sigma^{(n)}(t:t^{\perp})^2 {\sigma^{(c)}}^2},$$

$$b_n = D_t \log \frac{\sigma^{(n)}(t;t^{\perp})}{\sigma^{(n-1)}(t;t^{\perp})} - \zeta_c$$
$$= \zeta^{(n)}(t;t^{\perp}) - \zeta^{(n-1)}(t;t^{\perp}) - \zeta^{(c)}.$$

The proof follows from the definition of a's and b's, by calculation time derivatives, and:

$$P_n - P_{n-1} = \zeta^{(n+1)}(t;t^{\perp}) - 2\zeta^{(n)}(t;t^{\perp}) + \zeta^{(n-1)}(t;t^{\perp})$$
$$\cdots\cdots$$
$$P_3 - P_2 = \zeta^{(4)}(t;t^{\perp}) - 2\zeta^{(3)}(t;t^{\perp}) + \zeta^{(2)}(t;t^{\perp})$$
$$P_2 - P_1 = \zeta^{(3)}(t;t^{\perp}) - 2\zeta^{(2)}(t;t^{\perp}) + \zeta^{(1)}(t;t^{\perp})$$
$$P_1 - P_0 = \zeta^{(2)}(t;t^{\perp}) - 2\zeta^{(1)}(t;t^{\perp}) + \zeta^{(0)}(t;t^{\perp})$$

with

$$P_n = \zeta^{(n+1)}(t;t^{\perp}) - \zeta^{(n)}(t;t^{\perp}) - (\zeta^{(1)}(t;t^{\perp}) - \zeta^{(0)}(t;t^{\perp})) + P_0.$$

The total momentum should be invariant and thus

$$P_0 = -(\zeta^{(1)}(t;t^{\perp}) - \zeta^{(0)}(t;t^{\perp})) + p_0.$$

where p_0 is a constant corresponding to $\zeta^{(c)}$.

Let $2N \geq g+1$. For a hyperelliptic curve of genus g which has a point $(x'_1, y'_1) \in \Xi_{2N}$, $\mathcal{V}(u)$ is a periodic solution of the Toda lattice equation such that $\mathcal{V}(u) = \mathcal{V}(u+Nc)$ with $c = 2w(x'_1, y'_1)$.

Genus-1 example: For b_n we have the formula:

$$\zeta(u+v) - \zeta(u) - \zeta(v) = \frac{y(u) - y(v)}{x(u) - x(v)}.$$

For periodic Toda with period $N = 3$ and $N = 4$, we choose a simple case (note that in genus-1, every elliptic curve has points of any finite order, so the Ξ_n are never empty): $X : y^2 = x^3 - x$. Then the division polynomials are given by

$$\begin{aligned}
\psi_1 &= 1, \\
\psi_2 &= -2y, \\
\psi_3 &= 3x^4 - 6x^2 - 1, \\
\psi_4 &= -2y(x^2+1)(x^2+2x-1)(x^2-2x-1), \\
\psi_5 &= 32x^{14} - 187x^{12} - 64x^{11} + 2x^{10} + 320x^9 - 233x^8 \\
&\qquad + 320x^7 - 52x^6 - 64x^5 - 61x^4 + 50x^2 + 1.
\end{aligned}$$

We have a $N = 3$ and $g = 1$ for $x'_3 = (1/3)\sqrt{9+6\sqrt{3}}$ and a $N = 4$ and $g = 1$ soliton for $x'_3 = \sqrt{2}+1$ or a zero of ψ_4.

Poncelet and Toda curves as subvarieties of $\mathcal{M}_g$ and $\mathcal{M}_{N-1}$.

Poncelet's theorem is a classical result ot projective geometry:

Let C and D be two smooth conics generally situated in the projective plane so that C contains D.
For an integer $N > 2$, if there exists a closed N-polygon inscribed in C and circumscribed about D, then for every point P in C there exists an N-polygon whose vertices are on C and include P, and whose sides are tangent to D.

The result was recently connected with the John boundary problem [BurskiiZhedanov], and explicit formulas for the vertices of the polygon were given in the case of two parabolas, in terms of the $\wp$-function. The fact that the sequence of vertices is obtained by addition of a fixed point on an elliptic curve is then quite explicit. The genus-1 periodic Toda lattice was derived.

Let C be given by the homogeneous equation $y^2 = xz$, and parameterized by $[x, x^2, 1]$. The sequence of vertices of an N-polygon is given by N points $[x_i^{(0)}, x_i^{(0)^2}, 1]$ $(i = 1, \ldots, N)$.

Let D be given by $[x, y, z]A[x, y, z]^t = 0$,, where

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix},$$

and we assume $a_5 = 0$.
The dual conic D^* of D is given by $[X, Y, Z]A^{-1}[X, Y, Z]^t = 0$. A pair $(P, L) \in C \times D^*$, $P \in L$, satisfies

$$xX + yY + zZ = 0,$$

for $P = [x, y, z]$ and $L = [X, Y, Z]$. This incidence relation is given by the elliptic curve E_1

$$w^2 = \frac{1}{a_2 + a_4}[x, x^2, 1]A[x, x^2, 1]^t,$$

where

$$w = \frac{1}{\sqrt{\det A}} \left(h_1(x)\frac{Y}{X} - h_2(x) \right),$$

and $h(x)$'s are polynomials in x. [GriffithHarris] showed, by Cayley's determinantal condition, that Poncelet's problem is equivalent to finding the matrix A and a point (x, w) belonging to E_1 that satisfies the equation of Kiepert and Brioschi, $\psi_N((x, w)) = 0$.

For such A, Poncelet's condition is equivalent to the fact that for the vertex

$$P_n = [x_n, x_n^2, 1] \in C, \quad x_n = \wp((n-1)u_0+t), \quad (n = 1, \ldots, N)$$

satisfies the N-periodic Toda lattice for all t:

$$\begin{aligned} -\frac{d^2}{dt^2} \log[\wp(nu_0 + t) - \wp(u_0)] \\ = [\wp((n+1)u_0 + t) - \wp(u_0)] - 2[\wp(nu_0 + t) - \wp(u_0)] \\ +[\wp((n-1)u_0 + t) - \wp(u_0)], \end{aligned}$$

where

$$u = \int_\infty^{(x,w)} \frac{dx}{2w}, \qquad u_0 = \int_\infty^{(x_1^{(0)}, w_1^{(0)})} \frac{dx}{2w}.$$

Here $x_n^{(0)} = \wp((n-1)u_0)$ for every n.

Assuming that the periodic Toda lattice gives a generalized Poncelet condition in higher genus, we compute the Kac-van Moerbeke spectral curve to express the Poncelet closure algebraically.

The Lax matrix for the periodic Toda lattice is given by [KacvanMoerbeke]:

$$\mathcal{L} := \begin{bmatrix} b_1 & 1 & 0 & \cdots & a_N \hat{w}^{-1} \\ a_1 & b_2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{N-2} & b_{N-1} & 1 \\ \hat{w} & \cdots & \cdots & a_{N-1} & b_N \end{bmatrix}$$

The characteristic equation for $\mathcal{L}$ defines the hyperelliptic curve:

$$\det(\mathcal{L}-z) = -\left(\hat{w} + \frac{\prod_{i=1}^{N} a_i}{\hat{w}} - \mathcal{P}(z)\right) = 0,$$

which gives the affine curve of genus $N-1$,

$$\hat{X}_{N-1} \; : \; \hat{w}^2 - \mathcal{P}(z)\hat{w} + \prod_{i=1}^{N} a_i = 0.$$

Here $\mathcal{P}$ is given by

$$\mathcal{P}(z) := \Delta^{\text{per}}_{1,N}(z) - \Delta^{\text{per}}_{2,N-1}(z),$$

where

$$\Delta^{\text{per}}_{n,m} := \begin{vmatrix} b_m & 1 & 0 & \cdots & 0 \\ a_m & b_{m+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_{n-2} & b_{n-1} & 1 \\ 0 & \cdots & \cdots & a_{n-2} & b_n \end{vmatrix}.$$

Note: In [McKeanvanMoerbeke], the curves that support solutions of the N-Toda system are shown to be dense in moduli, by an analitic argument, non-vanishing of the differential of a map that takes the N-curves to their period matrix.

3. Thetanulls

In [PShaskaWijesiri] we were able to write equations for all the subloci of $\mathcal{M}_2$ that have a given automorphism group in terms of thetanulls. The method we used was partly aided by a Computer Algebra System (CAS); we expressed the branchpoints of an algebraic equation of the curve in terms of thetanulls,using classical Thomae's formulas; for example, we work out an equation for every curve of genus two:

$$y^2 = x\,(x-1)\,(x - \frac{\theta_1^2\theta_3^2}{\theta_2^2\theta_4^2})\,(x^2 \pm s\alpha\, x + \frac{\theta_1^2\theta_3^2}{\theta_2^2\theta_4^2}\,\alpha^2)$$

where

$$s = \frac{\theta_2^2\,\theta_3^2 + \theta_1^2\,\theta_4^2}{\theta_2^2\,\theta_4^2}, \qquad \alpha^2 + \frac{\theta_1^4 - \theta_2^4 + \theta_3^4 + \theta_4^4}{\theta_1^2\theta_2^2 - \theta_3^2\theta_4^2}\,\alpha + 1 = 0.$$

The θ_i are thetanulls corresponding to a Göpel system in the group of 16 points of period two of the Jacobian. We then use coordinates for $\mathcal{M}_2$ due to [Igusa] and their expressions in terms of the branchpoints of the curve due to [Shaska] to cut out the loci. We manage to apply this procedure to a cyclic curve of genus 3 because of the recent generalization of Thomae's formulas to $\mathbb{Z}_3$ curves [EisenmannFarkas], [Kopeliovich], [Nakayashiki], [Shepherd-Barron].

GAP recently enabled K. Magaard, S. Shpectorov, T. Shaska and H. Völklein (2002) to find all possible automorphism groups of curves of genus g as well as the equations of the curves. Method: monodromy, Hurwitz spaces classified by signature-group pairs; Galois action on function field.

Goal: give equations for the loci in terms of thetanulls (after classical ideas of Riemann, Krazer, and current work by H. Farkas and R.D.M. Accola)

Thanks to the determination of the automorphism group, we can handle the cases of hyperelliptic curves and curves of genus 3.

Sample results (method: determine the action of the groups on the space of holomorphic differentials):

Let X be a genus 3 hyperelliptic curve and $G = \text{Aut}(X)$. Then, $V_4 \subset G$ if and only if there are quarter periods $f_1, f_2, h_1, h_2 \in \text{Jac}(X)$ such that

i) the groups $H_f := < f_1, f_2 >$ and $H_h := < h_1, h_2 >$ are both isomorphic to $C_4 \times C_4$

ii) all elements of H_f and H_h are vanishing theta-nulls

iii) $H_f \cap H_h \cong C_2$.

Let X be a genus 3 non-hyperelliptic curve and $G = \mathrm{Aut}(X)$. Then, $V_4 \subset G$ if and only if there are quarter periods $f_1, f_2, h_1, h_2, g_1, g_2 \in \mathrm{Jac}(X)$ such that

i) $2f_1 = 2h_1 = 2g_1$.

ii) the groups $H_f := < f_1, f_2 >$, $H_h := < h_1, h_2 >$, $H_g := < g_1, g_2 >$ are all isomorphic to $C_4 \times C_4$

iii) all elements of H_f, H_h, H_g are vanishing theta-nulls

Let X be a genus 3 non-hyperelliptic curve, Ω its period matrix, and $G = \mathrm{Aut}(X)$ its group of automorphisms. Then, $C_3 \subset G$ if and only if there exist two 1/6–periods vanishing theta-nulls $f_1, f_2 \in \mathrm{Jac}(X)$ such that $3f_1 = 3f_2$ and $|2f_1, f_1 + f_2| = 1$.

4. AgM

Here X is the Klein quartic: $XY^3+YZ^3+ZX^3=0$. Jac$X$ is not just isogenous, but isomorphic over $\mathbb{Q}(e^{2\pi i/7})$ to the product of three isomorphic elliptic curves.

The classical arithmetic-geometric mean (AgM) for elliptic curves has recently been generalized to genus 3, the last genus for which such an algorithm is possible. Using, on the one hand, the algebraic construction of the genus-3 AgM [LehaviRitzenthaler], on the other, the dictionary between the period lattice and the algebraic representation of curves of genus 3 with split Jacobian [HoweLéprevostPoonen], [Farrington] compares the curve resulting from this algorithm with a construction for the AgM image of X using the curve's split Jacobian and the elliptic curve AgM.

For genus 2, the classical construction of the AgM by Richelot and again the period-matrix representation are used to show [PFarrington]:

Let E_0 and F_0 be elliptic curves over $\mathbb{C}$ with real roots and $G_0 \subset (E_0 \times F_0)[2]$ the graph of a group isomorphism ψ between $E_0[2]$ and $F_0[2]$ that is not induced by an isomorphism of the curves. Let C be the genus-2 curve whose Jacobian JC is isomorphic to the quotient polarized variety $(E_0 \times F_0)/G_0$ and C' be the AgM of C, with Jacobian JC'. Let E_1 and F_1 be the AgMs of the elliptic curves with respect to the subgroups corresponding under ψ and $G_1 \subset (E_1 \times F_1)[2]$ the group corresponding to G_0. The following diagram commutes:

$$\begin{array}{ccc} (E_0 \times F_0)/G_0 & \longrightarrow & \mathrm{J}C \\ \mathrm{AgM} \uparrow & & \uparrow \quad \uparrow \mathrm{AgM}\,. \\ (E_1 \times F_1)/G_1 & \longrightarrow & \mathrm{J}C' \end{array}$$