SMOOTH HYPERELLIPTIC COVERS AND SYSTEMS OF POLYNOMIAL EQUATIONS

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1. Introduction

Let $\mathbb{P}^1$ and X denote, respectively, the projective line and a fixed smooth projective curve of genus 1, both defined over $\mathbb{C}$. Choosing an arbitrary point $q \in X$ as its origin, the pair (X, q) becomes an elliptic curve, having an inverse homomorphism $[-1] : X \to X$ fixing $\omega_o := q \in X$, as well as three other half-periods, $\{\omega_1, \omega_2, \omega_3\} \subset X$. The quotient curve is isomorphic to $\mathbb{P}^1$, and $\varphi_X : X \to \mathbb{P}^1$ will denote the corresponding degree-2 projection, sending the triplet $(\omega_o, \omega_1, \omega_2)$ onto $\{\infty, 0, 1\} \subset \mathbb{P}^1$. The remaining half-period projects onto $\varphi_X(\omega_3) = \lambda \neq 0, 1$. Classically φ_X is represented, in affine coordinates, as the projection

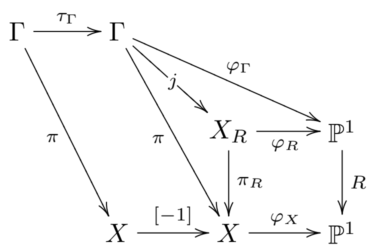
$$\big\{(x, y) \in \mathbb{C}^2,\, y^2 = x(x-1)(x-\lambda)\big\} \longrightarrow \mathbb{C}, \quad (x, y) \mapsto x.$$

More generally, we will consider smooth *hyperelliptic curves*, i.e.: projective curves of genus $g \geq 2$, having an involution $\tau_\Gamma : \Gamma \to \Gamma$, fixing exactly $2g+2$ (so-called *Weierstrass*) points. The quotient curve Γ/τ_Γ is therefore isomorphic to $\mathbb{P}^1$, and the corresponding degree-2 projection $\varphi_\Gamma : \Gamma \to \Gamma/\tau_\Gamma$ is ramified at those $2g+2$ points. As for the elliptic curve (X, q), fixing a triplet of *Weierstrass* points (p, p', p'') of Γ allows us to define φ_Γ as the unique degree-2 cover $\varphi_\Gamma : \Gamma \to \mathbb{P}^1$ sending (p, p', p'') onto $(\infty, 0, 1)$. As for φ_X, the projection φ_Γ affords a (so-called Rosenheim) affine equation

$$\big\{(t, v) \in \mathbb{C}^2,\, v^2 = t(t-1)\Pi_{i=1}^{2g-3}\big\} \longrightarrow \mathbb{C}, \quad (t, v) \mapsto t.$$

We will start studying and constructing all projections $\pi : \Gamma \to X$, called hereafter *hyperelliptic covers*, such that Γ is a smooth *hyperelliptic* curve (which we will usually mark with the choice of a triplet of *Weierstrass* points). Dropping the hyperelliptic condition, the genus of the cover would be (almost) completely independent of its degree; e.g.: according to a theorem of Riemann, for any effective divisor of even degree, $D = \sum_i m_i q_i$, and any $n \geq max\{m_i\}$, there exists a finite positive number of degree-n covers of X with discriminant D (and genus $g := \frac{1}{2}(degD + 1)$). Restricting instead to *hyperelliptic* covers changes radically the whole issue, as explained hereafter.

For any *hyperelliptic* cover $\pi : \Gamma \to X$, marked at a triplet of *Weierstrass* points (p, p', p''), we choose $q := \pi(p)$ as origin of X and let $[-1] : X \to X$ denote the corresponding inverse homomorphism. We then prove the equality $[-1] \circ \pi = \pi \circ \tau_\Gamma$, which in turn implies the existence of a degree-n projection $R : \mathbb{P}^1 \to \mathbb{P}^1$, fitting into the following commutative diagram:



where X_R is the fiber product of $R : \mathbb{P}^1 \to \mathbb{P}^1$ and $\varphi_X : X \to \mathbb{P}^1$, while $j : \Gamma \to X_R$ is the desingularization of X_R. It immediately follows that:

(1) at any *Weierstrass* point $w \in \Gamma$, the ramification index $ind_\pi(w)$ is odd;

(2) the ramification divisor Ram_π is τ_Γ-invariant;

(3) the genus and degree of π, say g and n, satisfy $g \leq 2n\text{-}1$;

(4) the discriminant $Disc_\pi$ is [-1]-invariant and has degree $2g\text{-}2$;

(5) $R(\infty) = \infty$ and $R : \mathbb{P}^1 \to \mathbb{P}^1$ has odd ramification index at $\{\infty, 0, 1\}$.

Conversely, consider a rational fraction $R := \frac{P}{Q}$ such that:

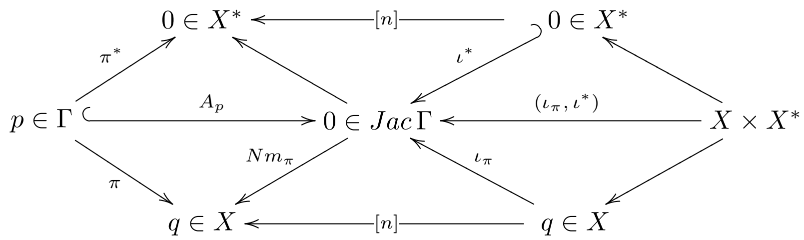
(1) P and Q are coprime, $degP = n$ and $degP\text{-}degQ > 0$ is odd;

(2) R has odd ramification index at $\{0, 1\}$ and $R(0), R(1) \in \{\infty, 0, 1, \lambda\}$.

Let X_R denote the fiber product of $\varphi_X : X \to \mathbb{P}^1$ with $R : \mathbb{P}^1 \to \mathbb{P}^1$, and let $j : \Gamma \to X_R$ denote its desingularization. Then, Γ is naturally equipped with the *Weierstrass* point $p := \varphi_\Gamma^{-1}(\infty)$, as well as two projections, $\varphi_\Gamma : \Gamma \to \mathbb{P}^1$ and $\pi : \Gamma \to X$, of degrees 2 and n respectively. Hence, $\pi : \Gamma \to X$ is a *hyperelliptic cover*, fitting in a commutative diagram as above.

In both cases Ram_π can be deduced from Ram_R (**3.4.**), implying in particular, that constructing *hyperelliptic* covers with given ramification divisor, reduces to finding rational fractions with given derivative.

In the general set up, the *hyperelliptic cover* $\pi : p \in \Gamma \to q \in X$ factors via the canonical *Abel* embedding $A_p : p \in \Gamma \to 0 \in Jac\,\Gamma$, followed by a homomorphism $Nm_\pi : 0 \in Jac\,\Gamma \to q \in X$, with kernel a $(g\text{-}1)$-dimensional abelian subvariety of $Jac\,\Gamma$, say $X^* \overset{\iota^*}{\hookrightarrow} Jac\,\Gamma$. Furthermore, by dualizing Nm_π we get a homomorphism, $\iota_\pi : q \in X \to 0 \in Jac\,\Gamma$, such that $Nm_\pi \circ \iota_\pi = [n] : X \to X$, the multiplication by n. Analogously, we obtain a projection $0 \in Jac\,\Gamma \to 0 \in X^*$, which composed with A_p defines a second morphism $\pi^* : p \in \Gamma \to 0 \in X^*$, completing the following commutative diagram.



In particular, for any genus-2 *hyperelliptic cover* the corresponding subabelian factor X^* is an elliptic curve, such that $Jac\,\Gamma$ is isogenous to $X \times X^*$ and π^* a supplementary degree-n *hyperelliptic cover*. The latter can be constructed as degree-n *hyperelliptic* covers with a degree-2 discriminant.

More generally, we are interested in constructing all degree-n *hyperelliptic* covers with arbitrary given discriminant D (satisfiying the latter restrictions 3) & 4)), and supplementary combinatorial data (see **4.3.**(1)). They can be effectively constructed in terms of the polynomial reduction method mentioned above (which goes back to H.Langes's work, as explained in [1]; see also [3] & [4] for the genus-2 case). We will actually produce a system of N polynomial equations in N variables ($N \leq 2n\text{-}2+\frac{1}{2}degD$), in the complement of a hypersurface of $\mathbb{C}^N$, whose solutions parameterize the isomorphism classes of the latter *hyperelliptic* covers, and give Rosenheim affine equations representing them.

There is yet another interesting family of *hyperelliptic* covers, $\pi : p \in \Gamma \to X$, loosely defined hereafter, for which we can propose a similar presentation. Recall that $A_p(\Gamma)$, the image of Γ by the Abel map, intersects the elliptic curve $\iota_\pi(X)$ at the origin $0 \in Jac\,\Gamma$. We will say that π is a *hyperelliptic tangential* cover, whenever the latter curves are tangent at $0 \in Jac\,\Gamma$. One can weaken the tangency condition as follows. For any $1 \leq d \leq g$, let $V_{d,p}$ denote the d-th *hyperosculating subspace* to $A_p(\Gamma)$ at $A_p(p) = 0 \in Jac\,\Gamma$; we will call $\pi : p \in \Gamma \to q \in X$ a *hyperelliptic d-osculating cover*, if and only if the tangent to $\iota_\pi(X)$ at 0 is contained in $V_{d,p} \backslash V_{d\text{-}1,p}$. These covers have been extensively studied and exist in arbitrary degree (or arbitrary genus), over any elliptic curve (cf. [7] & [6]).

For fixed elliptic curve X and degree $n \geq 2$, there can only exist a finite number of *hyperelliptic tangential covers*, all of them with genus g bounded as follows: $(2g+1)^2 \leq 8n+1$ (e.g.: [7], [5] and all the references in both articles). However, their existence was only proved when $2n\text{-}3 \leq (2g+1)^2$, leaving even the genus-2 case completely unanswered (for any $n > 14$). Similar results hold for the *hyperelliptic d-osculating covers*. As for the preceding family, given n, we will construct a system of N polynomial equations and N variables ($N \leq 3n+2$), parameterizing degree-n *hyperelliptic tangential covers* of the initial elliptic curve (X, q) (as well as similar results for the *d-osculating* case). Last but not least, we should stress that, although such a system may have no solution, an easy application of the Theorem of Bezout gives us an upper bound of the number of corresponding *hyperelliptic* covers.

2. Hyperelliptic covers of an elliptic curve - General properties

Let $\mathbb{P}^1$ and X denote, respectively, the projective line and a fixed smooth projective curve of genus 1, both defined over $\mathbb{C}$. Choosing an arbitrary point $q \in X$ as its origin, the pair (X, q) becomes an elliptic curve, having an inverse homomorphism $[-1] : X \to X$ fixing $\omega_o := q \in X$, as well as three other half-periods, $\{\omega_1, \omega_2, \omega_3\} \subset X$. The quotient curve is isomorphic to $\mathbb{P}^1$, and $\varphi_X : X \to \mathbb{P}^1$ will denote the corresponding degree-2 projection, sending the triplet $(\omega_o, \omega_1, \omega_2)$ onto $\{\infty, 0, 1\} \subset \mathbb{P}^1$. The remaining half-period projects onto $\varphi_X(\omega_3) = \lambda \neq 0, 1$.

Classically φ_X is represented, in affine coordinates, as follows. The equation $y^2 = x(x-1)(x-\lambda)$ defines a smooth affine plane cubic, which can be compactified inside $\mathbb{P}^1 \times \mathbb{P}^1$, by adding the unibranch singular point (∞, ∞). Up to desingularizing the resulting curve at (∞, ∞), we obtain an elliptic curve isomorphic to (X, q), equipped with a marked degree-2 projection, identified with φ_X:

$$\big\{(x, y) \in \mathbb{C}^2,\, y^2 = x(x-1)(x-\lambda)\big\} \longrightarrow \mathbb{C}, \quad (x, y) \mapsto x, \quad q \mapsto \infty.$$

Definition 2.1.

(1) *We will call hyperelliptic curve any projective curve of genus $g \geq 2$, having an involution $\tau_\Gamma : \Gamma \to \Gamma$, such that the quotient curve Γ/τ_Γ is isomorphic to $\mathbb{P}^1$. The corresponding degree-2 projection $\varphi_\Gamma : \Gamma \to \Gamma/\tau_\Gamma = \mathbb{P}^1$ is therefore ramified at $2g + 2$ (so-called Weierstrass) points.*

(2) *We obtain a (so-called Rosenheim) affine equation for φ_Γ as follows: choose a triplet of Weierstrass points (p, p', p'') and identify Γ/τ_Γ with $\mathbb{P}^1$, by projecting (p, p', p'') onto $(\infty, 0, 1)$. The equation $v^2 = t(t-1)\Pi_{j=1}^{2g-3}(t-\alpha_j)$, where $\{\alpha_j\}$ are the projections of the remaining $2g-3$ Weierstrass points, defines an affine curve which can be compactified inside $\mathbb{P}^1 \times \mathbb{P}^1$, by adding the unibranch singular point (∞, ∞). Up to desingularizing the resulting curve at (∞, ∞), we obtain a hyperelliptic curve isomorphic to Γ, equipped with a marked degree-2 projection, identified with φ_Γ:*

$$\big\{(t, v) \in \mathbb{C}^2,\, v^2 = t(t-1)\Pi_{j=1}^{2g-3}(t-\alpha_j)\big\} \longrightarrow \mathbb{C} \quad (t, v) \mapsto t, \quad p \mapsto \infty.$$

(3) *We will call $\pi : \Gamma \to X$ hyperelliptic cover, if and only if Γ is a hyperelliptic curve (and will usually mark it with the choices of a Weierstrass point $p \in \Gamma$ and $q := \pi(p)$ as origin of X).*

Proposition 2.2.

Any hyperelliptic cover $\pi : p \in \Gamma \to q \in X$ satisfies $[-1] \circ \pi = \pi \circ \tau_\Gamma$, and can be pushed down to a morphism $R : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$, fitting in the following commutative diagram:

$$\begin{array}{ccccc} p \in \Gamma & \xrightarrow{\tau_\Gamma} & p \in \Gamma & \xrightarrow{\varphi_\Gamma} & \infty \in \mathbb{P}^1 \\ \pi \downarrow & & \pi \downarrow & & \downarrow R \\ q \in X & \xrightarrow{[-1]} & q \in X & \xrightarrow{\varphi_X} & \infty \in \mathbb{P}^1 \end{array}$$

Proof. Recall that for all $q' \in X$, its inverse with respect to the group structure of (X, q), denoted $[-1](q')$, is the unique point such that $[-1](q') - q$ is linearly equivalent to $q - q'$. Recall also that for any $r' \in \Gamma$ the divisor $r' + \tau_\Gamma(r')$ is linearly equivalent to $2p$. In other words, there exists a meromorphic function $f : \Gamma \to \mathbb{P}^1$, with divisor of zeroes and poles equal to $(f) := (f)_0 - (f)_\infty = r' + \tau_\Gamma(r') - 2p$. Consider the corresponding norm function, $Nm_\pi(f) : X \to \mathbb{P}^1$, defined for any $z \in X \setminus \{q\}$ as $Nm_\pi(f)(z) := \Pi_{i=1}^{n} f(\pi(p_i(z)))$, where $\{\pi(p_i(z)), i = 1, \dots, n\} = f^{-1}(z) \subset \Gamma$. Its divisor is equal to $\big(Nm_\pi(f)\big) = q' + q'' - 2q$, where $q' := \pi(r')$ and $q'' := \pi(\tau_\Gamma(r'))$. Hence $q' - q$ is linearly equivalent to $q - q''$, implying that ($q'' = [-1](q')$, and) for all $r' \in \Gamma$, $\pi\big(\tau_\Gamma(r')\big) = [-1]\big(\pi(r')\big)$ as asserted. Classical results imply that π can be pushed down to a morphism between the quotients. We can also define $R(\alpha)$, for any $\alpha \in \mathbb{P}^1$, as the unique point in $\varphi_X\big(\pi\big(\varphi_\Gamma^{-1}(\alpha)\big)\big)$. ■

Corollary 2.3.

Let π be a degree-n hyperelliptic cover as above, Ram_π its ramification divisor, W_Γ its set of Weierstrass points, and for any $i = 0, \dots, 3$, let $m_{\pi,i}$ denote the number of Weierstrass points, other than p, lying over the half-period ω_i. Then:

(1) $\pi(W_\Gamma) \subset \{\omega_i\}$ *and, at any $w \in W_\Gamma$, π has odd ramification index $ind_\pi(w)$;*

(2) *Ram_π is τ_Γ-invariant and $degRam_\pi = 2g - 2$, where g is the genus of Γ;*

(3) $m_{\pi,0} + 1 \equiv m_{\pi,1} \equiv m_{\pi,2} \equiv m_{\pi,3} \equiv n(\text{mod}.2)$;

(4) *the genus and degree of π satisfy $\quad ind_\pi(p) \leq 2g - 1 \leq 4n - 3$;*

(5) *the discriminant $Disc_\pi$ is $[-1]$-invariant and has degree $2g - 2$.*

Moreover, for any $w \in W_\Gamma$ and morphism $R : \mathbb{P}^1 \to \mathbb{P}^1$ as in **2.2.**, *π and R have same ramification indices, $ind_\pi(w) = ind_R\big(\varphi_\Gamma(w)\big)$, at w and $\varphi_\Gamma(w)$. Last but not least, given $(p, p', p'') \in W_\Gamma^3$, there exists a unique projection $R : \mathbb{P}^1 \to \mathbb{P}^1$ such that $R \circ \varphi_\Gamma = \varphi_X \circ \pi$ and $\varphi_\Gamma\big((p, p', p'')\big) = (\infty, 0, 1)$.*

Proof.

1) Knowing that $[-1] \circ \pi = \pi \circ \tau_\Gamma$, one can apply classical results implying that π can be pushed down to a morphism between the quotients, fitting in the latter diagram. Furthermore, since $R\big(\varphi_\Gamma(p)\big) = \varphi_X\big(\pi(p)\big)$ and $ind_{\varphi_\Gamma}(p) = 2 = ind_{\varphi_X}(q)$, we easily deduce that $ind_\pi(p) = ind_R(\infty)$.
2) & 5) The equality $[-1] \circ \pi = \pi \circ \tau_\Gamma$ implies that W_Γ, the fixed-point set of

τ_Γ, projects onto $\{\omega_i\}$, the fixed-point set of $[-1]$. It also follows that Ram_π and $Disc_\pi$ are τ_Γ and $[-1]$ invariant, respectively. Applying the Hurwitz formula to π and φ_Γ we deduce that $deg(Disc_\pi) = deg\big(\pi(Ram_\pi)\big) = 2g-2$ and $\sharp W_\Gamma = deg(Ram_{\varphi_\Gamma}) = 2g+2$, respectively.
3) Each fiber $\pi^{-1}(\omega_i)$ being τ_Γ invariant, its subset of non-Weierstrass points is made of pairs of points; hence $n - m_{\pi,i} - \delta_{i,0} \equiv 0(\text{mod}.2)$.
4)We know that $ind_\pi(p) - 1 \leq deg(Ram_\pi) = 2g-2 = -4 + \sharp W_\Gamma = -3 + \sum_{i=o}^{3} m_{\pi,i}$, as well as $m_{\pi,i} + \delta_{i,0} \leq n$, for any $i = 0, \cdots, 3$. Hence $ind_\pi(p) \leq 2g-1 \leq 4n-3$.

At last, once $(p, p', p'') \in W_\Gamma^3$ is choosed, there exists a unique isomorphism $\Gamma/\tau_\Gamma \simeq \mathbb{P}^1$, identifying $\varphi_\Gamma\big((p, p', p'')\big)$ with $(\infty, 0, 1)$. The quotient curve $X/[-1]$ being already identified with $\mathbb{P}^1$, the uniqueness of R follows.■

3. Polynomial approach to hyperelliptic covers

Given the elliptic curve (X, q) and the degree-2 cover $\varphi_X : X \to \mathbb{P}^1$, we have associated in **2.2.**, to any smooth *hyperelliptic cover* $\pi : \Gamma \to X$, marked at a triplet $(p, p', p'') \in W_\Gamma^3$, a particular rational fraction $R = \frac{P}{Q}$. Conversely, we have the following result.

Proposition 3.1.

Given a projection $R : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$, such that $ind_R(\infty), ind_R(0)$ and $ind_R(1)$ are odd, and $R(0), R(1) \in \{\infty, 0, 1, \lambda\}$, there exists a unique smooth hyperelliptic cover $\pi : \Gamma \to X$, equipped with a triplet of Weierstrass points (p, p', p'') projecting onto $(\infty, 0, 1)$, such that $R \circ \varphi_\Gamma = \varphi_X \circ \pi$.

Sketch of proof. Choosing a projection $R : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$, with odd ramification indices at $(\infty, 0, 1)$, such that $R(0), R(1) \in \{\infty, 0, 1, \lambda\}$, is equivalent to choosing a rational fraction $R(t) = \frac{P(t)}{Q(t)}$, such that $degP - degQ$ is an odd positive integer, $P'Q - PQ'$ has odd multiplicities $ind_R(0)$ and $ind_R(1)$, and $t(t-1)$ divides $PQ(P-Q)(P-\lambda Q)$. Replacing the variable x by the rational fraction $R(t)$ in the equation $y^2 = x(x-1)(x-\lambda)$, multiplying it by $Q(t)^4$ and making the birational change of variable $w = yQ(t)^2$, gives the affine equation of the fiber product of $R : \mathbb{P}^1 \to \mathbb{P}^1$ with $\varphi_X : X \to \mathbb{P}^1$, i.e.: $w^2 = P(t)Q(t)\big(P(t) - Q(t)\big)\big(P(t) - \lambda Q(t)\big)$. The corresponding completion in $\mathbb{P}^1 \times \mathbb{P}^1$, say Γ, comes with a degree-2 cover $\varphi_\Gamma : (t, w) \in \Gamma \mapsto t \in \mathbb{P}^1$, ramified at the triplet $(p, p', p'') = \big((\infty, \infty), (0, 0), (1, 0)\big)$, as well as the projection $\pi : (t, w) \in \Gamma \mapsto (x, y) = (R(t), \frac{w}{Q(t)^2}) \in X$. The corresponding involution $\tau_\Gamma : (t, w) \mapsto (t, -w)$, fixes the triplet (p, p', p'') of unibranch points, and $\varphi_\Gamma\big((p, p', p'')\big) = (\infty, 0, 1)$. Hence, up to desingularizing Γ, we obtain a smooth *hyperelliptic cover*, fitting in a commutative diagram as above, and equipped with a triplet $(p, p', p'') \in W_\Gamma^3$, such that $\varphi_\Gamma\big((p, p', p'')\big) = (\infty, 0, 1)$. ■

Remark 3.2.

(1) The results **2.3.** and **3.1.** set up a one to one correspondence between degree-n isomorphism classes of smooth triply marked *hyperelliptic covers* $\big\{\pi : \Gamma \to X\big\}$, and pairs of coprime polynomials $\big\{P, Q)\big\}$, such that $degP = n$ and $R := \frac{P}{Q}$ satisfies the conditions of **3.1.**.

(2) According to **2.3.**, $degRam_\pi = 2g\text{-}2$ and Γ has $2g+2$ *Weierstrass points*, at any one of which π has odd ramification index. Hence, there must be at least $g+3$ ones with $ind_\pi = 1$. In particular, we may choose the above triplet $(p, p', p'') \in W_\Gamma^3$ without ramification, or equivalently, $R := \frac{P}{Q}$ with $ind_R(\infty) = ind_R(0) = ind_R(1) = 1$.

(3) Given such a pair (P, Q), the product $P(t)Q(t)\big(P(t)\text{-}Q(t)\big)\big(P(t)\text{-}\lambda Q(t)\big)$ can be uniquely factored as $t(t\text{-}1)A(t)B(t)^2$, where $B(t)$ is monic, A has odd degree and $t(t\text{-}1)A(t)$ has no multiple root. It follows that the affine curve $\{(t,v) \in \mathbb{C}^2, v^2 = t(t\text{-}1)A(t)\}$, completed as explained in **2.1.**(2), and equipped with the projection $(t,v) \mapsto (x,y) := \big(\frac{P(t)}{Q(t)}, \frac{vB(t)}{Q(t)^2)}\big)$, gives the smooth *hyperelliptic cover* of (X, q), uniquely associated to (P, Q).

Working locally with the corresponding equations one easily deduces the ramification divisor Ram_π, out of Ram_R, as follows.

Lemma 3.3.

Let $\pi : p \in \Gamma \to q \in X$ be the smooth hyperelliptic cover associated to the projection $R : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$, and $\varphi_\Gamma : p \in \Gamma \to \infty \in \mathbb{P}^1$ the corresponding degree-2 projection. Then, for any $\alpha \in \mathbb{P}^1$:

(1) *if $R(\alpha) \notin \{0, 1, \lambda, \infty\}$, the fiber $\varphi_\Gamma^{-1}(\alpha)$ has two points, say $r \neq \tau_\Gamma(r) \in \Gamma$, and $ind_\pi(r) = ind_\pi\big(\tau_\Gamma(r)\big) = ind_R(\alpha)$;*

(2) *if $R(\alpha) \in \{0, 1, \lambda, \infty\}$ and $ind_R(\alpha)$ is even, the fiber $\varphi_R^{-1}(\alpha)$ has two points, say $r \neq \tau_\Gamma(r) \in \Gamma$, and $ind_\pi(r) = ind_\pi\big(\tau_\Gamma(r)\big) = \frac{1}{2} ind_R(\alpha)$;*

(3) *if $R(\alpha) \in \{0, 1, \lambda, \infty\}$ and $ind_R(\alpha)$ is odd, there is a unique (Weierstrass) point in $\varphi_R^{-1}(\alpha)$, say $r = \tau_\Gamma(r) \in W_\Gamma$, and $ind_\pi(r) = ind_R(\alpha)$.*

Proposition 3.4.

Let $\pi : p \in \Gamma \to q \in X$ be the smooth hyperelliptic cover associated to $R : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$. Then, Ram_π can be deduced from Ram_R. More precisely, if

$$Ram_R = \sum_J l_j \gamma_j + \sum_K 2m_k \alpha_k + \sum_S (2r_s\text{-}1)\beta_s, \quad \textit{where}$$

$$\forall j \in J,\ R(\gamma_j) \notin \{0, 1, \lambda, \infty\}, \quad \textit{and} \quad \forall k \in K, \forall s \in S,\ R(\alpha_k), R(\beta_s) \in \{0, 1, \lambda, \infty\},$$

$$\textit{then,} \quad Ram_\pi = \sum_J l_j \varphi_\Gamma^{-1}(\gamma_j) + \sum_K 2m_k \varphi_\Gamma^{-1}(\alpha_k) + \sum_S (r_s\text{-}1)\varphi_\Gamma^{-1}(\beta_s).$$

In particular, its genus is $g = 1 + \sum_J l_j + \sum_K m_k + \sum_S (r_s\text{-}1)$.

According to the Hurwitz formula, we must have $deg(Ram_R) = 2n\text{-}2$ and $deg(Ram_\pi) = 2g\text{-}2$. We deduce the following characterization:

Corollary 3.5.

The genus of Γ *equals* 2 *if, and only if, one of the following conditions is satisfied:*

(1) *either,* R *has one point with ramification index* 3 *and* $2n$ - 4 *other points with ramification index* 2*, all of them projecting into* $\{\infty, 0, 1, \lambda\} \subset \mathbb{P}^1$;

(2) *or,* R *has* $2n$ - 2 *points with ramification index* 2*, all but one projecting into* $\{\infty, 0, 1, \lambda\} \subset \mathbb{P}^1$.

In case (1)*,* $Ram_\pi = 2p'$*, for some* $p' = \tau_\Gamma(p') \in W_\Gamma$*, and* $Disc_\pi = 2\pi(p')$*, while in case* (2)*,* $Ram_\pi = p' + \tau_\Gamma(p')$*, where* $\tau_\Gamma(p') \neq p'$ *and* $Disc_\pi = \pi(p') + [-1]\big(\pi(p')\big)$.

Definition 3.6.

(1) Let $\pi : \Gamma \to X$ be a smooth *hyperelliptic cover*, $W_{\pi,i} = W_\Gamma \cap \pi^{-1}(\omega_i)$ the subset of *Weierstrass points* projecting onto ω_i, and $m_{\pi,i} := \sharp W_{\pi,i}$ its cardinal ($i = 0, \cdots, 3$). We will call $(m_{\pi,i})$ the *Weierstrass type* of π.

(2) Each fiber $\pi^{-1}(\omega_i)$ being τ_Γ-invariant, its non-*Weierstrass points* come in (say $m^\vee_{\pi,i}$) pairs, and it must decompose as

$$\pi^{-1}(\omega_i) = \sum_{k_i=1}^{m_{\pi,i}} ind_\pi(p_{k_i})p_{k_i} + \sum_{s_i=1}^{m^\vee_{\pi,i}} ind_\pi(p_{s_i})\big(p_{s_i} + \tau_\Gamma(p_{s_i})\big)\,.$$

In particular, taking degrees we obtain a decomposition of $n := deg(\pi)$,

$$n = \sum_{k_i=1}^{m_{\pi,i}} ind_\pi(p_{k_i}) + \sum_{s_i=1}^{m^\vee_{\pi,i}} 2ind_\pi(p_{s_i})\,,$$

as a sum of $m_{\pi,i}$ odd positive integers, plus $m^\vee_{\pi,i}$ even positive integers. We will arrange the latter odd and even coefficients in increasing order, and denote $\vec{Ind}_{\pi,i} = (\vec{ind}_{\pi,W_i}, \vec{ind}_{\pi,W^\vee_i}) \in \mathbb{N}^{m_{\pi,i}} \times \mathbb{N}^{m^\vee_{\pi,i}}$ the corresponding pair of increasing sequences.

(3) We will call $\Big(Disc_\pi, \, , \big(\vec{Ind}_{\pi,i}\big)\Big)$ the *augmented discriminant of* π.

(4) Taking into account that $R \circ \varphi_\Gamma = \varphi_X \circ \pi$, the above decomposition of $\pi^{-1}(\omega_i)$ gives:

$$R^{-1}\big(\varphi_\Gamma(\omega_i)\big) = \sum_{k_i=1}^{m_{\pi,i}} ind_\pi(p_{k_i})\varphi_\Gamma(p_{k_i}) + \sum_{s_i=1}^{m^\vee_{\pi,i}} 2ind_\pi(p_{s_i})\varphi_\Gamma(p_{s_i})\,.$$

In other words, the vector $(\vec{Ind}_{\pi,i})$ also codifies the structure of the fibers of R over $\varphi_X(\{\omega_i\}) = \{\infty, 0, 1, \lambda\}$. We will call $\Big(Disc_R, \, , \big(\vec{Ind}_{\pi,i}\big)\Big)$ the *λ-augmented discriminant of* R.

Remark 3.7.

(1) Given two coprime polynomials P and Q, such that $\rho := degP - degQ$ is an odd positive integer, the morphism $R := \frac{P(t)}{Q(t)} : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$ and the corresponding *hyperelliptic cover* $\pi : p \in \Gamma \to q \in X$, have same degree $n := degP$ and same ramification indices $ind_\pi(p) = \rho = ind_R(\infty)$. On the other hand, since $w^2 = P(t)Q(t)\big(P(t) - Q(t)\big)\big(P(t) - \lambda Q(t)\big)$ defines a birational model of Γ, its genus satisfies $\rho \leq 2g - 1 \leq 4n - \rho$, with maximal genus if and only if $P(t)Q(t)\big(P(t) - Q(t)\big)\big(P(t) - \lambda Q(t)\big)$ has no multiple root.

(2) The ramification divisor Ram_R is equal to $(P'Q - PQ')_o + (\rho - 1)\infty$, where $(P'Q - PQ')_o$ denotes the degree-$(2n - 2)$ zero-divisor of $P'Q - PQ'$.

(3) We may have two rational fractions sharing the same discriminant and yet defining smooth *hyperelliptic covers* of different genus. In fact, the relation between $Disc_\pi$ and $Disc_R$ is many to one in both directions.

(4) However, there is a one to one correspondance between the augmented discriminants of π and R (**3.11.**).

The following straightforward **Lemmas** will help us in:

(1) finding all morphisms $R = \frac{P}{Q}$ with given discriminant D;

(2) linking the multiplicities of the roots of P, Q, $P - Q$, and $P - \lambda Q$, with the ramification indices of R over $\{\infty, 0, 1, \lambda\}$;

(3) deducing the *augmented discriminant* of π, out of the λ-*augmented discriminant* of R (**3.11.**).

At last, they will be instrumental in **4**, for the construction of all *hyperelliptic covers* with given *augmented discriminant.*

Lemma 3.8.

For any polynomial $A \in \mathbb{C}[t]$ *let* $disc(A)$ *denote its discriminant; i.e.: the resultant of* $A(t)$ *and its derivative* $A'(t)$. *Then,* $D(x,y) := disc(xP - yQ)$ *is a degree-*$(2n - 2)$ *form, with divisor of zeroes equal to* $Disc_R$*, the discriminant of the morphism* $R := \frac{P}{Q} : \mathbb{P}^1 \to \mathbb{P}^1$.

Lemma 3.9.

Let $\big(P(t), Q(t)\big)$ *be a pair of coprime polynomials as in* **3.7.**(1) *and* $\lambda \in \mathbb{C}\backslash\{0,1\}$. *Then,* $\alpha \in \mathbb{C}$ *is a root of multiplicity* $m \geq 2$ *of* $PQ(P - Q)(P - \lambda Q)$*, if and only if,* $\frac{P(\alpha)}{Q(\alpha)} \in \{0, 1, \lambda, \infty\}$ *and* α *is a root of multiplicity* $m - 1$ *of* $P'Q - PQ'$.

Lemma 3.10.

Let $\pi : p \in \Gamma \to q \in X$ *be the hyperelliptic cover, with odd ramification index* $\rho := degP - degQ$ *at* p*, associated to a rational fraction* $R = \frac{P}{Q}$ *as in* **3.7.**(1)*, and*

$P = A_1B_1^2$, $Q = A_oB_o^2$, $P\text{-}Q = A_2B_2^2$, $P\text{-}\lambda Q = A_3B_3^2$ denote the unique factorizations such that $\forall i = 0,\cdots,3$, B_i is monic and A_i has no multiple root. Then,

(1) *$\Gamma\setminus\{p\}$ is isomorphic to the affine curve $\big\{(t,v)\in\mathbb{C}^2,\ v^2=\Pi_iA_i\big\}$;*

(2) the *Weierstrass type* of π is equal to $(degA_i)$;

(3) *the genus of Γ, say g, satisfies $2g+1=\sum_i degA_i$.*

Furthermore, for any $p'\in\pi^{-1}(\omega_i)$, let m' denote the multiplicity of $A_iB_i^2$ at $t' := \varphi_\Gamma(p')$. Then, either m' is odd, $p'\in W_\Gamma$ and $ind_\pi(p') = m'$, or m' is even, $p'\notin W_\Gamma$ and $ind_\pi(p') = \frac{1}{2}m'$.

Proposition 3.11.

*Let $\pi:\Gamma\to X$ be the hyperelliptic cover associated to $R := \frac{P}{Q}$ (**3.7.**(1)), $(m_{\pi,i})$ its Weierstrass type and $(m^\vee_{\pi,i})\in\mathbb{N}^4$ such that $m_{\pi,i}+m^\vee_{\pi,i} = \sharp\,\pi^{-1}(\omega_i)$. Let also*

$$\vec{Ind}_{\pi,i} = (\vec{ind}_{\pi,W_i},\vec{ind}_{\pi,W_i^\vee})\in\mathbb{N}^{m_{\pi,i}}\times\mathbb{N}^{m^\vee_{\pi,i}},\quad (i=0,\cdots,3),$$

*denote the positive integer vector deduced from $\pi^{-1}(\omega_i)$, and codifying the corresponding decomposition of n (cf. **3.6.**(2)). Then, $\big(Disc_\pi,\ (\vec{Ind}_{\pi,i})\big)$, the augmented discriminant of π, can be deduced out of $\big(Disc_R,\ (\vec{Ind}_{\pi,i})\big)$, the λ-augmented discriminant of R, and vice-versa.*

Proof. Given the vector $(\vec{Ind}_{\pi,i})$, the discriminant Ram_π must be equal to

$$\sum_J l_j\varphi_\Gamma^{-1}(\gamma_j)+\sum_{i=0}^{3}\Big(\sum_{k_i=1}^{m_{\pi,i}}\big(ind_\pi(p_{k_i})\text{-}1\big)p_{k_i}+\sum_{s_i=1}^{m^\vee_{\pi,i}}\big(ind_\pi(p_{s_i})\text{-}1\big)\big(p_{s_i}+\tau_\Gamma(p_{s_i})\big)\Big),$$

where $R(\gamma_j)\notin\{\infty,0,1,\lambda\}$, for any $j\in J$, while $\pi(p_{k_i}) = \pi(p_{s_i}) = \omega_i$, for any k_i and s_i (cf. **3.6.**(2)). It also follows that Ram_R must be equal (cf. **3.4.**) to

$$\sum_J l_j\gamma_j+\sum_{i=0}^{3}\Big(\sum_{k_i=1}^{m_{\pi,i}}\big(ind_\pi(p_{k_i})\text{-}1\big)\varphi_\Gamma(p_{k_i})+\sum_{s_i=1}^{m^\vee_{\pi,i}}\big(2ind_\pi(p_{s_i})\text{-}1\big)\varphi_\Gamma(p_{s_i})\Big).$$

Projecting on X and $\mathbb{P}^1$ we end up obtaining that

$$Disc_\pi=\sum_J l_j\varphi_X^{-1}\big(R(\gamma_j)\big)+\sum_{i=0}^{3}\Big(\sum_{k_i=1}^{m_{\pi,i}}\big(ind_\pi(p_{k_i})\text{-}1\big)+\sum_{s_i=1}^{m^\vee_{\pi,i}}\big(2ind_\pi(p_{s_i})\text{-}2\big)\Big)\omega_i\,,$$

and

$$Disc_R=\sum_J l_jR(\gamma_j)+\sum_{i=0}^{3}\Big(\sum_{k_i=1}^{m_{\pi,i}}\big(ind_\pi(p_{k_i})\text{-}1\big)+\sum_{s_i=1}^{m^\vee_{\pi,i}}\big(2ind_\pi(p_{s_i})\text{-}1\big)\Big)\varphi_X(\omega_i)\,.$$

The latter formulae imply the relations,

$$\varphi_X(Disc_\pi + \sum_{i=0}^{3} m^\vee_{\pi,i}\omega_i) = Disc_R + \sum_J l_j R(\gamma_j)$$

and

$$\varphi_X^{-1}\big(Disc_R - \sum_{i=0}^{3} m^\vee_{\pi,i}\varphi_X(\omega_i)\big) = 2Disc_R - \sum_J l_j\varphi_X^{-1}\big(R(\gamma_j)\big)\,.$$

Whence, a one to one correspondance between the *augmented discriminants.* ■

Corollary 3.12.

Let $\pi : p \in \Gamma \to q \in X$ be the hyperelliptic cover with ramification index ρ at $p \in \Gamma$, associated to a rational fraction $R : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$ as in **3.7.**(1). *Then, the genus of Γ attains its minimal value $g := \frac{1}{2}(\rho+1)$, if and only if the following equivalent conditions are satisfied:*

i) $Ram_\pi = (\rho - 1)p$;
ii) $P'Q - PQ'$ has no multiple root and divides $PQ(P - Q)(P - \lambda Q)$.

In the latter case $(P'Q - PQ')^2$ divides $PQ(P - Q)(P - \lambda Q)\big) = (P'Q - PQ')^2T$, and outside $\pi^{-1}(\{p\})$ the projection $\pi : \Gamma \longrightarrow X$ is isomorphic to

$$(t, v) \in \{v^2 = T(t)\} \longmapsto \big(R(t), v.R'(t)\big) = (x, y) \in \{y^2 = x(x\text{-}1)(x\text{-}\lambda)\}\,.$$

Proof. Property **3.12.**(ii), coupled with **3.9.**, imply that all (simple) roots of $P'(t)Q(t) - P(t)Q'(t)$ are double roots of $P(t)Q(t)\big(P(t) - Q(t)\big)\big(P(t) - \lambda Q(t)\big)$. In other words, $PQ(P - Q)(P - \lambda Q) = (P'Q - PQ')^2T$, where T has only simple roots (again **3.9.**) and $degT = 4n - \rho - 2\big(2n - (\rho + 1)\big) = \rho + 2$. Hence, replacing $w = v(P'Q - PQ')$ in the equation $w^2 = P(t)Q(t)\big(P(t) - Q(t)\big)\big(P(t) - \lambda Q(t)\big)$, which defines a birational model for Γ (**3.7.**(1)), simplifies it to $v^2 = T(t)$ and defines Γ. It follows that $2g + 1 = degT = \rho + 2$ as asserted. We can also check that outside $\pi^{-1}(\{p\})$, π is given by the projection $(t, v) \longmapsto \big(R(t), v.R'(t)\big)$.

Conversely, assuming $g = \frac{1}{2}(\rho + 1)$ is equivalent to $Ram_\pi = (\rho - 1)p$, and implies (cf. **3.4.**) that $ind_R(t) = 2, \forall t \in \mathbb{P}^1 \setminus \{\infty\}$ in the support of Ram_R, and that all roots of $P'Q - PQ'$ must be simple and should lie in $R^{-1}\big(\{\infty, 0, 1, \lambda\}\big)$ as well. Hence (cf. **3.9.**) $\big(P'Q - PQ'\big)^2$ divides $PQ(P - Q)(P - \lambda Q)$. ■

Remark 3.13.

For $\rho = 3$, any hyperelliptic cover $\pi : p \in \Gamma \to q \in X$ as above, has genus $g = 2$ implying that $Jac\,\Gamma$ splits, up to isogeny, as a sum $X + X^*$, where $X^* \subset Jac\,\Gamma$ is an elliptic curve. Furthermore, π being ramified at $p \in \Gamma$ forces X^* to be tangent to the Abel image of (Γ, p) at the origin $A_p(p) = 0 \in Jac\,\Gamma$ (cf. [2]*). In other words, $\pi^* : p \in \Gamma \to 0 \in X^*$ is a hyperelliptic tangential cover (cf.* [7]*).*

4. HYPERELLIPTIC COVERS AND POLYNOMIAL EQUATIONS

4.1. According to **Proposition 2.3.** and **Proposition 3.1.**, there is a bijection between the set of *hyperelliptic covers* $\pi : p \in \Gamma \to q \in X_\lambda$, marked at $(p, p', p'') \in W_\Gamma^3$, and the set of projections $R : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$ with odd ramification indices at $\{\infty, 0, 1\}$, such that $R(0), R(1) \in \{\infty, 0, 1, \lambda\}$. Therefore, fixing in advance some properties of the latter covers is tantamount to putting further restrictions on the corresponding rational fractions. As for the ones with fixed degree n, odd ramification index $\rho = ind_\pi(p)$ and minimal genus $g = \frac{1}{2}(\rho + 1)$ (cf. **3.12.**), they make a finite subset, say $H(n, \rho)$, for which we have the following basic result.

Proposition 4.2.
There exists a system of $2n + 1 - \rho$ polynomial equations in $2n + 1 - \rho$ variables, such that $H(n, \rho)$ parameterizes its set of isolated solutions.

Proof. Any $\pi \in H(n, \rho)$, corresponds to a unique pair of coprime polynomials, of degrees n and $n - \rho$ respectively: P unitary equal to $P(t) = t^n + \sum_{i=0}^{n-1} \alpha_i t^i$ and $Q(t) = \sum_{j=0}^{n-\rho} \beta_j t^j$, satisfying **3.7.**(1) & **3.12.**, as explained hereafter.

Dividing $PQ(P - Q)(P - \lambda Q)$ by $P'Q - PQ'$ gives a remainder $S(t) = \sum_{k=0}^{2n-\rho-2} s_k t^k$, of degree strictly smaller than $2n - 1 - \rho$, with coefficients $\{s_k\}$ depending polynomially on those of P and Q. Assuming $P'Q - PQ'$ divides $PQ(P - Q)(P - \lambda Q)$ is equivalent to the system $\{s_k(\alpha_i, \beta_j) = 0,\ k = 0, \dots, 2n - 2 - \rho\}$, of $2n - 1 - \rho$ polynomial equations in the $2n + 1 - \rho$ variables $\{\alpha_i, \beta_j\}$.

We must also assume $P'Q - PQ'$ without multiple roots, implying the factorization $PQ(P - Q)(P - \lambda Q) = (P'Q - PQ')^2 T$ (**3.9.**). Adding the supplementary equations $T(0) = 0 = T(1)$, which reflect the conditions $R(0) = R(1) = 0$, we thus obtain a system of $2n + 1 - \rho$ polynomial equations in $2n + 1 - \rho$ variables.

Conversely, any pair (P, Q) of polynomials, $P(t) = t^n + \sum_{i=0}^{n-1} \alpha_i t^i$, and $Q(t) = \sum_{j=0}^{n-\rho} \beta_j t^j$, satisfying the latter system of equations, as well as the open conditions $\{degQ = n - \rho,\ disc(P'Q - PQ') \neq 0,\ PGCD(P, Q) = 1\}$, give rise to a degree-$n$ morphism $R = \frac{P}{Q}$ satisfying **3.7.**(1) & **3.12.** Hence, the corresponding *hyperelliptic cover* belongs to $H(n, \rho)$ ■

Definition 4.3.

(1) Given $n, g \in \mathbb{N}^*$ and a $[-1]$-invariant degree-$(2g - 2)$ effective divisor

$$D = \sum_J l_j \varphi_X^{-1}(\gamma_j) + \sum_{i=0}^{3} a_i \omega_i \,,$$

and for any $i = 0, \cdots, 3$, a pair of increasing sequences of odd and even positive integers $(\vec{ind}_i, \vec{ind}_i^\vee) := \big((2h_{i,k_i} + 1), (2g_{i,s_i})\big) \in \mathbb{N}^{m_i} \times \mathbb{N}^{m_i^\vee}$, of lengths $(m_i, m_i^\vee)$, codifying a decomposition of n,

$$\sum_{k_i=1}^{m_i} (2h_{i,k_i} + 1) + \sum_{s_i=1}^{m_i^\vee} 2g_{i,s_i} = n \,,$$

such that

$$\sum_{k_i=1}^{m_i} 2h_{i,k_i} + \sum_{s_i=1}^{m_i^\vee} (2g_{i,s_i} - 1) = a_i \,.$$

We remark that $(m_i) \in \mathbb{N}^4$ satisfies $m_o + 1 \equiv m_1 \equiv m_2 \equiv m_3 \equiv n(\mathrm{mod}.2)$. We will let $H\Big(n, D, \big((\vec{ind}_i, \vec{ind}_i^\vee)\big)\Big)$ denote the moduli space of degree-n *hyperelliptic covers* π, with augmented discriminant $\Big(D, \big((\vec{ind}_i, \vec{ind}_i^\vee)\big)\Big)$. In other words, $(\vec{ind}_i, \frac{1}{2}\vec{ind}_i^\vee)$ gives, for any $i = 0, \cdots, 3$, the ramification indices of π, at the Weierstrass and non-Weierstrass points of $\pi^{-1}(\omega_i)$.

(2) Let $\pi : p \in \Gamma \to q \in X$ be a *hyperelliptic cover* and consider the canonical Abel embedding $A_p : p \in \Gamma \to 0 \in Jac\,\Gamma$ and the homomorphism $\iota_\pi : q \in X \to 0 \in Jac\,\Gamma$ (cf. **1**) . We will call π a *hyperelliptic tangential cover*, if and only if $A_p(\Gamma)$ and $\iota_\pi(X)$ are tangent at $0 \in Jac\,\Gamma$ (cf. [7]).

(3) For any $1 \leq d \leq g := dim(Jac\,\Gamma)$, let $V_{d,p}$ denote the d-th osculating subspace to $A_p(\Gamma)$ at 0 (cf. [6]). We will call $\pi : p \in \Gamma \to q \in X$ a *hyperelliptic d-osculating cover*, if and only if the tangent to $\iota_\pi(X)$ at 0 is contained in $V_{d,p} \backslash V_{d-1,p}$. For $d = 1$ we recover the *hyperelliptic tangential covers*.

(4) For any $n, d \in \mathbb{N}^*$ and $(m_i) \in \mathbb{N}^4$, such that $m_o + 1 \equiv m_1 \equiv m_2 \equiv m_3 \equiv n(\mathrm{mod}.2)$, we will let $HypOsc\big(n, d, (m_i)\big)$ denote the moduli space of smooth degree-n *hyperelliptic d-osculating covers*, having *Weierstrass type* (m_i).

The families $H\Big(n, D, \big((\vec{ind}_i, \vec{ind}_i^\vee)\big)\Big)$ and $HypOsc\big(n, 1, (m_i)\big)$, of *hyperelliptic covers*, are classically known to be finite. We will prove that they can also be parameterized by suitable polynomial systems, with as many equations as variables. As for the moduli spaces $HypOsc\big(n, d, (m_i)\big)$, with $d \geq 2$, all known families have dimension d-1 (cf. [6]), and we will parameterize them via polynomial systems of $3n + m_o + 1$ equations in $3n + m_o + d$ variables.

Proposition 4.4.
For any data $\Big(n, D, \big((\vec{ind}_i, \vec{ind}_i^\vee)\big)\Big)$ as in **4.3.**, *there exists a polynomial system of $N := 2n + 2 + \sum_J l_j$ equations, in an open dense subset of $\mathbb{C}^N$, such that its set of solutions parameterizes the moduli space $H\Big(n, D, \big((\vec{ind}_i, \vec{ind}_i^\vee)\big)\Big)$.*

Proof. Up to choosing a triplet of *Weierstrass points* (cf. **3.1.**& **3.2.**(2)), any class $\pi \in H\Big(n, D, \big((\vec{ind}_i, \vec{ind}_i^\vee)\big)\Big)$ corresponds to a unique rational morphism $R = \frac{P}{Q} : \mathbb{P}^1 \to \mathbb{P}^1$, such that $degR = n$ (i.e.: P and Q are coprime polynomials), $R(\infty) = \infty$, $\{R(0), R(1)\} \subset \{\infty, 0, 1, \lambda\}$ and $ind_R(\infty) = ind_R(0) = ind_R(1) = 1$. Moreover, $\big((\vec{ind}_i, \vec{ind}_i^\vee)\big)$ gives the multiplicities of all roots of Q, P, P-Q and P-λQ:

$$Q = A_o B_o^2 \ = \Pi_{K_o}(t - \alpha_{o,k_o})^{2h_{o,k_o}+1}\Pi_{S_o}(t - \beta_{o,s_o})^{2g_{o,s_o}}$$

$$P = cA_1 B_1^2 = c\Pi_{K_1}(t - \alpha_{1,k_1})^{2h_{1,k_1}+1}\Pi_{S_1}(t - \beta_{1,s_1})^{2g_{1,s_1}}$$

$$P - Q = cA_2B_2^2 = c\Pi_{K_2}(t-\alpha_{2,k_2})^{2h_{2,k_2}+1}\Pi_{S_2}(t-\beta_{2,s_2})^{2g_{2,s_2}}$$

$$P - \lambda Q = cA_3B_3^2 = c\Pi_{K_3}(t-\alpha_{3,k_3})^{2h_{3,k_3}+1}\Pi_{S_3}(t-\beta_{3,s_3})^{2g_{3,s_3}} \,.$$

It follows that $Disc_\pi$ must contain $\sum_{i=0}^{3} a_i\omega_i$, since for any $i = 0, \cdots, 3$,

$$\sum_{k_i=1}^{m_i} 2h_{i,k_i} + \sum_{s_i=1}^{m_i^\vee}(2g_{i,s_i} - 1) = a_i \,.$$

We still need $Disc_\pi$ to contain $\sum_J l_j\varphi_X^{-1}(\gamma_j)$, which amounts to $Disc_R$ containing $\sum_J l_j\gamma_j$. This last condition translates, by **Lemma 3.8.**, to condition (4) below. All in all, we have $1+\sum_{i=o}^{3}(m_i+m_i^\vee)$ variables, and the following equations:

(1) $cA_1B_1^2 - A_oB_o^2 \; = cA_2B_2^2\,;$

(2) $cA_1B_1^2 - \lambda A_oB_o^2 = cA_3B_3^2\,;$

(3) $R(0) = R(1) = 0\,;$

(4) $\Pi_J\big(y - \gamma_j\big)^{l_j}$ divides $disc(P - yQ)\,.$

Each side in (1) and (2), has c as highest coefficient. Identifying the other ones gives us $2n$ polynomial equations in our variables. Taking into account (3) and (4), gives $2+deg\Big(\Pi_J\big(y - R(t_j)\big)^{l_j}\Big) = 2+\sum_J l_j$ more equations, adding to $2n+2+\sum_J l_j$. At last, it is enough to check that $1 + \sum_{i=o}^{3}(m_i + m_i^\vee) = 2n + 2 + \sum_J l_j$.

Conversely, given $c \in \mathbb{C}^*$ and a set of unitary polynomials $\{A_i, B_i\}$, subject to the latter system of equations, plus the open condition $resultant(A_1B_1^2, A_oB_o^2) \neq 0$, the corresponding morphism $R := \frac{P}{Q} : \mathbb{P}^1 \to \mathbb{P}^1$, has $degR = n$ and corresponds to a unique *hyperelliptic cover* π, having $\Big(n, D, \big((\vec{ind}_i, \vec{ind}_i^\vee)\big)\Big)$ as augmented discriminant. ∎

Let $\pi : p \in \Gamma \to q \in X$ a degree-n *hyperelliptic cover* of *Weierstrass type* $(m_{\pi,i}) \in \mathbb{N}^4$, associated to the rational fraction $R := \frac{P}{Q}$. Consider the canonical factorizations of $P = cA_1B_1^2$, $Q = cA_1B_1^2$, $P - Q = cA_2B_2^2$ and $P - \lambda Q = cA_3B_3^2$, with unitary polynomials, (A_i) and (B_i), such that for any $i = 0, \cdots, 3$, $degA_i = m_i$ and $degB_i^2 = n - m_i - \delta_{i,o}$. Recall that its genus satisfies $2g+1 = \sum_i m_i = \sum_i degA_i$, and outside $\pi^{-1}(q) \subset \Gamma$ (cf. **3.2.**), the projection π is isomorphic to

$$(t, v) \in \Big\{v^2 = c^3\Pi_iA_i\Big\} \quad \mapsto \quad \Big(R(t), \frac{v\Pi_iB_i}{Q^2}\Big) = (x, y) \in \Big\{y^2 = x(x\text{-}1)(x\text{-}\lambda)\Big\}\,.$$

We will let z denote hereafter, the local coordinate of X at q, defined as $z := \frac{P}{yQ}$. The following technical **Lemmas** will help us prove a result for $HypOsc\big(n, d, (m_i)\big)$, analogous to **4.4.**.

Lemma 4.5.

The function $\kappa_o := \frac{v}{A_oB_o} : \Gamma \to \mathbb{P}^1$ *is anti-*τ_Γ*-invariant (i.e.:* $\kappa_o \circ \tau_\Gamma = -\kappa_o$ *), holomorphic outside* $\pi^{-1}(q)$*, has order* $2deg(A_oB_o) - (2g+1) \geq -1$ *at* $p \in \Gamma$*, and a pole of same order as* $\frac{1}{z} = \frac{yQ}{P}$*, at any other point of* $\pi^{-1}(q)$*.*

Lemma 4.6.

Any anti-τ_Γ*-invariant meromorphic function* $\kappa : \Gamma \to \mathbb{P}^1$*, holomorphic outside* $\pi^{-1}(q)$*, having a pole of order* $2d-1$ *at* $p \in \Gamma$*, and a pole of same order as* $\frac{1}{z} = \frac{yQ}{P}$*, at any point other point of* $\pi^{-1}(q)$*, is equal to* $\frac{vM}{A_oB_o}$*, for a unique polynomial* $M(t)$ *of degree* $degM = deg(A_oB_o) - g + d - 1$*.*

Lemma 4.7.

Let $\kappa = \frac{vM}{A_oB_o} : \Gamma \to \mathbb{P}^1$ *be as in* **4.6.***. Then,* $\kappa - \frac{1}{z}$ *has a pole of order* $2d-1$ *at* $p \in \Gamma$*, and no other pole over* $\pi^{-1}(q)$*, if and only if* A_oB_o *divides* $MA_1B_1 - B_2B_3$*.*

Proposition 4.8.
For any $n, d \geq 1$ *and* $(m_i) \in \mathbb{N}^4$ *such that* $m_o \equiv m_1 \equiv m_2 \equiv m_3 \equiv n(\text{mod}.2)$*, there exists a polynomial system of* $N := 2n + 1 + \frac{1}{2}(n + m_o + 1)$ *equations, in an open dense subset of* $\mathbb{C}^{N+d-1}$*, such that its set of solutions parameterizes the moduli space* $HypOsc\big(n, d, (m_i)\big)$*.*

Proof. Consider $c \in \mathbb{C}^*$, two arbitrary sequences of unitary polynomials, (A_i) and (B_i), such that for any $i = 0, \cdots, 3$, $degA_i = m_i$ and $degB_i^2 = n - m_i - \delta_{i,o}$, and a polynomial M of degree $degM = deg(A_oB_o) - g + d - 1$. These data depend upon $2n + 1 + deg(A_oB_o) + d = 2n + d + \frac{1}{2}(n + m_o + 1)$ variables, and we ask them to satisfy the following set of $2n + 1 + \frac{1}{2}(n + m_o + 1)$ equations:

$$cA_1B_1^2 - A_oB_o^2 = cA_2B_2^2, \quad cA_1B_1^2 - \lambda A_oB_o^2 = cA_3B_3^2$$

and

$$t(t-1) \text{ divides } \Pi_iA_i \text{ and } A_oB_o \text{ divides } MA_1B_1 - B_2B_3 .$$

Let $P := cA_1B_1^2$, $Q := A_oB_o^2$ and $R := \frac{P}{Q}$, and assume further the open conditions

$$c \neq 0 \quad disc(\Pi_iA_i) \neq 0 \quad \text{and} \quad resultant(P, Q) \neq 0 .$$

Then, $R := \frac{P}{Q}$ is a degree-n morphism, with an associated *hyperelliptic cover* $\pi : p \in \Gamma \to q \in \mathbb{P}^1$ isomorphic (outside $\pi^{-1}(q)$), to

$$(t, v) \in \Big\{v^2 = c^3\Pi_iA_i\Big\} \quad \mapsto \quad \Big(R(t), \frac{v\Pi_iB_i}{Q^2}\Big) = (x, y) \in \Big\{y^2 = x(x-1)(x-\lambda)\Big\} .$$

Besides having *Weierstrass type* (m_i), the meromorphic function $\kappa := \frac{vM}{A_oB_o}$ satisfies all properties quoted in **4.6.**, implying that π is indeed a *hyperelliptic d-osculating cover*. Conversely, let $\pi \in HypOsc\big(n, d, (m_i)\big)$ be a degree-n *hyperelliptic d-osculating cover* of *Weierstrass type* $(m_i) \in \mathbb{N}^4$, associated to a rational fraction $R := \frac{P}{Q}$ as in **3.1.**. Consider the canonical factorizations of $P = cA_1B_1^2$, $Q = cA_1B_1^2$, $P - Q = cA_2B_2^2$ and $P - \lambda Q = cA_3B_3^2$, with unitary polynomials, (A_i) and (B_i), such that for any $i = 0, \cdots, 3$, $degA_i = m_i$ and $degB_i^2 = n - m_i - \delta_{i,o}$. Needless to say that they satisfy the equations

$$cA_1B_1^2\text{-}A_oB_o^2 = cA_2B_2^2, \quad cA_1B_1^2\text{-}\lambda A_oB_o^2 = cA_3B_3^2$$

and

$$t(t\text{-}1) \quad \text{divides} \quad \Pi_i A_i \,.$$

Moreover, there must be a meromorphic function satisfying properties **4.6.**, which must be uniquely expressed as $\frac{vM}{A_oB_o}$ for a unique polynomial $M(t)$ of degree $deg(A_oB_o)\text{-}\,g{+}d\text{-}\,1$. In other words, any class $\pi \in HypOsc\big(n, d, (m_i)\big)$ corresponds to a unique solution of the latter systems of equations (and open conditions). ■

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